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ON MINIMAL ISOTROPIC TORI IN $\mathbb{C}P^3$

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Abstract: We show that one of the classes of minimal tori in $\mathbb{C}P^3$ is determined by the smooth periodic solutions to the sinh-Gordon equation. We also construct examples of such surfaces in terms of Jacobi elliptic functions.

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1. Introduction

Isotropic submanifolds in a symplectic space are a natural generalization of Lagrange submanifolds. A surface $\Sigma \subset \mathbb{C}P^3$ is called *isotropic* if $\omega|_\Sigma = 0$, where ω is the Fubini–Study symplectic form on $\mathbb{C}P^3$. We will define isotropic surfaces as the image of the composition $\mathscr{H} \circ r : \mathbb{R}^2 \to \mathbb{C}P^3$, where $r : \mathbb{R}^2 \to S^7 \subset \mathbb{C}^4$ is the horizontal mapping into the unit sphere, and $\mathscr{H} : S^7 \to \mathbb{C}P^3$ is the Hopf projection. Assume moreover that, in the coordinates (x, y), the metric on the surface has the conformal form $ds^2 = 2e^{v(x,y)}(dx^2+dy^2)$. The isotropy of Σ, the horizontality of the metric r, and the conformality of the metric imply that

$$\langle r_x, r\rangle = \langle r_y, r\rangle = \langle r_x, r_y\rangle = 0, \quad \langle r_x, r_x\rangle = \langle r_y, r_y\rangle = 2e^{v(x,y)};$$

moreover, $|r| = 1$, where $\langle\cdot,\cdot\rangle$ is the Hermitian product in $\mathbb{C}^4$. Thus, the unit vectors $r, \frac{r_x}{|r_x|}, \frac{r_y}{|r_y|} \in \mathbb{C}^4$ are pairwise orthogonal in the Hermitian sense. Complement this collection of vectors by the normal vector $n(x, y)$ to the Hermitian frame

$$R = \begin{pmatrix} r \\ \frac{r_x}{|r_x|} \\ \frac{r_y}{|r_y|} \\ n \end{pmatrix} \in SU(4).$$

Introduce matrices $A, B \in su(4)$ such that

$$R_x = AR, \quad R_y = BR. \tag{1}$$

In what follows, we assume that the image of $\mathscr{H} \circ r$ is a minimal immersed torus; i.e., $\mathscr{H} \circ r$ is a doubly periodic mapping with some lattice of periods.

Lemma 1. *The matrices $A, B \in su(4)$ have the form*

$$A = \begin{pmatrix} 0 & \sqrt{2}e^{\frac{v}{2}} & 0 & 0 \\ -\sqrt{2}e^{\frac{v}{2}} & \frac{k_1}{2}ie^{-v} & \frac{1}{2}(ik_2e^{-v} - v_y) & e^{-\frac{v}{2}}(V_1 + iV_3) \\ 0 & \frac{1}{2}(ik_2e^{-v} + v_y) & -\frac{k_1}{2}ie^{-v} & e^{-\frac{v}{2}}(V_2 + iV_4) \\ 0 & -e^{-\frac{v}{2}}(V_1 - iV_3) & -e^{-\frac{v}{2}}(V_2 - iV_4) & 0 \end{pmatrix},$$

$$B = \begin{pmatrix} 0 & 0 & \sqrt{2}e^{\frac{v}{2}} & 0 \\ 0 & \frac{k_2}{2}ie^{-v} & \frac{1}{2}(-ik_1e^{-v} + v_x) & e^{-\frac{v}{2}}(V_2 + iV_4) \\ -\sqrt{2}e^{\frac{v}{2}} & -\frac{1}{2}(ik_1e^{-v} + v_x) & -\frac{k_2}{2}ie^{-v} & -e^{-\frac{v}{2}}(V_1 + iV_3) \\ 0 & -e^{-\frac{v}{2}}(V_2 - iV_4) & e^{-\frac{v}{2}}(V_1 - iV_3) & 0 \end{pmatrix},$$

