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INTEGRO-LOCAL LIMIT THEOREMS FOR COMPOUND RENEWAL PROCESSES UNDER CRAMÉR'S CONDITION. I

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Abstract: We obtain integro-local limit theorems in the phase space for compound renewal processes under Cramér's moment condition. These theorems apply in a domain analogous to Cramér's zone of deviations for random walks. It includes the zone of normal and moderately large deviations. Under the same conditions we establish some integro-local theorems for finite-dimensional distributions of compound renewal processes.

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1. Introduction

Consider an initial random vector $\boldsymbol{\xi}_1 = (\tau_1, \zeta_1)$ and a sequence $\boldsymbol{\xi} = (\tau, \zeta)$, $\boldsymbol{\xi}_2 = (\tau_2, \zeta_2)$, $\boldsymbol{\xi}_3 = (\tau_3, \zeta_3), \dots$, independent of $\boldsymbol{\xi}_1$, of independent and identically distributed random nondegenerate vectors, where $\tau_1 \geq 0$ and $\tau > 0$. Let

$$T_0 = Z_0 = 0, \quad T_n := \sum_{j=1}^{n} \tau_j, \quad Z_n := \sum_{j=1}^{n} \zeta_j \quad \text{for } n \geq 1.$$

Put

$$\eta(t) := \min\{k \geq 0 : T_k > t\}, \quad \nu(t) := \max\{k \geq 0 : T_k \leq t\}$$

for $t \geq 0$. It is clear that

$$\nu(t) = \eta(t) - 1.$$

For $t \geq 0$ we have the *undershoot* $\gamma(t)$ and *overshoot* $\chi(t)$ at level t of the walk $\{T_k\}$,

$$\gamma(t) := t - T_{\nu(t)}, \quad \chi(t) := T_{\eta(t)} - t,$$

so that

$$\tau_{\eta(t)} = \gamma(t) + \chi(t).$$

The *compound renewal process* $Z(t)$ is defined as

$$Z(t) := Z_{\nu(t)} \quad \text{for } t \geq 0 \quad \text{and } Z(0) = \zeta_1 \mathbf{I}_{\{\tau_1 = 0\}}.$$

Along with $Z(t)$ we study the process

$$Y(t) := Z_{\eta(t)} = Z(t) + \zeta_{\eta(t)} \quad \text{for } t \geq 0,$$

which we also call the compound renewal process. The trajectories of $Z(t)$ and $Y(t)$ are right-continuous. This article establishes integro-local limit theorems for $Z(t)$ and $Y(t)$. In the domain of large deviations they differ slightly.

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The standard generally accepted model of compound renewal processes assumes that the time τ_1 and the size ζ_1 of the first jump have a joint distribution different in general from the joint distribution of (τ, ζ); see [1, 2]. For example, this happens for compound renewal processes with stationary increments. If $(\tau_1, \zeta_1) =_d (\tau, \zeta)$ then the process $Z(t)$ is called *homogeneous*, and otherwise *inhomogeneous*, so that for $\tau_1 = 0$ the process $Z(t)$ is a homogeneous compound renewal process with the initial value $Z(0) = \zeta_1$.

If $\tau_1 = \tau \equiv 1$ then $Z(t)$ for $t = 0, 1, 2, \dots$ becomes the *random walk* generated by the sequence of sums of independent random variables $\{\zeta_k\}$. The integro-local theorems for sums of random vectors can be found in [3–6].

If $\zeta_1 = \zeta \equiv 1$ then $Z(t) = \nu(t)$ becomes a *simple* renewal process. For these the distribution of $Z(t)$ coincides up to 1 with the distribution of the time $\eta(t)$ of the first crossing of level t by the walk $\{T_k\}$. Since $\tau_j \geq 0$, it follows that

$$\mathbf{P}(Z(t) \geq n) = \mathbf{P}(Y(t) > n) = \mathbf{P}(\eta(t) > n) = \mathbf{P}(T_n \leq t), \tag{1.1}$$

reducing the study of the distribution of $Z(t)$ and $Y(t)$ again to the distribution of sums of independent and identically distributed random variables. This object is well understood in the domains of both normal and large deviations, together with integro-local theorems; see [5, Chapter 9; 7, Chapter VIII] for instance.

