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ON THE CENTRALIZER DIMENSION AND LATTICE OF GENERALIZED BAUMSLAG–SOLITAR GROUPS

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Abstract: A *generalized Baumslag–Solitar group* (a GBS group) is a finitely generated group G acting on a tree so that all vertex and edge stabilizers are infinite cyclic groups. Each GBS group is the fundamental group $\pi_1(\mathbb{A})$ of some labeled graph $\mathbb{A}$. We describe the centralizers of elements and the centralizer lattice. Also, we find the centralizer dimension for GBS groups if $\mathbb{A}$ is a labeled tree.

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Introduction

A finitely generated group G acting on a tree so that all vertex and edge stabilizers are infinite cyclic groups is called a *generalized Baumslag–Solitar group* (GBS group). By the Bass–Serre Theorem, G is representable as $\pi_1(\mathbb{A})$, the fundamental group of a graph of groups $\mathbb{A}$ [1] whose vertex and edge groups are infinite cyclic.

With each GBS group G, we can associate a labeled graph $\mathbb{A}$, a particular case of a graph of groups. Such a labeled graph corresponds to an action of G on a tree and defines a presentation of G (more details on labeled graphs and their properties are given in [2]). If $\mathbb{A}$ is a tree of groups then the group G can be constructed by means of the construction of an amalgamated product from infinite cyclic groups.

As was observed by Robinson in [3], the GBS groups occupy central positions in combinatorial group theory due to the following properties: Noncyclic GBS groups are exactly those finitely generated groups of cohomological dimension 2 having a commensurable cyclic group; GBS groups are coherent (each finitely generated subgroup admits a finite presentation).

Let G be a group and let M be a subset of G. Denote by $C(M)$ the *centralizer* of M in G:

$$C(M) = \{g \in G \mid g^{-1}mg = m \text{ for all } m \in M\}.$$

Suppose that a group G has a strictly decreasing chain of centralizers $C_1 \supset C_2 \supset \cdots \supset C_d$ of length d, i.e., a chain contains exactly d elements, but does not contain such a chain of length $d+1$. Then the *centralizer dimension* $\operatorname{cdim}(G)$ equals d. If there is no such number d then we put $\operatorname{cdim}(G) = \infty$. More complete information on the centralizer dimensions of groups can be found in [4].

It was observed in [5] that $\operatorname{cdim}(G)$ coincides with the height of the centralizer lattice and proved that for each $m \geq 1$ the class of groups CD_m of a given centralizer dimension m is axiomatized by a universal formula of the first-order logic. Therefore, the description of the centralizer dimension of GBS groups can contribute to the study of the universal theory of GBS groups and the solution of the question of their universal equivalence.

In this article we describe the centralizers of elements, the centralizer lattice, and find the centralizer dimension of the group $\pi_1(\mathbb{A})$ if $\mathbb{A}$ is a labeled tree.

§ 1. *Z*-Maximal Subtrees

A *graph* A is the vertex set $V(A)$, the edge set $E(A)$, the mappings $\partial_0, \partial_1 : E(A) \to V(A)$, are sending an edge to its beginning and end, and an inversion $\bar{\ } : E(A) \to E(A)$ such that $\partial_0(\bar{e}) = \partial_1(e)$,

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and $\partial_1(\bar{e}) = \partial_0(e)$, $\bar{\bar{e}} = e$, $\bar{e} \neq e$. An *edge path* is a sequence of edges $p = (e_1, e_2, \ldots, e_k)$ such that $\partial_0(e_{i+1}) = \partial_1(e_i)$ for $i = 1, 2, \ldots, k-1$.

If A is a tree then for every two vertices a and b there exists a unique shortest path with beginning a and end b. We will refer to this path as *geodesic* and denote it by $p(a, b)$.

With every GBS group G, we can associate a *labeled graph* $\mathbb{A} = (A, \lambda)$, where A is a finite connected graph and $\lambda : E(A) \to \mathbb{Z} \setminus \{0\}$ are labels on the edges of A. The label $\lambda_e = \lambda(e)$ on an edge e with beginning at a vertex v defines the embedding $\alpha_e : e \to v^{\lambda(e)}$ of the cyclic edge group $\langle e \rangle$ into the cyclic vertex group $\langle v \rangle$ (more details on labeled graphs and their properties can be found in [2]).

The *fundamental group* $\pi_1(\mathbb{A})$ *of a labeled graph* $\mathbb{A} = (A, \lambda)$ is defined by generators and defining relations. Denote by $\overline{A}$ the graph obtained from A by identifying e and $\bar{e}$. The maximal subtree T in $\overline{A}$ defines the presentation of the group $\pi_1(\mathbb{A})$

$$\left\langle \begin{array}{ccc} g_v,\ v \in V(\overline{A}) & \mid & g_{\partial_0(e)}^{\lambda(e)} = g_{\partial_1(e)}^{\lambda(\bar{e})},\ e \in E(T) \\ t_e,\ e \in E(\overline{A}) \setminus E(T) & \mid & t_e^{-1} g_{\partial_0(e)}^{\lambda(e)} t_e = g_{\partial_1(e)}^{\lambda(\bar{e})},\ e \in E(\overline{A}) \setminus E(T) \end{array} \right\rangle.$$

For different maximal subtrees, the corresponding presentations define isomorphic groups. If A is a tree then $\pi_1(\mathbb{A})$ admits the presentation

$$\left\langle g_v,\ v \in V(\overline{A}) \mid g_{\partial_0(e)}^{\lambda(e)} = g_{\partial_1(e)}^{\lambda(\bar{e})},\ e \in E(T) \right\rangle.$$

In what follows, we for convenience denote by v the vertex of the graph as well as the corresponding generator g_v of the fundamental group. To each subtree T of a tree A, there naturally corresponds the labeled graph $\mathbb{T}$, where the natural homomorphism $\pi_1(\mathbb{T}) \to \pi_1(\mathbb{A})$ is an embedding.

If two labeled graphs $\mathbb{A}$ and $\mathbb{B}$ define isomorphic GBS groups $\pi_1(\mathbb{A}) \cong \pi_1(\mathbb{B})$ and $\pi_1(\mathbb{A})$ is not isomorphic to $\mathbb{Z}$ and $\mathbb{Z}^2$ or to the Klein bottle group $\langle a, b \mid a^{-1}ba = b^{-1} \rangle$ then there exists a finite sequence of *expansions* and *collapses* (Fig. 1) joining $\mathbb{A}$ and $\mathbb{B}$ [6] (in Fig. 1, to each edge there correspond two integers $\lambda(e)$ and $\lambda(\bar{e})$). A labeled graph is called *reduced* if it does not admit collapses (for a labeled tree, this means that there are no edges with labels ± 1).

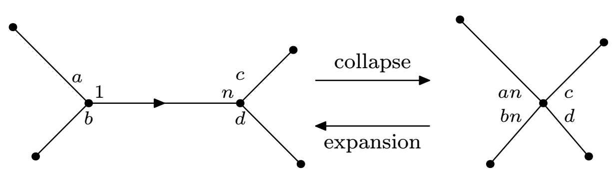

Fig. 1. Collapse and expansion

Let a group G be the amalgamated product of groups A and B with amalgamated subgroups $K \subseteq A$ and $L \subseteq B$. A sequence $c_1, c_2, \ldots, c_n$, $n \geq 0$, of elements in G is called *reduced* (see [7, Chapter 4, § 2]) if

(1) each c_i lies in one of the factors A or B;

(2) consecutive c_i and c_{i+1} lie in different factors;

(3) if $n > 1$ then no c_i lies in K and L;

(4) if $n = 1$ then $c_1 \neq 1$.