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$k_1, k_2 \in \mathbb{R}$, $V_i(x,y)$ *are some real doubly periodic functions. The compatibility condition*

$$A_y - B_x + [A,B] = 0 \tag{2}$$

for (1) *is equivalent to the system of equations:*

$$-V_2V_3 + V_1V_4 = 0, \tag{3}$$

$$4e^{v} - e^{-2v}\bigl(k_1^2 + k_2^2\bigr) - 2e^{-v}\bigl(V_1^2 + V_2^2 + V_3^2 + V_4^2\bigr) + \Delta v = 0, \tag{4}$$

$$k_2V_3 - k_1V_4 + e^{v}(V_{1y} - V_{2x}) = 0, \tag{5}$$

$$-k_2V_1 + k_1V_2 + e^{v}(V_{3y} - V_{4x}) = 0, \tag{6}$$

$$-k_1V_3 - k_2V_4 + e^{v}(V_{2y} + V_{1x}) = 0, \tag{7}$$

$$k_1V_1 + k_2V_2 + e^{v}(V_{4y} + V_{3x}) = 0. \tag{8}$$

We can distinguish two natural cases when this system amounts to a single equation. If $V_i = 0$ then $v(x,y)$ satisfies the Tzitzeica equation

$$\Delta v + 4e^{v(x,y)} - \bigl(k_1^2 + k_2^2\bigr)e^{-2v(x,y)} = 0.$$

In this case, the vector n is constant and the surface Σ lies in some complex projective plane $\mathbb{C}P^2 \subset \mathbb{C}P^3$. Thus, Σ is a Lagrange surface in $\mathbb{C}P^2$. This case was studied particularly in many works (see, for example, [1–8]), and we will not address it here.

The second natural case is distinguished by the conditions

$$k_1 = k_2 = 0, \quad V_i = p_i = \text{const}, \ i = 1,2, \quad p_3 = p_4 = 0. \tag{9}$$

Then $v(x,y)$ satisfies the sinh-Gordon equation

$$\Delta v + 4e^{v(x,y)} - 2\bigl(p_1^2 + p_2^2\bigr)e^{-v(x,y)} = 0. \tag{10}$$

Equation (10) appears, for example, in the theory of surfaces of constant mean curvature in $\mathbb{R}^3$ [9] and also in mathematical physics (see, for example, [10]). Smooth periodic and quasiperiodic solutions to (10) were studied in [11].

Give an example of an isotropic minimal surface corresponding to (10). Let $c > 0$ and p_1 and p_2 be real numbers such that $c^2 > 2\bigl(p_1^2 + p_2^2\bigr)$, $p_1 \neq 0$. Put

$$\gamma_1 = \frac{c + \sqrt{c^2 - 2\bigl(p_1^2 + p_2^2\bigr)}}{2}, \quad \gamma_2 = \frac{c - \sqrt{c^2 - 2\bigl(p_1^2 + p_2^2\bigr)}}{2}.$$

Let $\pm\alpha_1$ and $\pm\alpha_2$ be the roots of the equation

$$\frac{\alpha^4}{2} - c\alpha^2 + p_2^2 = 0, \tag{11}$$

where $\alpha_1 > \alpha_2 > 0$.

We have

Theorem 1. *The mapping* $\mathscr{H} \circ r$, *where*

$$r = \bigl(F_1(x)e^{i(G_1(x)+\alpha_1 y)}, F_1(x)e^{-i(G_1(x)+\alpha_1 y)}, F_2(x)e^{i(G_2(x)+\alpha_2 y)}, F_2(x)e^{-i(G_2(x)+\alpha_2 y)}\bigr), \tag{12}$$

defines a minimal isotropic immersion of $\mathbb{R}^2$ *in* $\mathbb{C}P^3$. *Here*

$$F_1 = \sqrt{\frac{2e^{v(x)} - \alpha_2^2}{2\bigl(\alpha_1^2 - \alpha_2^2\bigr)}}, \quad F_2 = \sqrt{\frac{\alpha_1^2 - 2e^{v(x)}}{2\bigl(\alpha_1^2 - \alpha_2^2\bigr)}}, \tag{13}$$

$$G_1 = 2p_1p_2\int_0^x \frac{dz}{\alpha_1\bigl(\alpha_2^2 - 2e^{v(z)}\bigr)}, \quad G_2 = 2p_1p_2\int_0^x \frac{dz}{\alpha_2\bigl(\alpha_1^2 - 2e^{v(z)}\bigr)}, \tag{14}$$

$$e^{v(x)} = \gamma_1 - (\gamma_1 - \gamma_2)sn^2\left(\sqrt{2\gamma_1}x, \sqrt{\frac{\gamma_1 - \gamma_2}{\gamma_1}}\right),$$