The integro-local theorems for $Z(T)$ and $Y(T)$ as $T \to \infty$ in the domain of normal deviations in the case of independent nonlattice τ and ζ with finite second moments are established in [5], and in general under certain additional conditions in [8].

Some rough analogs of integro-local theorems, the local large deviation principles for finite-dimensional distributions, are established in [6, 9].

Assume henceforth, unless stated otherwise, that Cramér's condition holds in the following form:

$[\mathbf{C}_0]$ $\mathbf{E}e^{v|\boldsymbol{\xi}_1|} < \infty$ and $\mathbf{E}e^{v|\boldsymbol{\xi}|} < \infty$ for some $v > 0$.

Moreover, assume that the random vector $\boldsymbol{\xi} = (\tau, \zeta)$ is nonlattice. To avoid repetition, we omit these two conditions in our main statements.

Therefore, the principal object of our study is the sharp asymptotics for the integro-local probability

$$\mathbf{P}(Z(T) \in \Delta[x)), \quad \mathbf{P}(Y(T) \in \Delta[x)), \tag{1.2}$$

where $\Delta[x)$ is the half-open interval $[x,\ x + \Delta)$, while $\Delta = \Delta_T \to 0$ sufficiently slowly as $T \to \infty$ and $\alpha = \frac{x}{T}$ lies in some compact set (closed interval) K to be specified below.

It is not difficult to see, as we illustrate in Corollary 3.1 and remarks on it, that the knowledge of the asymptotics for (1.2) enables us to obtain integral limit theorems for compound renewal processes without much effort.

The asymptotics for (1.2) is a very complicated object. Its study requires integro-local theorems for the renewal measure corresponding to the sequence (τ_j, ζ_j). This enables us to obtain rather complete results, while the approach based on the asymptotics for the renewal measure is apparently the only one possible. No other approach to limit theorems is clear now, including integral theorems, for compound renewal processes in the domain of large deviations.

The statements of our main results involve a series of functions whose meanings and properties we should know to understand the nature of the laws established. Section 2 introduces the basic notation, defines the required functions, and studies their properties. Section 3 presents the statements of the main results and some comments. A considerable part of the proof of the main integro-local theorem for the process $Z(t)$ consists in proving an integro-local theorem for the renewal measure corresponding to the random walk $\{(T_n, Z_n)\}$ done in Section 4. Section 5 contains a proof of the main theorem and its generalizations to the case that the distribution of the initial vector $\boldsymbol{\xi}_1$ depends on T. This generalization is used in Section 6, which establishes an integro-local theorem for the process $Y(t)$ and integro-local theorems for finite-dimensional distributions of compound renewal processes.

2. Preliminaries

Below A, D, λ, and μ stand for functions of both one and two variables. In order to distinguish between them easier, we highlight some functions of two variables using a semibold font. Put

$$\psi(\lambda,\mu) := \mathbf{E}e^{\lambda\tau+\mu\zeta}, \quad \psi_1(\lambda,\mu) := \mathbf{E}e^{\lambda\tau_1+\mu\zeta_1},$$

$$\mathbf{A}(\lambda,\mu) := \ln\psi(\lambda,\mu), \quad \mathbf{A}_1(\lambda,\mu) := \ln\psi_1(\lambda,\mu), \quad (\lambda,\mu)\in\mathbb{R}^2;$$

$$\mathscr{A} := \{(\lambda,\mu) : \mathbf{A}(\lambda,\mu)<\infty\}, \quad \mathscr{A}_1 := \{(\lambda,\mu) : \mathbf{A}_1(\lambda,\mu)<\infty\}.$$

Clearly, in accordance with condition $[\mathbf{C}_0]$, the interiors $(\mathscr{A})$ and $(\mathscr{A}_1)$ of $\mathscr{A}$ and $\mathscr{A}_1$ contain the point $(\lambda,\mu)=(0,0)$ and are the domains of analyticity of $\mathbf{A}(\lambda,\mu)$ and $\mathbf{A}_1(\lambda,\mu)$ respectively.

All vectors and matrices, when they are denoted by one symbol, are also highlighted by using a semibold font.