It was also proved there each element $g \in G$ is representable as $g = c_1 \cdot c_2 \cdot \ldots \cdot c_n$ of the product of the elements of some reduced sequence $c_1, c_2, \ldots, c_n$, and if $n \geq 1$ then $g \neq 1$ in G. The number n is called the *length* of g with respect to the given decomposition of G into an amalgamated product and denoted by $|g|$. Observe that if $g \in B *_{K=L} C$ is represented by a reduced sequence then $|g \cdot h| = |g|$ for all $h \in K = L$.

A sequence $c_1, c_2, \ldots, c_n$, $n \geq 0$, is called *cyclically reduced* if its every cyclic permutation is reduced. Each $g \in G$ is representable in the reduced form $g = u \cdot v \cdot u^{-1}$, where $v = c_1 \cdot c_2 \cdot \ldots \cdot c_n$ for a suitable cyclically reduced sequence $c_1, c_2, \ldots, c_n$.

Unless otherwise specified, we assume further that $\mathbb{A}$ is a reduced labeled tree and $G = \pi_1(\mathbb{A})$ is the corresponding GBS group.

If e is an edge of a tree A then G has the representation

$$G \simeq B *_{K=L} C,$$

where $B = \pi_1(\mathbb{A}_1)$, $C = \pi_1(\mathbb{A}_2)$, and $K \simeq L \simeq \mathbb{Z}$, so that A_1 and A_2 are two components of A without the edge e. Such a representation will be called the *e-decomposition* of G. Further, the equality $G = B *_{K=L} C$ means in the article that this decomposition is the e-decomposition for a suitable edge e. If $g \in G$ is represented by a reduced (cyclically reduced) sequence with respect to the e-decomposition of G then we will write briefly that g is *e-reduced* (*e-cyclically reduced*).

The product of two e-reduced elements $g = c_1 \cdot c_2 \cdot \ldots \cdot c_n$ and $h = h_1 \cdot h_2 \cdot \ldots \cdot h_m$ gives the element $g \cdot h$ which is representable by the reduced sequence $c_1, \ldots, c_{l-1}, c_l', h_{n-l+1}, \ldots, h_m$, where $c_{l+1} \cdot \ldots \cdot c_n \cdot h_1 \cdot \ldots \cdot h_{n-l} = b \in K = L$ and $c_l' = c_l \cdot b$. In this case, we say that the word $c_1 \cdot \ldots \cdot c_n \cdot h_1 \cdot \ldots \cdot h_m$ *was subject to $n - l$ reductions.*

We say that $g = c_1 \cdot c_2 \cdot \ldots \cdot c_n \in G$ is *reduced* for $n \geq 2$, $c_i = a_i^{\alpha_i}$, and $a_i \in V(A)$, if c_i and c_{i+1}, $i = 1, \ldots, k-1$, do not lie in $\langle a \rangle$, $a \in V(A)$. This definition agrees with the notion of e-reducedness since, for every $e \in E(A)$, the element g can be written down in the e-reduced form $g = b_1 \cdot b_2 \cdot \ldots \cdot b_k$, so that $b_1 = c_1 \ldots c_{i_1}$, $b_2 = c_{i_1} \ldots c_{i_2}$, $\ldots$, and $b_k = c_{i_{k-1}} \ldots c_n$.

Similarly, we say that $g = c_1 \cdot c_2 \cdot \ldots \cdot c_n \in G$, $n \geq 1$, $c_i = a_i^{\alpha_i}$, $a_i \in V(A)$ *is cyclically reduced* if $g = c_1 \cdot c_2 \cdot \ldots \cdot c_n$ is reduced, where c_1 and c_n do not lie in $\langle a \rangle$, $a \in V(A)$. Note that a reduced element $g = c_1 \cdot c_2 \ldots c_n$ is equal to 1 only if $n = 1$ and $c_1 = 1$ (see, for example, [7]).

The propositions "$g = c_1 \cdot \ldots \cdot c_n \in G$ is reduced" (cyclically reduced) and "e-reduced" (e-cyclically reduced) have different meanings. The second is used only for a fixed decomposition of G into an amalgamated product. Using both notions turns out more convenient than relying upon one of them.

In accordance with [6], call an element *elliptic* if it is conjugate to an element in $\langle a \rangle$ for some $a \in V(A)$; otherwise, the element is called *hyperbolic*. An elliptic element is called a *vertex* element if it lies in $\langle a \rangle$ for some $a \in V(A)$.

Define the modular homomorphism $\Delta : G \to \mathbb{Q}^*$. Given $g \in G$, choose an arbitrary nontrivial elliptic element $a \in G$. Then, for some integers m and n not equal to 0, we have $g^{-1} a^m g = a^n$. In this case, we put $\Delta(g) = \frac{m}{n}$. It is not hard to prove that this definition is correct. The modular homomorphism plays an important role in the study of GBS groups. If A is a tree then $\Delta(\pi_1(\mathbb{A})) = \{1\}$ (details and proofs can be found, for example, in [6]).

Lemma 1. *Let g and h be vertex elements of a group G. For $[g, h] = 1$ it is necessary and sufficient that $h = a^k$ and $g = a^l$ for some $k, l \in \mathbb{Z}$, $a \in V(A)$.*

Proof. Sufficiency is obvious. Prove necessity. Let $a, b \in V(A)$ be vertices in A such that $g \in \langle a \rangle$, $h \in \langle b \rangle$. Denote by t the last vertex of the path $p(a, b)$ (starting from a) whose degree is equal to g. Denote by e the last edge of the geodesic $p(b, t)$ and consider the e-decomposition of the group G. By construction (Fig. 2), $g = t^m \notin K = L$, $g \in \pi_1(\mathbb{A}_2)$, $h \in \pi_1(\mathbb{A}_1)$, $h \notin \pi_1(\mathbb{A}_2)$. So the element $h^{-1} \cdot g^{-1} \cdot h \cdot g$ is e-reduced and cannot be 1; a contradiction. The lemma is proved.

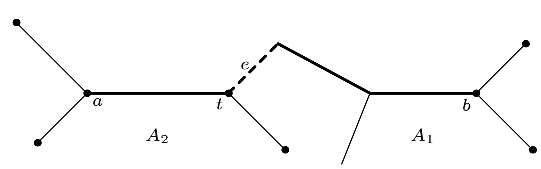

Fig. 2. The e-decomposition in Lemma 1

Proposition 2. *Let T be a subtree in A. Then*

$$Z(\pi_1(\mathbb{T})) = \bigcap_{a \in V(T)} \langle a \rangle \simeq \mathbb{Z}.$$

Proof. The inclusion $\supseteq$ is obvious. Prove the reverse inclusion $\subseteq$. Let $g = c_1c_2 \dots c_n \in Z(\pi_1(\mathbb{T}))$ be reduced. Then, for each $a \in V(T)$,

$$c_n^{-1} \dots c_2^{-1}c_1^{-1}a^{-1}c_1c_2 \dots c_n a = 1.$$

Equality is possible only if the element $c_1^{-1}a^{-1}c_1$ lies in $\langle a\rangle$ and is equal to the degree of a. Since $\Delta(G) = \{1\}$, we have $c_1^{-1}a^{-1}c_1 = a^{-1}$. Arguing similarly, we obtain $[c_j, a] = 1$, $1 \le j \le n$. Now the claim follows from Lemma 1. The proposition is proved.

If T_1 and T_2 are subtrees in a tree A then denote by $\langle T_1, T_2\rangle$ the minimal subtree containing T_1 and T_2.

Proposition 3. $Z(\pi_1(\mathbb{T}_1)) \cap Z(\pi_1(\mathbb{T}_2)) = Z(\pi_1(\langle \mathbb{T}_1, \mathbb{T}_2\rangle))$.

Proof. The inclusion $\supseteq$ is obvious. The reverse inclusion $\subseteq$ can be proved on using Proposition 2 by analogy to Lemma 1. Proposition 3 is proved.