$sn(z,k)$ *is a Jacobi elliptic function.*

Note that $\mathscr{H} \circ r : \mathbb{R}^2 \to \mathbb{C}P^3$ is periodic in y for $\frac{\alpha_2}{\alpha_1} \in \mathbb{Q}$. Let τ be the period of $v(x)$. Then $\mathscr{H} \circ r$ is periodic in x if $G_1(\tau), G_2(\tau) \in \pi\mathbb{Q}$.

2. Proofs of Lemma 1 and Theorem 1

Equations (1) and the condition $A, B \in su(4)$ imply that A and B look as

$$A = \begin{pmatrix} 0 & \sqrt{2}e^{\frac{v}{2}} & 0 & 0 \\ -\sqrt{2}e^{\frac{v}{2}} & if_1 & a_1+ib_1 & a_2+ib_2 \\ 0 & -a_1+ib_1 & if_2 & a_3+ib_3 \\ 0 & -a_2+ib_2 & -a_3+ib_3 & -i(f_1+f_2) \end{pmatrix},$$

$$B = \begin{pmatrix} 0 & 0 & \sqrt{2}e^{\frac{v}{2}} & 0 \\ 0 & ig_1 & c_1+id_1 & c_2+id_2 \\ -\sqrt{2}e^{\frac{v}{2}} & -c_1+id_1 & ig_2 & c_3+id_3 \\ 0 & -c_2+id_2 & -c_3+id_3 & -i(g_1+g_2) \end{pmatrix},$$

where $a_i(x,y)$, $b_i(x,y)$, and $c_i(x,y)$ are real functions. Furthermore, from (1) we obtain the equalities

$$r_{xx} = \sqrt{2}e^{\frac{v}{2}}(a_2+ib_2)n + \left(\frac{v_x}{2} + if_1\right)r_x + (a_1+ib_1)r_y - 2e^v r,$$

$$r_{yy} = \sqrt{2}e^{\frac{v}{2}}(c_3+id_3)n + (id_1 - c_1)r_x + \left(\frac{v_y}{2} + ig_2\right)r_y - 2e^v r,$$

$$r_{xy} = \sqrt{2}e^{\frac{v}{2}}(a_3+ib_3)n + (-a_1+ib_1)r_x + \left(\frac{v_x}{2} + if_2\right)r_y,$$

$$r_{xy} = \sqrt{2}e^{\frac{v}{2}}(c_2+id_2)n + \left(\frac{v_y}{2} + ig_1\right)r_x + (c_1+id_1)r_y.$$

Consequently, $c_2 = a_3$, $d_2 = b_3$, $b_1 = g_1$, $a_1 = -\frac{v_y}{2}$, $c_1 = \frac{v_x}{2}$, $d_1 = f_2$,

$$\Delta r = i(f_1+f_2)r_x + i(g_1+g_2)r_y - 4e^v r + \sqrt{2}e^{\frac{v}{2}}(a_2+c_3+i(b_2+d_3))n. \tag{15}$$

Since the image of $\mathscr{H}\circ r : \mathbb{R}^2 \to \mathbb{C}P^3$ is a minimal surface, its horizontal lifting $r : \mathbb{R}^2 \to S^7$ is minimal in S^7. The mapping r defines a minimal surface if and only if $\Delta_{LB} r = -2r$, where Δ_{LB} is the Laplace–Beltrami operator for the induced metric $ds^2 = 2e^{v(x,y)}(dx^2+dy^2)$ on the surface (see, for example, [12]) or, equivalently, $\Delta r = -4e^v r$. Consequently, from (15) we obtain $f_2 = -f_1$, $g_2 = -g_1$, $c_3 = -a_2$, and $d_3 = -b_2$. Thus, A and B take the form