Given a function $F(u,v)$ of two variables u and v, indicate with lower indices (1) and (2) the derivatives with respect to the first and second arguments; for instance

$$F'_{(1)}(u_1,v_1) = \frac{\partial}{\partial u}F(u,v_1)\bigg|_{u=u_1}, \quad F''_{(1)}(u_1,v_1) = \frac{\partial^2}{\partial u^2}F(u,v_1)\bigg|_{u=u_1},$$

$$F''_{(2)}(u_1,v_1) = \frac{\partial^2}{\partial v^2}F(u_1,v)\bigg|_{v=v_1}, \quad F''_{(2,1)}(u_1,v_1) = \frac{\partial}{\partial u}\frac{\partial}{\partial v}F(u,v)\bigg|_{(u,v)=(u_1,v_1)}.$$

Denote by $\mathbf{F}'=\mathbf{F}'(u,v)$ and $\mathbf{F}''=\mathbf{F}''(u,v)$ respectively the vector

$$\mathbf{F}'=\mathbf{F}'(u,v)=(F'_{(1)}(u,v),F'_{(2)}(u,v))$$

and the matrix

$$\mathbf{F}''=\mathbf{F}''(u,v)=\|F''_{(i,j)}(u,v)\|_{i,j=1,2}.$$

Denote by $|\mathbf{F}''|$ the determinant of $\mathbf{F}''$.

2.1. The deviation function for $\boldsymbol{\xi}=(\tau,\zeta)$ and some of its properties. We use the available integro-local theorems for the sums

$$\mathbf{S}_n := \sum_{j=1}^{n}\boldsymbol{\xi}_j = (T_n,\ Z_n);$$

for instance, see [6, § 2.9] or Theorem 4.2 below. The *deviation function*

$$\Lambda(\theta,\alpha) := \sup_{(\lambda,\mu)}\{\lambda\theta+\mu\alpha-\mathbf{A}(\lambda,\mu)\} \tag{2.1}$$

corresponding to the random vector $\boldsymbol{\xi}=(\tau,\zeta)$ plays an important role. As the Legendre transform of the lower-continuous convex function $\mathbf{A}(\lambda,\mu)$, it is also convex and lower-continuous.

Along with the sets $\mathscr{A}_1$ and $\mathscr{A}$ we need the domain $\mathscr{L}$ of analyticity of $\Lambda(\theta,\alpha)$. This domain consists of the points (θ,α) for which the system of equations

$$\begin{cases} \mathbf{A}'_{(1)}(\lambda,\mu)=\theta, \\ \mathbf{A}'_{(2)}(\lambda,\mu)=\alpha \end{cases} \tag{2.2}$$

for the coordinates of (λ,μ) at which the upper bound in (2.1) is attained has a solution $(\boldsymbol{\lambda}(\theta,\alpha),\boldsymbol{\mu}(\theta,\alpha))$ belonging to $(\mathscr{A})$, so that $\mathscr{L}=\{\mathbf{A}'(\lambda,\mu):(\lambda,\mu)\in(\mathscr{A})\}$. Since the function $\mathbf{A}(\lambda,\mu)$ is strictly convex in $(\mathscr{A})$, this solution is always unique; see [10, Chapter 1] for instance. This means that the conditions

$$(\theta,\alpha)\in\mathscr{L} \quad \text{and} \quad (\boldsymbol{\lambda}(\theta,\alpha),\boldsymbol{\mu}(\theta,\alpha))\in(\mathscr{A})$$

are equivalent. It is clear that the point $(\theta, \alpha) = (a_\tau, a_\zeta)$, with $a_\tau := \mathbf{E}\tau$ and $a_\zeta := \mathbf{E}\zeta$, lies in $\mathscr{L}$ and $(\boldsymbol{\lambda}(a_\tau, a_\zeta), \boldsymbol{\mu}(a_\tau, a_\zeta)) = (0, 0) \in (\mathscr{A})$.