Let $h \in G$ be a hyperbolic element. Denote by S_h the minimal subtree such that $h \in \pi_1(\mathbb{S}_h)$.

Corollary 4. *Let $g \in G$ be a hyperbolic element and let $a \in C_G(g)$ be an elliptic element. Then $a \in Z(\pi_1(\mathbb{S}_g))$.*

Proof. See the proof of Proposition 2.

Even if T_1 is a proper subtree in T_2 then $Z(\pi_1(\mathbb{T}_2))$ may be equal to $Z(\pi_1(\mathbb{T}_1))$. For example, the tree of groups $\mathbb{R}(k, s)$ (Fig. 3) is such that $Z(\pi_1(\mathbb{A})) = Z(\pi_1(\mathbb{R}(k, s))) = \langle v_1^{3^{k-1}}\rangle = \langle v_k^{2^{k-1}}\rangle$, where $A = \langle v_i$, $1 \le i \le k\rangle$. Adjusting the parameters k and s, we can change the number of subtrees with coinciding centers of the corresponding fundamental groups and the number of subtrees for which the centers of the fundamental groups are pairwise distinct. We will use this later for proving the main theorem.

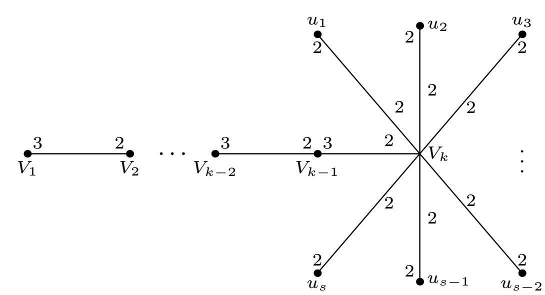

Fig. 3. The tree of groups $\mathbb{R}(k, s)$

Let T be a subtree in A. Proposition 2 implies that $Z(\pi_1(\mathbb{T})) \simeq \langle g\rangle$ is cyclic for a suitable vertex element $g \in G$. Denote by T_g the maximal subtree in A such that $Z(\pi_1(\mathbb{T}_g)) = \langle g\rangle$. Each of these subtrees will be called *Z-maximal*.

Proposition 5. *The definition of T_g is correct, and T_g is the greatest subtree with the above-mentioned condition.*

Proof. If there are two maximal subtrees T_1 and T_2: $\langle g\rangle = Z(\pi_1(\mathbb{T}_1)) = Z(\pi_1(\mathbb{T}_2))$, then $g \in Z(\pi_1(\mathbb{T}_1)) \cap Z(\pi_1(\mathbb{T}_2)) = Z(\pi_1(\langle \mathbb{T}_1, \mathbb{T}_2\rangle))$ and $\langle \mathbb{T}_1, \mathbb{T}_2\rangle \supset \mathbb{T}_1$. Thus, by Proposition 3, $Z(\pi_1(\langle \mathbb{T}_1, \mathbb{T}_2\rangle)) \subseteq Z(\pi_1(\mathbb{T}_1)) = \langle g\rangle$. Hence, $Z(\pi_1(\langle \mathbb{T}_1, \mathbb{T}_2\rangle)) = \langle g\rangle$. Since $\langle T_1, T_2\rangle \supset T_1$, we get a contradiction.

Remark 6. Let X be the set of the centers of all subtrees of a tree A and let Y be the set of all Z-maximal subtrees in A. The mapping $\mathscr{T} : X \to Y$, $Z(\pi_1(\mathbb{T})) = \langle g\rangle \to T_g$ is a bijection.

Remark 7. Each Z-maximal subtree in T_g is generated by the vertices a for which there is $k \in \mathbb{Z}$ with $g = a^k$.

Proof. By Proposition 2, some power of every vertex in T_g is equal to g. Conversely, if a is a vertex such that $g = a^k$ for suitable k then, arguing as in Lemma 1, we can see that $g \in Z(\pi_1(\langle T_g, a\rangle))$. By Proposition 3, $Z(\pi_1(\langle T_g, a\rangle)) = Z(\pi_1(T_g)) \cap Z(\pi_1(a)) \subseteq Z(\pi_1(T_g)) = \langle g\rangle$. Thus, $Z(\pi_1(\langle T_g, a\rangle)) = \langle g\rangle$. By the maximality of T_g, we obtain $a \in V(T_g)$. Remark 7 is proved.

§ 2. The Centralizers of Elements

Lemma 8. *Let r be a hyperbolic e-cyclically reduced element of the group $G = B *_{K=L} C$, $|r| > 1$. Put*

$$g = r^s b, \quad h = r^k c, \quad k \neq 0, \quad s \neq 0, \quad b, c \in K = L.$$

If $[g,h] = 1$ then there exist e-cyclic reduced $r_1 \in G$, $b_1, c_1 \in C_{K=L}(r_1)$ and integers $s_1 \neq 0$, $k_1 \neq 0$ such that

$$g = r_1^{s_1} b_1, \quad h = r_1^{k_1} c_1.$$

Proof. If $s = k$ then $g = r^s b = hc^{-1}b$. After the redenotation $r_1 = h$, $s_1 = k_1 = 1$, $c_1 = 1$, $b_1 = c^{-1}b$, we get what was required because

$$1 = [g,h] = [r_1 b_1, r_1 c_1].$$

If $s \neq k$ then we may assume without loss of generality that $s > k$ (otherwise, replace g by h). We may also assume that $s > 0$ (otherwise, replace r by r^{-1}). Rewrite $[r^s b, r^k c] = 1$ with more detail:

$$b^{-1} \cdot r^{-s} \cdot c^{-1} \cdot r^{s-k} \cdot b \cdot r^k \cdot c = 1.$$

If $k > 0$ then reductions in the subword $r^{s-k} \cdot b \cdot r^k$ are impossible since r is e-cyclically reduced and $b \in K = L$. Hence, reductions begin in the subword $r^{-s} \cdot c^{-1} \cdot r^{s-k}$. Therefore, $r^{-1} \cdot c^{-1} \cdot r = c_0 \in K = L$, and since $K = L \cong \mathbb{Z}$, we have $[r, c] = 1$ because $\Delta(G) = \{1\}$ and, similarly, $[r, b] = 1$.

If $k < 0$ and there is no full reduction of the subword $r^{-1} \cdot c^{-1} \cdot r$ or $r \cdot b \cdot r^{-1}$ (if there is at least one such reduction then act as above) then $s - k = 1$; consequently, $k = s - 1 \geq 0$; a contradiction. Lemma 8 is proved.

Proposition 9. *Suppose that hyperbolic elements g and h in G commute. Then there exist e-cyclically reduced $r \in G$, $w \in G$, and $b, c \in L = K$ for a suitable e-edge decomposition $G = B *_{K=L} C$ such that $|r| > 1$,*

$$g = w^{-1}(r^s b)w, \quad h = w^{-1}(r^k c)w, \quad k \neq 0, \ s \neq 0,$$

and $b, c \in C_{K=L}(r)$.

Proof. Let us demonstrate that $E(S_g \cap S_h) \neq \varnothing$. If $E(S_g \cap S_h) = \varnothing$ then either $V(S_g \cap S_h) = \varnothing$ or $V(S_g \cap S_h) \neq \varnothing$.

In the first case, g and h lie in the different factors of a suitable f-decomposition of G and so they cannot commute.

In the second case, $V(S_g \cap S_h) = \{v\}$. Choose an edge f with beginning v so that $g \in D$, where D is a component of the f-decomposition $G = D *_{M=R} F$. Suppose that $h = h_1 \cdot h_2 \dots h_n$, $n \geq 2$, is e-reduced. The equality $[g,h] = 1$ takes the form

$$g^{-1} \cdot h_n^{-1} \dots h_1^{-1} \cdot g \cdot h_1 \dots h_{n-1} \cdot h_n = 1.$$

If $h_1 \notin D$ and $h_n^{-1} \notin D$ then the left-hand side of the equality is e-reduced and cannot be 1. If $h_1 \notin D$ and $h_n^{-1} \in D$ then $g^{-1} \cdot h_n^{-1} \in D$ but $g^{-1} \cdot h_n^{-1} \notin F$ since $g \notin \langle v\rangle$.