$$A = \begin{pmatrix} 0 & \sqrt{2}e^{\frac{v}{2}} & 0 & 0 \\ -\sqrt{2}e^{\frac{v}{2}} & if_1 & ig_1 - \frac{v_y}{2} & a_2+ib_2 \\ 0 & ig_1 + \frac{v_y}{2} & -if_1 & a_3+ib_3 \\ 0 & -a_2+ib_2 & -a_3+ib_3 & 0 \end{pmatrix},$$

$$B = \begin{pmatrix} 0 & 0 & \sqrt{2}e^{\frac{1}{2}v} & 0 \\ 0 & ig_1 & -if_1 + \frac{v_x}{2} & a_3+ib_3 \\ -\sqrt{2}e^{\frac{v}{2}} & -f_1 - \frac{v_x}{2} & -ig_1 & -a_2-ib_2 \\ 0 & -a_3+ib_3 & a_2-ib_2 & 0 \end{pmatrix}.$$

Relation (2) implies the equations

$$f_{1y} + f_1 v_y - g_{1x} - g_1 v_x = 0, \quad g_{1y} + g_1 v_y + f_{1x} + f_1 v_x = 0,$$

which are equivalent to

$$\partial_y(f_1 e^v) - \partial_x(g_1 e^v) = 0, \quad \partial_y(g_1 e^v) + \partial_x(f_1 e^v) = 0.$$

Since f_1, g_1, and υ are doubly periodic, we have

$$f_1 = \frac{k_1}{2}e^{-\upsilon}, \quad g_1 = \frac{k_2}{2}e^{-\upsilon}, \quad k_1, k_2 \in \mathbb{R}.$$

Introduce the notations:

$$V_1 = a_2 e^{\frac{\upsilon}{2}}, \quad V_2 = a_3 e^{\frac{\upsilon}{2}}, \quad V_3 = b_2 e^{\frac{\upsilon}{2}}, \quad V_4 = b_3 e^{\frac{\upsilon}{2}}.$$

Then the matrices A and B take the form of Lemma 1, and (2) is equivalent to (3)–(8). Lemma 1 is proved.

Turn to proving Theorem 1. We will assume that (9) and (10) are fulfilled, the function υ depends only on x, and put $p_3 = p_4 = 0$. It follows from (10) that υ satisfies the equation

$$(\upsilon')^2 = -8e^{\upsilon} - 4e^{-\upsilon}\big(p_1^2 + p_2^2\big) + 8c, \tag{16}$$

where c is a real constant. We will search for r_j and n_j in the form

$$r_j = R_j(x)e^{i\beta_j y}, \quad n_j = N_j(x)e^{i\beta_j y},$$

where $R_j(x)$ and $N_j(x)$ are complex-valued functions, $\beta_j \in \mathbb{R}$. Relation (1) implies the equalities

$$-4p_1 N_j + \sqrt{2}(2R_j'' + 4e^{\upsilon}R_j - R_j'\upsilon') = 0, \tag{17}$$

$$4p_2 N_j - \beta_j i\sqrt{2}(2R_j' - R_j\upsilon') = 0, \tag{18}$$

$$\sqrt{2}ip_2\beta_j R_j + 2e^{\upsilon}N_j' + \sqrt{2}p_1 R_j' = 0, \tag{19}$$

$$4p_1 N_j + \sqrt{2}\big(R_j\big(4e^{\upsilon} - 2\beta_j^2\big) + R_j'\upsilon'\big) = 0, \tag{20}$$

$$2ie^{\upsilon}\beta_j N_j - \beta_j i\sqrt{2}p_1 R_j + \sqrt{2}p_2 R_j' = 0. \tag{21}$$