Suppose that $(\theta, \alpha) \in \mathscr{L}$. Then

$$\Lambda(\theta, \alpha) = \boldsymbol{\lambda}(\theta, \alpha)\theta + \boldsymbol{\mu}(\theta, \alpha)\alpha - \mathbf{A}(\boldsymbol{\lambda}(\theta, \alpha), \boldsymbol{\mu}(\theta, \alpha)), \tag{2.3}$$

$$\Lambda'_{(1)}(\theta, \alpha) = \boldsymbol{\lambda}(\theta, \alpha) + \boldsymbol{\lambda}'_{(1)}(\theta, \alpha)\theta + \boldsymbol{\mu}'_{(1)}(\theta, \alpha)\alpha$$
$$-\mathbf{A}'_{(1)}(\boldsymbol{\lambda}(\theta, \alpha), \boldsymbol{\mu}(\theta, \alpha))\boldsymbol{\lambda}'_{(1)}(\theta, \alpha) - \mathbf{A}'_{(2)}(\boldsymbol{\lambda}(\theta, \alpha), \boldsymbol{\mu}(\theta, \alpha))\boldsymbol{\mu}'_{(1)}(\theta, \alpha).$$

By (2.2), this yields

$$\Lambda'_{(1)}(\theta, \alpha) = \boldsymbol{\lambda}(\theta, \alpha). \tag{2.4}$$

Similarly,

$$\Lambda'_{(2)}(\theta, \alpha) = \boldsymbol{\mu}(\theta, \alpha), \tag{2.5}$$

so that

$$\Lambda'(\theta, \alpha) := (\Lambda'_{(1)}(\theta, \alpha), \Lambda'_{(2)}(\theta, \alpha)) = (\boldsymbol{\lambda}(\theta, \alpha), \boldsymbol{\mu}(\theta, \alpha)). \tag{2.6}$$

If τ and ζ are independent then

$$\mathbf{A}(\lambda, \mu) = A^{(\tau)}(\lambda) + A^{(\zeta)}(\mu),$$

where

$$A^{(\tau)}(\lambda) := \ln \mathbf{E}e^{\lambda\tau}, \quad A^{(\zeta)}(\mu) := \ln \mathbf{E}e^{\mu\zeta}.$$

Therefore, $\mathbf{A}'_{(1)}(\lambda, \mu)$ is independent of μ and $\mathbf{A}'_{(2)}(\lambda, \mu)$ is independent of λ, so the domains $(\mathscr{A})$ and $\mathscr{L}$ are rectangular.

2.2. The deviation function for the renewal measure (second deviation function). Along with the deviation function $\Lambda(\theta, \alpha)$, we need the *second deviation function* $\mathbf{D}_\Lambda(\theta, \alpha)$ defined as

$$\mathbf{D}_\Lambda(\theta, \alpha) := \inf_{r>0} r\Lambda\left(\frac{\theta}{r}, \frac{\alpha}{r}\right).$$

It arises naturally when we study the asymptotics for the renewal measure corresponding to the random walk $\{T_n, Z_n\}$ and is crucial in its description; see Theorem 4.1 below. The properties of the function $\mathbf{D}_\Lambda(\theta, \alpha)$ are thoroughly understood; see [6, § 2.9; 11] for instance. It is convex, semiadditive, and linear along each ray from 0. In order for $\mathbf{D}_\Lambda(\theta, \alpha)$ to better fit the term the "rate function for the renewal measure," we slightly modify it by extending as lower semicontinuity to the boundary of the effective domain

$$\mathscr{D}^{<\infty} := \{(\theta, \alpha) : \mathbf{D}_\Lambda(\theta\alpha) < \infty\} \tag{2.7}$$

of $\mathbf{D}_\Lambda$; the deviation function in LDP is necessarily lower semicontinuous, while the function $\mathbf{D}_\Lambda$ in general is not; see [6, § 2.9] for instance. Denote this amended version by $\mathbf{D}(\theta, \alpha)$.

The following is established in [6, § 2.9]; see also [11, 9]:

The function $\mathbf{D}(\theta, \alpha)$ admits representation

$$\mathbf{D}(\theta, \alpha) = \sup_{(\lambda,\mu)\in\mathscr{A}^{\le 0}} \{\lambda\theta + \mu\alpha\} = \sup_{(\lambda,\mu)\in\partial\mathscr{A}^{\le 0}} \{\lambda\theta + \mu\alpha\}, \tag{2.8}$$

where $\mathscr{A}^{\le 0} := \{(\lambda, \mu) : \mathbf{A}(\lambda, \mu) \le 0\}$ and ∂B is the boundary of B.