If $h_1 \in D$ then $h_1 \in \langle v\rangle$. Note that $h_1^{-1} \cdot g \cdot h_1 \notin M = R \cong \mathbb{Z}$; otherwise, $h_1^{-1} \cdot g \cdot h_1 \in \langle v\rangle$, and hence $g \in \langle v\rangle$; a contradiction.

Hence, $E(S_g \cap S_h) \neq \varnothing$. Let $e \in E(S_g \cap S_h)$ and let $G = B *_{K=L} C$ be the e-decomposition. Suppose that $g = c_1 \cdot c_2 \cdot \dots \cdot c_n \in G$, $n \geq 2$, is e-reduced, and $h = t_1 \cdot t_2 \cdot \dots \cdot t_m \in G$, $m \geq 2$, is e-reduced. Assume that $g = x^{-1}ux$, u is e-cyclically reduced and $h = x^{-1}vx$, v is e-reduced. Since $[u, v] = 1 \Leftrightarrow [g, h] = 1$, conjugating by suitable w, assume that g is e-cyclically reduced and $|g| > 1$. In particular, $|g|$ is even. Without loss of generality (up to the inversion of g), we may assume that $c_1 \in B$, $c_n \in C$. Consider variants for h. Up to conjugacy, we obtain the two variants of a representation for the equality $[g, h] = 1$.

Reductions in the word $c_1 \dots c_n t_1 \dots t_m c_n^{-1} \dots c_1^{-1} t_m^{-1} \dots t_1^{-1} = 1$ can happen only at the junction of c_1^{-1} and t_m^{-1} in the case of $t_1 \in B$, $t_m \in B$ (the case of $t_1 \in C$ and $t_m \in C$ is analogous). This variant is impossible since, in this case, the number of reductions is at most m, and the equality 1 requires $n+m$ reductions.

Reductions in the word $c_1 \dots c_n t_1 \dots t_m c_n^{-1} \dots c_1^{-1} t_m^{-1} \dots t_1^{-1} = 1$ begin only in the middle at the junction $t_m c_n^{-1}$ in the case of $t_1 \in B$, $t_m \in C$ (the case of $t_1 \in C$ and $t_m \in B$ is similar). Under such reduction, either g^{-1} or h is completely reduced. Let $m \geq n$. Then $h = t_1 \dots t_m = t_1 \dots t'_{m-n} \cdot g = h_1 \cdot g$, where $t'_{m-n} = t_{m-n} \cdot d$ for some $d \in K = L$. Moreover, $[h, g] = 1 \Leftrightarrow [h_1, g] = 1$. This makes it possible to prove the proposition by induction on $m + n$.

The induction base. Since $n, m \geq 2$, it follows that $n + m = 4$ is the minimal variant. We have

$$c_1 c_2 t_1 t_2 c_2^{-1} c_1^{-1} t_2^{-1} t_1^{-1} = 1.$$

If $t_1 \in B$ and $t_2 \in C$ then reductions begin at the junction of t_2 and c_2^{-1} and last to the end. It follows that $g = h \cdot b$, $b \in K = L$, and b lies in $C_{K=L}(r)$ by Lemma 8. The case of $t_1 \in C$ and $t_2 \in B$ is similar.

The induction step. Arguing as above, we obtain $h = h_1 \cdot g$ for $m \geq n$. If $|h_1| > 1$ then we can apply the induction hypothesis for the pair h_1, g. The elements g and h_1 have the form

$$g = w^{-1}(r^s b)w, \quad h_1 = w^{-1}(r^k c)w.$$

Then $h = h_1 \cdot g = w^{-1}(r^{k+s} cb)w$ and $k + s \neq 0$ since otherwise h is conjugate to an elliptic element. An application of Lemma 7 finishes the proof in this case.

If $|h_1| = 1$ then $[g, h_1] = 1$ only in the case when $h_1 \in C_{K=L}(g)$; then $h = g \cdot b$, $b \in C_{K=L}(g)$. Proposition 9 is proved.

Lemma 10. *Let u and v be hyperbolic elements of a group G such that $|u| > |v| > 1$ with respect to the decomposition $G = B *_{K=L} C$. Moreover, there are no reductions in the words $u \cdot v$ and $v \cdot u$ and $v \cdot u = c_1 \cdot u \cdot v$ for some $c_1 \in K = L$. Then $u = w^q \cdot a_1$ and $v = w^p \cdot a_2$ for suitable $a_i \in C_{K=L}(w)$, $i = 1, 2$.*

Proof. Proceed by induction on the remainder of the division of $|u|$ by $|v|$.

The induction base. Suppose that $|v|$ divides $|u|$. The condition $v \cdot u = c_1 \cdot u \cdot v$ implies that $u = u_1 \cdot v$. Reduction gives $u_1 = u_2 \cdot v$, etc. We obtain $u = a \cdot v^k$ for appropriate $a \in K = L$. Inserting this in the initial equality, we infer

$$v \cdot a \cdot v^k \cdot v^{-1} \cdot v^{-k} \cdot a^{-1} \cdot v^{-k} = c_1;$$

moreover,

$$\underline{v \cdot a \cdot v^{-1}} \cdot a^{-1} = c_1.$$

Reductions can begin only in the underlined word; hence, $v \cdot a \cdot v^{-1} \in K = L$. Since $\Delta(G) = 1$, we get $v \cdot a \cdot v^{-1} = a$.

The induction step. As earlier, observe that u ends with a power of v. Let $u = h \cdot v^k$; moreover, $|u| = k \cdot |v| + |h|$ and $|h| < |v|$. After inserting this in the equality, we find $v \cdot h = c_1 \cdot h \cdot v$, and hence we can use the induction assumption $v = w^p \cdot a_2$, $h = w^q \cdot a_1$, $a_i \in C_{K=L}(w)$. Then $u = h \cdot v^k = w^{q+kp} \cdot a_1 \cdot a_2^k$. Lemma 10 is proved.

Lemma 11. *Suppose that r_1 and r_2 are e-cyclically reduced in the group $G = B *_{K=L} C$, $|r_1|, |r_2| > 1$. If*

$$r_1^s \cdot b_1 = r_2^k \cdot b_2, \quad b_i \in C_{K=L}(r_i), \quad i = 1, 2, \ k \neq 0, \ s \neq 0,$$

and r_1 and r_2 are not representable as

$$r_i = h_i^{m_i} \cdot a_i, \quad a_i \in C_{K=L}(h_i),\ i = 1, 2,$$

then $k = s$ and $r_1 = r_2 \cdot b$, $b \in C_{K=L}(r_2)$.

PROOF. If $|r_1| = |r_2|$ then $k = s$. Then the equality immediately gives $r_1 = r_2 \cdot b$ and, as in the proof of Lemma 10, it is not hard to understand that $b \in C_{K=L}(r_2)$.

Let $|r_1| > |r_2|$. Then $s < k$. The equality $r_1^s \cdot b_1 = r_2^k \cdot b_2$ implies that $r_1 = r_2^q \cdot u$, where u is the beginning of r_2; i.e., $r_2 = u \cdot r_3$. Inserting this in the initial inequality, we infer

$$u \cdot \left(r_2^q \cdot u\right)^{s-1} \cdot b_1 = r_2^{k-q} \cdot b_2.$$

Reducing u from the left, we see that r_3 begins with u too. Reducing u as many times as possible, we get $r_2 = u^p \cdot v$, $|v| < |u|$. Note that the right-hand side of the initial equality ends with $u \cdot v \cdot b_2$, and the left-hand side, by $v \cdot u \cdot b_1$. Thus, $c \cdot u \cdot v \cdot b_2 = v \cdot u \cdot b_1$. Putting $c_1 = b_1^{-1} \cdot c \cdot b_2$, we get into the conditions of Lemma 10 and obtain a contradiction; hence, $|r_1| = |r_2|$. Lemma 11 is proved.