Put $R_j(x) = F_j(x)e^{iG_j(x)}$, where $F_j(x)$ and $G_j(x)$ are real functions. Then from (18) and (21) we obtain

$$F_j(x) = s_j\sqrt{p_2^2 - \beta_j^2 e^{\upsilon(x)}}, \quad G_j'(x) = \frac{\beta_j p_1 p_2}{p_2^2 - \beta_j^2 e^{\upsilon(x)}},$$

$$N_j(x) = \frac{\beta_j p_1 R_j(x) + ip_2 R_j'(x)}{\beta_j\sqrt{2}e^{\upsilon}(x)},$$

where s_j and β_j are some constants. Straightforward calculations can show that (17), (19), and (20) hold if and only if β_j satisfy the equality

$$\frac{\beta_j^4}{2} - c\beta_j^2 + p_2^2 = 0.$$

Put $\beta_1 = \alpha_1$, $\beta_2 = -\alpha_1$, $\beta_3 = \alpha_2$, and $\beta_4 = -\alpha_2$, where $\pm\alpha_1$ and $\pm\alpha_2$ are the roots of (11). The equalities

$$|r| = 1, \quad \langle r, r_x\rangle = \langle r, r_y\rangle = \langle r_x, r_y\rangle = 0$$

give the following relations on the constants s_k, $k = 1, \dots, 4$, α_j, $j = 1, 2$:

$$\big(s_1^2 + s_2^2\big)\alpha_1^2 + \big(s_3^2 + s_4^2\big)\alpha_2^2 = 0,$$

$$\big(p_2^2 + p_4^2\big)\big(s_1^2 + s_2^2 + s_3^2 + s_4^2\big) = \frac{\alpha_1^2\alpha_2^2}{2}\big(s_1^2 + s_2^2 + s_3^2 + s_4^2\big) = 1,$$

$$\big(s_1^2 - s_2^2\big)\alpha_1 + \big(s_3^2 - s_4^2\big)\alpha_2 = 0, \quad \big(s_1^2 - s_2^2\big)\alpha_1^3 + (s_3^2 - s_4^2\big)\alpha_2^3 = 0.$$

The last system implies that $s_2 = \pm s_1$ and $s_4 = \pm s_3$. We will assume that $s_2 = s_1$ and $s_4 = s_3$ (the other cases are reduced to this by means of an automorphism of $\mathbb{C}P^3$). This yields

$$s_1 = \pm \frac{1}{\alpha_1\sqrt{\alpha_2^2 - \alpha_1^2}}, \quad s_3 = \pm \frac{i}{\alpha_2\sqrt{\alpha_2^2 - \alpha_1^2}},$$

and so (13) and (14). Straightforward calculations show that, for r defined by (12), the matrix R belongs to $SU(4)$. Find smooth periodic solutions to (16). Perform the change $h(x) = e^{v(x)}$; then (16) takes the form

$$(h')^2 + 8h(h-\gamma_1)(h-\gamma_2) = (h')^2 + 8h^3 - ch^2 + 4h\big(p_1^2 + p_2^2\big) = 0, \tag{22}$$

where γ_1 and γ_2 are the roots of the equation $\gamma^2 - c\gamma + \frac{1}{2}\big(p_1^2 + p_2^2\big) = 0$. The identity $(sn(x,k)')^2 = (1 - sn^2(x,k))(1 - k^2 sn^2(x,k))$ (see [13]), where $sn(x,k) = \sin\varphi$, φ is the inverse function to

$$\omega(\varphi) = \int\limits_0^{\varphi} \frac{dt}{\sqrt{1 - k^2\sin^2 t}}, \quad 0 < k < 1,$$

implies that (22) has a solution of the form

$$h(x) = e^{v(x)} = \gamma_1 - \left(\gamma_1 - \gamma_2)sn^2(\sqrt{2\gamma_1}x, \sqrt{\frac{\gamma_1 - \gamma_2}{\gamma_1}}\right).$$

The function $v(x)$ is periodic with period $\tau = \frac{1}{\sqrt{2\gamma_1}}\int_0^{2\pi} \frac{dt}{\sqrt{1-k^2\sin^2 t}}$.

Note that $\alpha_1^2 > 2\gamma_1 \geq 2e^{v(x)} \geq 2\gamma_2 > \alpha_2^2$. Therefore, $\alpha_2^2 - 2e^{v(x)} \neq 0$, $\alpha_1^2 - 2e^{v(x)} \neq 0$, and so G_1 and G_2 are smooth functions. Theorem 1 is proved.

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