Representation (2.8) enables us to find another very important characterization of $\mathbf{D}(\theta, \alpha)$. Observe beforehand that, due to the positive homogeneity of $\mathbf{D}$,

$$\mathbf{D}(\theta, \alpha) = \theta\mathbf{D}\left(1, \frac{\alpha}{\theta}\right) \tag{2.9}$$

for $\theta > 0$, and therefore the function $\mathbf{D}(\theta, \alpha)$ of two variables is completely determined by the values of the function

$$D(\alpha) := \mathbf{D}(1, \alpha) \tag{2.10}$$

of one variable. The function $D(\alpha)$ turns out the deviation function for a compound renewal process. It is in terms of $D(\alpha)$ that we state the main results of this article. By (2.8),

$$D(\alpha) = \sup_{(\lambda,\mu)\in\partial\mathscr{A}^{\leq 0}} \{\mu\alpha + \lambda\}. \tag{2.11}$$

Observe first of all that since $\mathbf{E}\tau > 0$, $\mathbf{A}(0,0) = 0$, and $\mathbf{A}'_{(1)}(0,0) > 0$, the set $\mathscr{A}^{\leq 0}$ was always nonempty interior. To describe the boundary $\partial\mathscr{A}^{\leq 0}$, consider the level μ section $\mathscr{A}_\mu$ of $\mathscr{A}$,

$$\mathscr{A}_\mu := \{\lambda : \mathbf{A}(\lambda, \mu) < \infty\},$$

and put

$$\mu^+ := \sup\{\mu : \mathscr{A}_\mu \neq \varnothing\}, \quad \mu^- := \inf\{\mu : \mathscr{A}_\mu \neq \varnothing\}.$$

Denote by $\mathscr{M}$ the set of values of μ with $\mathbf{A}(\lambda, \mu) < \infty$ for some λ:

$$(\mu^-, \mu^+) \subset \mathscr{M} := \{\mu : \min_\lambda \mathbf{A}(\lambda, \mu) < \infty\} \subset [\mu^-, \mu^+].$$

If $\mu \in \mathscr{M}$ then the function $\mathbf{A}(\lambda, \mu)$ increases strictly with λ from $-\infty$ to ∞ and therefore the values

$$\begin{aligned} A(\mu) &:= \inf\{\lambda : \mathbf{A}(-\lambda, \mu) \leq 0\} = -\sup\{\lambda : \mathbf{A}(\lambda, \mu) \leq 0\} > -\infty, \\ A^\infty(\mu) &:= \inf\{\lambda : \mathbf{A}(-\lambda, \mu) < \infty\} = -\sup\{\lambda : \mathbf{A}(\lambda, \mu) < \infty\} > -\infty \end{aligned} \tag{2.12}$$

are defined. Obviously,

$$(-A(\mu), \mu) \in \partial\mathscr{A}^{\leq 0}, \quad (-A^\infty(\mu), \mu) \in \partial\mathscr{A},$$

so the curve $\lambda = -A(\mu)$ in $\mathbb{R}^2$ is a parametrization of $\partial\mathscr{A}^{\leq 0}$, while $\lambda = -A^\infty(\mu)$ is a parametrization of the boundary of $\mathscr{A}$ for $\mu \in \mathscr{M}$. If for $\mu \in \mathbb{R}$ there is no λ with $\mathbf{A}(\lambda, \mu) < \infty$ then put $A(\mu) = \infty$ and $A^\infty(\mu) = \infty$. This way we extend the function $A(\mu)$ to the whole axis; furthermore, $A(\mu) = \infty$ for $\mu \notin \mathscr{M}$ (note that $\mu^\pm$ are discontinuity points, and the values $A(\mu^\pm)$ can be finite), while we can express (2.11) as

$$D(\alpha) = \sup_\mu \{\mu\alpha - A(\mu)\}. \tag{2.13}$$

This means that $D(\alpha)$ is the Legendre transform of $A(\mu)$.

2.3. The basic function $A(\mu)$ and its properties. Connection with the deviation function $D(\alpha)$ for GRP. The function $A(\mu)$ defined in (2.12) will play a key role below. We call it the *basic function* for the compound renewal processes. Let us present its main properties.