Corollary 12. *Let $g \in G = B *_{K=L} C$ be a hyperbolic element. Then*

$$C_G(g) = w^{-1} \cdot (\langle r \rangle \times \langle b \rangle) \cdot w,$$

where r is e-cyclically reduced, $|r| > 1$, and r is not representable as $h^m \cdot a$, $a \in C_{K=L}(h)$, and $\langle b \rangle = C_{K=L}(r)$.

PROOF. Let $h \in C_G(g)$ be a hyperbolic element. By Proposition 9,

$$g = w^{-1}(r^s \cdot b_0)w, \quad h = w^{-1}(r^k \cdot c)w, \quad k \neq 0,\ s \neq 0,\ b_0, c \in C_G(r).$$

Then $C_G(g) = w^{-1} \cdot C_G(r^s \cdot b_0) \cdot w$. We may assume that r is not representable as $h_1^m \cdot a$, $a \in C_{K=L}(h_1)$. Otherwise, we can take h_1 as r. If $h \in C_G(r^s \cdot b_0)$ then, by Proposition 9, $r^s \cdot b_0 = r_1^{s_1} \cdot b_1$ and $h = r_1^{s_2} \cdot b_2$, $b_1, b_2 \in C_{K=L}(r_1)$; by Lemma 11, $r_1 = r \cdot b_3$, $b_3 \in C_{K=L}(r)$, and hence $h \in \langle r \rangle \times \langle b \rangle$. The reverse inclusion is obvious.

Lemma 13. *Let g be a vertex element of G. Then $C_G(g) = \pi_1(\mathbb{T}_h)$ for a suitable Z-maximal subtree T_h.*

PROOF. Let V_g be the set of the vertices a for which there exists a number k such that $a^k = g$. Denote by T the tree on these vertices. Let $Z(\pi_1(\mathbb{T})) = \langle h \rangle = Z(\pi_1(\mathbb{T}_h)) \supseteq \langle g \rangle$. Then, by Remark 7, $T \subseteq T_h$ (the equality can fail since $\langle g \rangle$ can fail to be the center of the fundamental group of some tree). We obtain $g = h^k$ and $\pi_1(\mathbb{T}_h) \subseteq C_G(g)$.

Conversely, let $g' = c_1 \cdot c_2 \cdot \ldots \cdot c_n \in C_G(g)$ be reduced. Then, by Corollary 4, we get $[c_i, g] = 1$ and, by Lemma 1, c_i coincides with the degree of some vertex in V_g; hence, $g' \in \pi_1(\mathbb{T}) \subseteq \pi_1(\mathbb{T}_{g_1})$. The lemma is proved.

§ 3. Intersections of Centralizers

We showed in § 2 that the centralizer of a single element g is equal either to $w^{-1}(\langle r \rangle \times \langle b \rangle)w$, where r is e-cyclically reduced and is not a power of its subword, $|r| > 1$, $b \in C_{K=L}(r) \simeq \mathbb{Z}$ if g is a hyperbolic element, or $w^{-1} \cdot \pi_1(\mathbb{T}_h) \cdot w$ if g is an elliptic element. What variants are there for the stabilizers of finite collections of elements?

Lemma 14. *Let g_1 and g_2 be hyperbolic elements of G. Then $C_G(g_1, g_2) = C_G(g_1) \cap C_G(g_2)$ is equal either to $C_G(g_1) = C_G(g_2)$ or to $w^{-1} \cdot Z(\pi_1(\mathbb{T})) \cdot w$ for a suitable Z-maximal subtree T, $w \in G$.*

PROOF. Proposition 9 shows that

$$C_G(g_i) = w_i^{-1}(\langle r_i \rangle \times \langle b_i \rangle)w_i, \quad i = 1, 2.$$

If the centralizers intersect then there are $s, t, k, l \in \mathbb{Z}$ such that

$$w_1^{-1}\big(r_1^s \cdot b_1^t\big)w_1 = w_2^{-1}\big(r_2^k \cdot b_2^l\big)w_2.$$

It follows from [7, Chapter 4, Theorem 2.8] that e-cyclically reduced elements are conjugate if and only if one can be obtained from the other by a cyclic permutation up to conjugacy by an element of the amalgamated part. Hence, for some cyclic permutation $r_3 = v \cdot u = u^{-1} \cdot r_2 \cdot u$ of the element $r_2 = u \cdot v$ and $c \in K = L$, we have

$$r_1^s b_1^t = c^{-1} \cdot r_3^k \cdot b_2^l \cdot c.$$

If $r_4 = c^{-1} \cdot r_3 \cdot c$ then r_4 is represented by an e-cyclically reduced sequence (after correcting the first and last factors r_3 with the use of c^{-1} and c respectively) and

$$r_1^s b_1^t = r_4^k \cdot b_2^l.$$

If $k \neq 0$ and $s \neq 0$, then by Lemma 11 $k = s$ and $r_1 = r_4 \cdot b$, $b \in C_{K=L}(r_1)$. So,

$$r_1 = r_4 \cdot b = (u \cdot c)^{-1} \cdot r_2 \cdot (u \cdot c) \cdot b,$$

$$C_G(g_1) = (u \cdot c \cdot w_1)^{-1} \cdot (\langle r_2\rangle \times \langle b_2\rangle) \cdot (u \cdot c \cdot w_1).$$

If $r_2^s \cdot b_2^t = w^{-1} \cdot \big(r_2^k \cdot b_2^l\big) \cdot w$; then, as was shown above, $k = s$, and we can demonstrate that $t = l$. Therefore, $w \in C_G\big(r_2^s \cdot b_2^t\big)$ and, by Proposition 9, we obtain $w = r_2^p \cdot b_2^q$. If instead of w we consider $u \cdot c \cdot w_1 \cdot w_2^{-1}$ then $C_G(g_1) = C_G(g_2)$.

If $k = s = 0$ then $C_G(g_1) \cap C_G(g_2)$ contains only elements of the form $w_1^{-1} b_1^t w_1 = w_2^{-1} b_2^l w_2$ for some $t, l \in \mathbb{Z}$. By Lemma 13, $C_G\big(b_1^t\big)$ is equal to $w^{-1}\pi_1(\mathbb{T})w$ for a suitable Z-maximal subtree T and $w \in G$. Lemma 14 is proved.

Lemma 15. *Let $g \in G$ be a hyperbolic element and let T be a Z-maximal subtree of A. Then $C_G(g) \cap u^{-1} \cdot \pi_1(\mathbb{T}) \cdot u$ is equal either to $C_G(g) = w^{-1}(\langle r\rangle \times \langle b\rangle)w$ or to $v^{-1} \cdot Z(\pi_1(\mathbb{T}_1)) \cdot v$ for a suitable Z-maximal subtree T_1 and $v \in G$.*

Proof. By Proposition 9, $C_G(g) = w^{-1}(\langle r\rangle \times \langle b\rangle)w$. Up to conjugacy, we may assume that $u = 1$. Then the intersection takes the form

$$w^{-1} \cdot (\langle r\rangle \times \langle b\rangle) \cdot w \cap \pi_1(\mathbb{T}).$$

Consider the case of $w \in \pi_1(\mathbb{T})$. Suppose that the intersection contains $r^k \cdot b^s \in \pi_1(\mathbb{T})$. Since r is e-cyclically reduced, either $r \in \pi_1(\mathbb{T})$ or $k = 0$.