Lemma 2.1. *The function $A(\mu)$ is lower-continuous and convex (inside the effective domain), presenting the Legendre transform of the deviation function $D(\alpha)$:*

$$A(\mu) = \sup_\alpha \{\mu\alpha - D(\alpha)\}. \tag{2.14}$$

PROOF. Observe beforehand that we always have

$$\mathbf{A}(-A(\mu), \mu) \leq 0 \quad \text{provided that } A(\mu) < \infty. \tag{2.15}$$

Indeed, the definition in (2.12) implies that in the case $A(\mu) < \infty$ there exists a sequence $\lambda_n < -A(\mu)$ with $\lambda_n \to -A(\mu)$ as $n \to \infty$ and $\mathbf{A}(\lambda_n, \mu) \leq 0$; i.e., $(\lambda_n, \mu) \in (\mathscr{A}^{\leq 0})$. But $\mathbf{A}(\lambda, \mu)$ is lower-continuous inside the effective domain. Therefore,

$$0 \geq \mathbf{A}(\lambda_n, \mu) \to \mathbf{A}(-A(\mu), \mu) \quad \text{as } n \to \infty,$$

justifying (2.15).

Let us establish the convexity of $A(\mu)$, meaning that

$$A(p\mu_1 + q\mu_2) \le pA(\mu_1) + qA(\mu_2) \tag{2.16}$$

for all $\mu_1, \mu_2 \in \mathbb{R}$ as well as $p \ge 0$ and $q \ge 0$ with $p + q = 1$. If $A(\mu_1) = \infty$ or $A(\mu_2) = \infty$ then (2.16) is satisfied. If $A(\mu_1) < \infty$ and $A(\mu_2) < \infty$ then (2.15) yields

$$\mathbf{A}(-A(\mu_1), \mu_1) \le 0, \quad \mathbf{A}(-A(\mu_2), \mu_2) \le 0.$$

Since $\mathbf{A}(\lambda, \mu)$ is a convex function, we have

$$\mathbf{A}(-pA(\mu_1) - qA(\mu_2), p\mu_1 + q\mu_2) \le p\mathbf{A}(-A(\mu_1), \mu_1) + q\mathbf{A}(-A(\mu_2), \mu_2) \le 0. \tag{2.17}$$

Since the left-hand side of (2.17) is nonpositive, by (2.12) the value of the first argument of $\mathbf{A}(\cdot, \cdot)$ on the left-hand side of (2.17) is at most the value of $-A(\cdot)$ of the second argument of $\mathbf{A}(\cdot, \cdot)$; i.e.,

$$-pA(\mu_1) - qA(\mu_2) \le -A(p\mu_1 + q\mu_2).$$

This justifies (2.16).

Let us establish now that the function $A(\mu)$ is lower-continuous. For the convex function $A(\mu)$ it suffices to verify lower continuity only at the discontinuity points $\mu^{\pm}$. Put

$$A_+ := \lim_{\mu \uparrow \mu^+} A(\mu).$$

If $A_+ = \infty$ then the convexity of $A(\mu)$ yields $A(\mu^+) = \infty = A_+$. Assume now that $A_+ < \infty$. Since the function $\mathbf{A}(\lambda, \mu)$ defining $\mathscr{A}^{\le 0}$ is convex and lower-continuous, it follows that $\mathscr{A}^{\le 0}$ is convex and closed. Consequently, the horizontal ray $L_+ := \{(\lambda, \mu^+) : \lambda \le -A_+\}$ lies in $\mathscr{A}^{\le 0}$. By (2.12), L_+ is a part of the boundary $\partial\mathscr{A}^{\le 0}$ of $\mathscr{A}^{\le 0}$. This implies that $A(\mu^+) = A_+$, meaning the lower continuity of $A(\mu)$ at the right discontinuity point μ^+. The lower continuity of $A(\mu)$ at the left discontinuity point μ^- is verified similarly.

Since $A(\mu)$ and $D(\alpha)$ are lower-continuous convex functions, from (2.13) we infer (2.14); for instance, see [12, § 12]. The proof of Lemma 2.1 is complete. □

The function $A(\mu)$ possesses many other properties of the logarithm of the Laplace transform of some distribution: $A(0) = 0$ and $A(\mu) \to \infty$ as $|\mu| \to \infty$ if the random variable ζ has distinct values (so that $D(0) = \sup\{-A(\mu)\} < \infty$), and for a homogeneous compound renewal process $Z(t)$ we have

$$a := A'(0) = \frac{a_\zeta}{a_\tau} \sim \mathbf{E}\frac{Z(T)}{T}, \tag{2.18}$$

$$\sigma^2 := A''(0) = \frac{1}{a_\tau}\mathbf{E}(\zeta - a\tau)^2 \sim \mathbf{D}\frac{Z(T)}{T} \quad \text{as } T \to \infty. \tag{2.19}$$

Moreover,

$$A(\mu) \sim \frac{1}{T} \ln \mathbf{E}e^{\mu Z(T)} \quad \text{as} \quad T \to \infty \tag{2.20}$$

under weak assumptions. The listed properties (2.18)–(2.20) of $A(\mu)$ remain valid in the inhomogeneous case under the conditions of "admissible inhomogeneity." These properties are obtained in [13] from the representations (2.12) and (2.13) together with the limit relations established, for instance, in [5].

Thus, as regards the compound renewal process $Z(t)$ the basic function $A(\mu)$ plays the same role as the logarithm of the Laplace transform of the distribution of a random variable does with respect to the corresponding random walk; for random walks (2.18)–(2.20) hold as exact equalities rather than asymptotically.

It is obvious that we always have $A^\infty(\mu) \le A(\mu)$. Since $(0, 0) \in (\mathscr{A})$ with $\mathbf{A}(0, 0) = 0$ and $A(0) = 0$, it follows that $A^\infty(\mu) < A(\mu)$ in a neighborhood of the point $\mu = 0$. Take the maximal interval (μ_-, μ_+)

containing the point $\mu = 0$ such that $A^\infty(\mu) < A(\mu)$ for all $\mu \in (\mu_-, \mu_+)$. In the domain $\mu \in (\mu_-, \mu_+)$ we have that $(-A(\mu), \mu) \in (\mathscr{A})$ and $\lambda = A(\mu)$ is a unique solution to the equation

$$\mathbf{A}(-\lambda, \mu) = 0. \tag{2.21}$$

Therefore, by the Implicit Function Theorem the solution $\lambda = A(\mu)$ is an analytic function of μ. Refer to (μ_-, μ_+) as the *main domain of analyticity* of the function $A(\mu)$.

The principal difference between the basic function $A(\mu)$ and the logarithm of the Laplace transform is that the values $\mu_\pm$ and $\mu^\pm$ need not coincide and in the domain $(\mu_\pm, \mu^\pm)$ the function $A(\mu)$ can be finite but not analytic. If on the boundary $\partial\mathscr{A}$ the value of $\mathbf{A}(\lambda, \mu)$ for $\mu \in (\mu_\pm, \mu^\pm)$ oscillates near 0, then $A(\mu)$ and $A^\infty(\mu)$ may coincide on some closed intervals in $(\mu_\pm, \mu^\pm)$ and differ on some open intervals. On the interval with $A(\mu) > A^\infty(\mu)$ the function $A(\mu)$ is once again analytic, but outside these intervals it is not in general.

Furthermore, take a point $\mu(\alpha)$ at which the supremum in (2.13) is attained. If $A(\mu)$ is differentiable at $\mu(\alpha)$ then the value of $\mu(\alpha)$ is a solution to the equation

$$A'(\mu) = \alpha. \tag{2.22}$$

Since $A'(\mu)$ is monotone increasing, this equation has a unique solution

$$\mu(\alpha) = (A')^{(-1)}(\alpha), \tag{2.23}$$

which is the inverse function of $A'(\mu)$. Put $\alpha_\pm := A'(\mu_\pm \mp 0)$. Then $\mu(\alpha)$ is obviously analytic in the domain (α_-, α_+). Moreover, [13] established the relation $\mu(\alpha) = D'(\alpha)$, hence

$$D(\alpha) = \alpha\mu(\alpha) - A(\mu(\alpha)) = \int_a^\alpha \mu(v)\, dv; \tag{2.24}$$

this implies that $D(\alpha)$ is also analytic in (α_-, α_+).