If $r \in \pi_1(\mathbb{T})$ then $S_r \subseteq T$. Therefore, $\langle b\rangle = Z(\pi_1(\mathbb{S}_r))$ lies in $\pi_1(\mathbb{T})$, and the lemma is proved.

If $k = 0$ then the intersection is conjugate to a power of b. Find the minimal s such that $b^s \in \pi_1(\mathbb{T})$. This s is characterized by the fact that $T_{b^s} \cap T$ is nonempty. Then the desired intersection is equal to $w^{-1} \cdot \langle b^s\rangle \cdot w = w^{-1} \cdot Z(\pi_1(\mathbb{T}_{b^s})) \cdot w$.

There has remained the case when $w \notin \pi_1(\mathbb{T})$. In this case, $w^{-1} \cdot r^k \cdot b^s \cdot w$ is e-reduced and can lie in $\pi_1(\mathbb{T})$ only if $k = 0$. For each s, we can find a prefix u_s such that $w = u_s \cdot v_s$ and $w^{-1} \cdot b^s \cdot w = v_s^{-1} \cdot b^s \cdot v_s$ is e-reduced. Find the minimal s such that $v_s^{-1} \cdot b^s \cdot v_s \in \pi_1(\mathbb{T})$. Then the intersection is equal to $v_s^{-1} \cdot \langle b^s\rangle \cdot v_s = v_s^{-1} \cdot Z(\pi_1(\mathbb{T}_{b^s})) \cdot v_s$. Lemma 15 is proved.

Lemma 16. *Suppose that T_1 and T_2 are $\mathbb{Z}$-maximal subtrees and $v, w \in G$. Then*

$$v^{-1} \cdot \pi_1(\mathbb{T}_1) \cdot v \cap w^{-1} \cdot Z(\pi_1(\mathbb{T}_2)) \cdot w = u^{-1} \cdot Z(\pi_1(\mathbb{S})) \cdot u$$

for a suitable $\mathbb{Z}$-maximal subtree $S \supseteq T_2$, $u \in G$.

Proof. Consider $Z(\pi_1(\mathbb{T}_2)) \cap u^{-1} \cdot \pi_1(\mathbb{T}_1) \cdot u$ for $u \notin \pi_1(\mathbb{T}_1)$. Let a be the generator of the cyclic group $Z(\pi_1(\mathbb{T}_2)) \simeq \mathbb{Z}$. Then $u^{-1} \cdot g \cdot u \in Z(\pi_1(\mathbb{T}_2))$, and hence $g = u \cdot a^l \cdot u^{-1} \in \pi_1(\mathbb{T}_1)$ for suitable l. Such inclusion is possible only if $a^l \in C_G(u)$, and, by Corollary 4, $a^l \in Z(\pi_1(\mathbb{S}_u))$ and $g = a^l$. Thus,

$$Z(\pi_1(\mathbb{T}_2)) \cap u^{-1} \cdot \pi_1(\mathbb{T}_1) \cdot u = \langle a^l\rangle = Z(\pi_1(\mathbb{S}))$$

for a Z-maximal subtree $S = \langle S_u, T_2\rangle$. Lemma 16 is proved.

Lemma 17. *Let g_1 and g_2 be elliptic elements of G conjugate to vertex elements h_1 and h_2 and let $T = T_{h_1}$ and $S = T_{h_2}$ be Z-maximal subtrees. Then $C_G(g_1) \cap C_G(g_2) = w^{-1}\pi_1(\mathbb{T})w \cap u^{-1}\pi_1(\mathbb{S})u$ is equal either to $v^{-1}\pi_1(\mathbb{R})v$, where $R = T \cap S$, or to $v^{-1}Z(\pi_1(\mathbb{P}))v$ for a suitable Z-maximal subtree P, $v \in G$.*

Proof. Up to conjugacy, we may assume that $u = 1$. By Lemma 13, the intersection is equal to $w^{-1} \cdot \pi_1(\mathbb{T}) \cdot w \cap \pi_1(\mathbb{S})$. Let $w = v_1 \cdot v_2$, so that $v_1 \in \pi_1(\mathbb{T})$ and $v_2 \in \pi_1(\mathbb{S})$. Then the intersection looks as $v_2^{-1} \cdot (\pi_1(\mathbb{T}) \cap \pi_1(\mathbb{S})) \cdot v_2$. If $T \cap S = R \neq \varnothing$ then the intersection is equal to $v_2^{-1} \cdot (\pi_1(\mathbb{R})) \cdot v_2$.

If $T \cap S = \varnothing$ then $g \in \pi_1(\mathbb{T}) \cap \pi_1(\mathbb{S})$ only if g is a vertex element and, by Lemma 1, we obtain $h_1 = a^k$, $g = a^l = b^s$, $h_2 = b^m$ for suitable vertices $a \in V(T)$ and $b \in V(S)$. So, $g \in Z(\langle a \rangle) \cap Z(\langle b \rangle)$, and, by Proposition 3, $g \in Z(\pi_1(\mathbb{P}))$, where P is the minimal Z-maximal subtree containing a and b.

We are left with the case that w is not representable as $w = v_1 \cdot v_2$ so that $v_1 \in \pi_1(\mathbb{T})$ and $v_2 \in \pi_1(\mathbb{S})$. However, $w = w_1 \cdot w_2 \cdot w_3$, and so w_1 and w_3 are maximal subwords with $w_1 \in \pi_1(\mathbb{T})$, $w_3 \in \pi_1(\mathbb{S})$, and $w_2 \cdot w_3 \notin \pi_1(\mathbb{T})$, $w_1 \cdot w_2 \notin \pi_1(\mathbb{S})$. Let $h_1 \in \pi_1(\mathbb{T})$ and $h_2 \in \pi_1(\mathbb{S})$ be such that

$$g = w^{-1} \cdot h_1 \cdot w = h_2 \in w^{-1} \cdot \pi_1(\mathbb{T}) \cdot w \cap \pi_1(\mathbb{S}).$$

Then

$$w_3^{-1} \cdot w_2^{-1} \cdot \left(w_1^{-1} \cdot h_1 \cdot w_1\right) \cdot w_2 \cdot \left(w_3 \cdot h_2^{-1}\right) = 1.$$

The last equality holds only if $h_3 = w_1^{-1} \cdot h_1 \cdot w_1$ is a vertex element. The again from the equality it follows that $h_3 \in C_G(w_2)$, and, by Corollary 4, we get $h_3 \in Z(\pi_1(\mathbb{S}_{w_2}))$. Moreover, it is easy to see that $h_3 \in \pi_1(\mathbb{T})$ and $h_3 \in \pi_1(\mathbb{S})$. Hence, the initial intersection equals

$$w_3^{-1} \cdot (Z(\pi_1(\mathbb{S}_{w_2})) \cap \pi_1(\mathbb{T}) \cap \pi_1(\mathbb{S})) \cdot w_3.$$

The proof of the lemma follows now from Lemma 16.

Lemma 18. *Let g be a hyperbolic element of G and let T be a $\mathbb{Z}$-maximal subtree, $v \in G$. Then*

$$C_G(g) \cap v^{-1} \cdot Z(\pi_1(\mathbb{T})) \cdot v = v^{-1} \cdot Z(\pi_1(\mathbb{S})) \cdot v$$

for a suitable $\mathbb{Z}$-maximal subtree $S \supset T$.

Proof. Consider $Z(\pi_1(\mathbb{T})) \cap v \cdot C_G(g) \cdot v^{-1}$. For $w \in G$ suitable, we have

$$v \cdot C_G(g) \cdot v^{-1} = w^{-1} \cdot (\langle r \rangle \times \langle b \rangle) \cdot w;$$

moreover, $Z(\pi_1(\mathbb{T})) \simeq \mathbb{Z}$. Let $w^{-1}r^k bw \in Z(\pi_1(\mathbb{T}))$; then $|w^{-1}r^k bw| = 1$, and by reducedness we obtain $k = 0$. Thus, $|w^{-1}b^l w| = 1$, i.e., $w^{-1}b^l w = c$, where b^l and c are vertex elements. Since $\Delta(G) = \{1\}$, we have $b^l \in C_G(w)$, and hence $b^l \in Z(\pi_1(\mathbb{S}_w)) = Z(\pi_1(\mathbb{S}_1))$ for a suitable $\mathbb{Z}$-maximal subtree S_1.

Furthermore, $Z(\pi_1(\mathbb{T})) \cap Z(\pi_1(\mathbb{S}_1)) = Z(\pi_1(\langle \mathbb{T}, \mathbb{S}_1 \rangle))$ by Corollary 3. Lemma 18 is proved.

Lemma 19. *Let T_a and T_b be $\mathbb{Z}$-maximal subtrees and $v, w \in G$. Then*

$$v^{-1} \cdot Z(\pi_1(\mathbb{T}_a)) \cdot v \cap w^{-1} \cdot Z(\pi_1(\mathbb{T}_b)) \cdot w = u^{-1} \cdot Z(\pi_1(\mathbb{T})) \cdot u,$$

where T is a $\mathbb{Z}$-maximal subtree containing T_a and T_b, $u \in G$.

Proof. Consider $Z(\pi_1(\mathbb{T}_a)) \cap u^{-1} \cdot Z(\pi_1(\mathbb{T}_b)) \cdot u$, where $u = w \cdot v^{-1}$. By definition, $Z(\pi_1(\mathbb{T}_a)) = \langle a \rangle$ and $Z(\pi_1(\mathbb{T}_b)) = \langle b \rangle$. Then $a^k = u^{-1}b^l u$. Since a^k and b^l are vertex elements and $\Delta(G) = \{1\}$, we have $a^k = b^l$ and $b^l \in Z(\pi_1(\mathbb{S}_u))$. Thus, $c = a^k = b^l$ enters into the intersection if it lies in

$$Z(\pi_1(\mathbb{T}_a)) \cap Z(\pi_1(\mathbb{T}_b)) \cap Z(\pi_1(\mathbb{S}_u)) = Z(\pi_1(\langle \mathbb{T}_a, \mathbb{T}_b, \mathbb{S}_u \rangle)) = Z(\pi_1(\mathbb{T})).$$

The reverse inclusion is obvious. Lemma 19 is proved.

§ 4. Embeddings of Centralizers

We have proved that centralizers in G are of the three types:

$$v^{-1} \cdot \pi_1(\mathbb{T}) \cdot v, \quad T \text{ is a } Z\text{-maximal subtree,} \tag{1}$$

$$w^{-1} \cdot (\langle r \rangle \times \langle b \rangle) \cdot w, \tag{2}$$

$$u^{-1} \cdot Z(\pi_1(\mathbb{S})) \cdot u, \quad S \text{ is a } Z\text{-maximal subtree.} \tag{3}$$

For describing the centralizer lattice of a group G, we must know when one centralizer is included in another.

Proposition 20. *Let T_a and T_b be Z-subtrees. Then $w^{-1} \cdot Z(\pi_1(\mathbb{T}_a)) \cdot w \supset v^{-1} \cdot Z(\pi_1(\mathbb{T}_b)) \cdot v$ if and only if $T_a \subset T_b$, $b \in Z(\pi_1(\mathbb{S}_{wv^{-1}}))$.*

Proof. The inclusion has the form $u^{-1} \cdot \langle a \rangle \cdot u \supset \langle b \rangle$ and is equivalent to the equality $b = u^{-1} a^k u$ for a suitable integer k. This equality implies that $b = a^k \in Z(\pi_1(\mathbb{S}_u))$. Hence, the initial inclusion has the form $Z(\pi_1(\mathbb{T}_a)) \supset Z(\pi_1(\mathbb{T}_b))$. If $v \in V(T_a)$ then, by Remark 7, $v^l = a$, and so $a^k = v^{kl} = b$; consequently, $v \in V(T_b)$. Therefore, $T_a \subset T_b$. The reverse inclusion is obvious. Proposition 20 is proved.

Remark 21. Let g_1 and g_2 be hyperbolic elements of G. Then, by Lemma 14, $C_G(g_1) \supseteq C_G(g_2)$ if and only if $C_G(g_1) = C_G(g_2)$.

Proposition 22. *Let g be a hyperbolic element of G and let T_a is a Z-maximal subtree. Then*

$$C_G(g) = w^1(\langle r \rangle \times \langle b \rangle)w \supset v^{-1} \cdot Z(\pi_1(\mathbb{T}_a)) \cdot v$$

if and only if $a \in Z(\pi_1(\langle \mathbb{S}_r, \mathbb{S}_{vw^{-1}} \rangle))$.

Proof. Consider the reverse inclusion

$$vw^{-1} \cdot (\langle r \rangle \times \langle b \rangle) \cdot wv^{-1} \supset Z(\pi_1(\mathbb{T}_a)) = \langle a \rangle.$$

It is equivalent to the equality $vw^{-1} r^k b^l wv^{-1} = a$. After e-reduction, the left-hand side has the form $u^{-1} r_1^k b^l u$, where r_1 is obtained from r by a cyclic permutation; consequently, the comparison of the lengths gives $k = 0$. The equality takes the form $vw^{-1} b^l wv^{-1} = a$; therefore, $a = b^l \in Z(\pi_1(\mathbb{S}_{vw^{-1}}))$. Moreover, $b \in Z(\pi_1(\mathbb{S}_r))$, and so $a \in Z(\pi_1(\mathbb{S}_r))$. The reverse inclusion is obvious.

Proposition 23. *Let T_1 and T_2 be Z-subtrees and let $v_1, v_2 \in G$ be such that the element $v_2 v_1^{-1}$ is reduced in G and no prefix of v_i lies in $\pi_1(\mathbb{T}_i)$, $i = 1, 2$. Then $v_1^{-1} \cdot \pi_1(\mathbb{T}_1) \cdot v_1 \supset v_2^{-1} \cdot \pi_1(\mathbb{T}_2) \cdot v_2$ if and only if $T_1 \supset \langle T_2, S_{v_2 v_1^{-1}} \rangle$.*

Proof. The claim follows from the fact that

$$\pi_1(\mathbb{T}_1) \supset \left(v_2 v_1^{-1}\right)^{-1} \cdot \pi_1(\mathbb{T}_2) \cdot \left(v_2 v_1^{-1}\right)$$

only if $T_1 \supset \langle T_2, S_{v_2 v_1^{-1}} \rangle$. The proposition is proved.

Proposition 24. *Let T be a Z-maximal subtree and let g be a hyperbolic element of G. Then $v^{-1} \cdot \pi_1(\mathbb{T}) \cdot v \supset C_G(g) = w^{-1}(\langle r \rangle \times \langle b \rangle)w$ if and only if $T \supseteq \langle S_r, S_{vw^{-1}} \rangle$.*

Proof. The equivalent inclusion $\pi_1(\mathbb{T}) \supset vw^{-1}(\langle r \rangle \times \langle b \rangle)wv^{-1}$ implies that $vw^{-1} r^k wv^{-1} \in \pi_1(\mathbb{T})$ for each $k \in \mathbb{Z}$. Since r is e-cyclically reduced, after reductions, the word $vw^{-1} r^k wv^{-1}$ takes the form $u^{-1} r_1^k u$, where $wv^{-1} = u_1 \cdot u$, u_1 is a prefix of maximal length reducing with a part of the word r^k and r_1 is a cyclic permutation of r. After reductions, the word $u^{-1} r_1^k u$ is e-reduced and lies in $\pi_1(\mathbb{T})$. Consequently, $T \supseteq S_{u^{-1} r_1^k u}$; therefore, $T \supseteq S_r = S_{r_1}$ and $T \supseteq S_{vw^{-1}}$. The last inclusion holds because vw^{-1} differs from u only by an element of S_r. The proof of the converse stems easily from the fact that $b \in \pi_1(S_r)$.

Proposition 25. *Let T_1 and T_a be Z-maximal subtrees and let $v, w \in G$ be such that wv^{-1} is reduced in G and no prefix of wv^{-1} lies in $\pi_1(\mathbb{T}_1)$. Then $w^{-1} \cdot \pi_1(\mathbb{T}_1) \cdot w \supset v^{-1} \cdot Z(\pi_1(\mathbb{T}_a)) \cdot v$ if and only if $T_1 \cap T_a \neq 0$ and $a \in Z(\pi_1(\mathbb{S}_{wv^{-1}}))$.*

PROOF. The equivalent inclusion $vw^{-1} \cdot \pi_1(\mathbb{T}_1) \cdot wv^{-1} \supset Z(\pi_1(\mathbb{T}_a))$ is tantamount to $a \in vw^{-1} \cdot \pi_1(\mathbb{T}_1) \cdot wv^{-1}$. Then $a = (wv^{-1})^{-1} \cdot g \cdot wv^{-1}$ and the right-hand side is reduced. This equality is possible only if g is a vertex element. Consequently, $a = g = b^k \in Z(\pi_1(\mathbb{S}_{wv^{-1}}))$ for suitable $b \in V(T_1)$. By Remark 7, $b \in V(T_a)$, and hence $T_1 \cap T_a \neq 0$. The proof of the converse easily follows from what has already been proved. The proposition is proved.

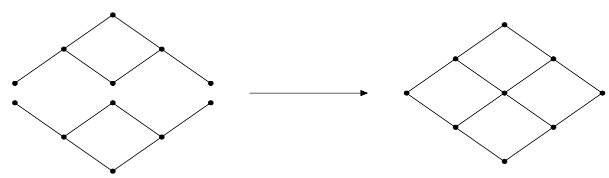

Fig. 4. The union of the lattices $\mathfrak{T}$ and $\breve{\mathfrak{T}}$

REMARK 26. The centralizer lattice of $\pi_1(\mathbb{A})$ is described in Propositions 20–25. Up to conjugacy, the centralizers of types (1) and (3) constitute a sublattice in this lattice which is isomorphic to the union of the lattice of Z-maximal subtrees $\mathfrak{T}$ and the lattice $\breve{\mathfrak{T}}$ dual to it with identified minimal and maximal elements (Fig. 4).

§ 5. Centralizer Dimension

Calculate the centralizer dimension of a group G.

Theorem 27. *Let $\mathbb{A}$ be a reduced labeled tree and let $\pi_1(\mathbb{A}) = G$. Then $\operatorname{cdim}(G) \leq 2 \cdot |V(A)| - 1$ and the number $\operatorname{cdim}(G)$ is odd. For each odd number l from 3 to $2 \cdot n - 1$, there exists a reduced labeled tree $\mathbb{B}$ on n vertices such that $\operatorname{cdim}(\pi_1(\mathbb{B})) = l$.*

PROOF. Every chain of centralizers $C_1 \supset C_1 \supset \dots \supset C_d$ consists of centralizes of three types. The centralizers of type (1) cannot have numbers greater than the centralizers of types (2) and (3); similarly, all centralizers of type (2) are located "to the left" of each centralizer of type (3). Moreover, two consecutive centralizers (2) cannot meet either. Observe also that if C_i is of type (2) then it is not of type (3) and not of type (1); furthermore, if $a \in V(A)$ then $C_G(a) = \langle a \rangle$ has both type (1) and type (3). However, apart from the subgroups conjugate to vertex subgroups, no centralizers possess this property.

Note that the results of § 4 imply that the chains of embedded centralizers of types (1) and (3) correspond to chains of embedded Z-maximal subtrees of A and hence are finite.

Thus, every maximal chain of centralizers has the form

$$(1) \supset \dots \supset (1) \supset (2) \supset (3) \supset \dots \supset (3)$$

or

$$(1) \supset \dots \supset (1) \supset (3) \supset \dots \supset (3).$$

In the latter case, the first centralizer of type (3) can also be a centralizer of type (1).

The longest chain of centralizers of type (1) is not longer than the maximal chain of embedded Z-maximal subtrees. The last chain in turn contains at most $|V(A)|$ elements. The situation with chains of centralizers of type (3) but, in this case, the decrease in the chain corresponds to the increase of Z-maximal subtrees. It remains to observe that if the last element in the chain of centralizers of type (1) has type (3) then a chain of the second kind is realized (without a centralizer of type (2)) and it consists of at most $2 \cdot |V(A)| - 1$ elements.

If the chain contains a centralizer of type (2) then there are at most $|V(A)| - 1$ centralizers of type (1) since $\mathbb{Z}$ does not contain $\mathbb{Z} \times \mathbb{Z}$. Similarly, in this case, there are at most $|V(A)| - 1$ centralizers of type (1) because a centralizer of type (2) does not include a subgroup conjugate to the vertex group. As a result, we conclude that the length of the chain of the first kind is at most $2 \cdot (|V(A)| - 1) + 1 = 2 \cdot |V(A)| - 1$.

Now, we provide some examples. Fix $n \geq 2$ and consider the labeled tree $\mathbb{R}(k, s)$ (see Fig. 3) on $k + s = n$ vertices. It is clear from the proof of the first part that $\mathrm{cdim}(\pi_1(\mathbb{A}))$ is odd. Find a chain of maximal length of Z-maximal subtrees. To this end, observe firstly that if a Z-maximal subtree T contains an edge joining u_i and v_k for some $i = 1, \dots, s$ then all vertices $u_1, u_2, \dots, u_s$ get into T. Denote by T_i the subtree $R(k, s)$ generated by the vertices $v_i, v_{i+1}, \dots, v_k, u_1, u_2, \dots, u_s$.

From this we can see that the chain of embedded subtrees

$$T_1 \supset T_2 \supset \dots \supset T_{k-1} \supset \langle v_k \rangle$$

is a maximal chain of embedded Z-maximal subtrees. Hence, it describes a chain of centralizes of maximal length. To it there correspond the two chains of centralizers: the chain of centralizes of type (1)

$$\pi_1(\mathbb{T}_1) \supset \pi_1(\mathbb{T}_2) \supset \dots \supset \pi_1(\mathbb{T}_{k-1}) \supset \pi_1(\langle v_k \rangle) = \langle v_k \rangle$$

and the chain of centralizers of type (3)

$$\begin{array}{ccccccccc} Z(\pi_1(\langle v_k \rangle)) & \supset & Z(\pi_1(\mathbb{T}_{k-1})) & \supset & \cdots & \supset & Z(\pi_1(\mathbb{T}_2)) & \supset & Z(\pi_1(R(k,s))) \\ \shortparallel & & \shortparallel & & & & \shortparallel & & \shortparallel \\ \langle v_k \rangle & \supset & \left\langle v_k^2 \right\rangle & \supset & \cdots & \supset & \left\langle v_k^{2^{k-2}} \right\rangle & \supset & \left\langle v_k^{2^{k-1}} \right\rangle. \end{array}$$

Moreover, the minimal element of the first chain coincides with the maximal element of the second chain. Therefore, $\mathrm{cdim}(\pi_1(\mathbb{R}(k, s))) = 2k - 1$ does not depend on s.

Note that $\mathrm{cdim}(G) = 1$ only if G is abelian and the reduced labeled graph on $n \geq 2$ vertices has nonabelian fundamental group. Theorem 27 is proved.

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