By the argument above, we infer from (2.9) and (2.10) that $D(\theta, \alpha)$ is finite, see (2.7), in the cone

$$\mathscr{D}^{<\infty} = \left\{(\theta, \alpha) : D\left(\frac{\alpha}{\theta}\right) < \infty\right\}$$

and analytic in the cone

$$\mathscr{D} := \left\{(\theta, \alpha) : \frac{\alpha}{\theta} \in (\alpha_-, \alpha_+)\right\}. \tag{2.25}$$

Putting

$$\lambda(\alpha) := -A(\mu(\alpha)), \tag{2.26}$$

we deduce from (2.21) and (2.24) that

$$D(\alpha) = \lambda(\alpha) + \alpha\mu(\alpha), \quad \mathbf{A}(\lambda(\alpha), \mu(\alpha)) = 0, \quad (\lambda(\alpha), \mu(\alpha)) \in (\mathscr{A}) \tag{2.27}$$

for $\alpha \in (\alpha_-, \alpha_+)$; i.e., as $\alpha \in (\alpha_-, \alpha_+)$ grows, the point $(\lambda(\alpha), \mu(\alpha))$ moves toward the boundary $\partial\mathscr{A}^{\le 0}$, passing through the point $(0,0)$ when $\alpha = a$. The pair $(\lambda(\alpha), \mu(\alpha))$, like $(-A(\mu), \mu)$, determines a parametrization of the boundary of $\mathscr{A}^{\le 0}$.

By (2.9) and (2.24), for $(\theta, \alpha) \in \mathscr{D}$ we have

$$\begin{aligned} \mathbf{D}'_{(1)}(\theta, \alpha) &= D\left(\frac{\alpha}{\theta}\right) - \frac{\alpha}{\theta} D'\left(\frac{\alpha}{\theta}\right) = \frac{\alpha}{\theta}\mu\left(\frac{\alpha}{\theta}\right) - A\left(\mu\left(\frac{\alpha}{\theta}\right)\right) - \frac{\alpha}{\theta}\mu\left(\frac{\alpha}{\theta}\right) \\ &= -A\left(\mu\left(\frac{\alpha}{\theta}\right)\right) = \lambda\left(\frac{\alpha}{\theta}\right), \end{aligned} \tag{2.28}$$

$$\mathbf{D}'_{(2)}(\theta, \alpha) = D'\left(\frac{\alpha}{\theta}\right) = \mu\left(\frac{\alpha}{\theta}\right). \tag{2.29}$$

The resulting relations are analogs of (2.4) and (2.5).

Since we study the asymptotics for the probability (1.2) in the domain of normalized deviations

$$\alpha := \frac{x}{T} \in (\alpha_-, \alpha_+), \tag{2.30}$$

it is natural to call the latter *Cramér's domain* by analogy with the domains of analyticity arising in the classical limit theorems for random walks. However, in contrast to those classical theorems, it is not always possible to obtain asymptotics for (1.2) in the whole domain (2.30). In some cases the domain must be restricted. If

$$\lambda_+ := \sup\{\lambda : A^{(\tau)}(\lambda) < \infty\} > D(0)$$

then the restrictions are not necessary. Consider the case $\lambda_+ \le D(0)$. In order to describe the required restriction, we need the properties of $\lambda(\alpha)$ (see the definition in (2.26)) which by (2.27) and (2.28) satisfies

$$\lambda(\alpha) = D(\alpha) - \alpha\mu(\alpha) = \mathbf{D}'_{(1)}(\theta, \alpha)\Big|_{\theta=1}. \tag{2.31}$$

Lemma 2.2. (i) *The function $\lambda(\alpha)$ reaches its maximal value at $\alpha = 0$, with $\lambda(0) = D(0)$. It increases, not always strictly, for $\alpha < 0$ and decreases for $\alpha > 0$, with $\lambda(a) = 0$.*

(ii) *Take $\lambda_+ \le D(0)$ and the maximal closed interval $[\beta_-, \beta_+]$ on which $\lambda(\alpha) \ge \lambda_+$, so that*

$$\lambda(\alpha) < \lambda_+ \quad \text{for } \alpha \notin [\beta_-, \beta_+],\ 0 \in [\beta_-, \beta_+].$$

If $a > 0$ then $\beta_+ \in [0, a)$. A similar claim holds in the case $a < 0$.

Proof. The first claim follows from Lemma 2.1 of [13]. The second claim follows because $\lambda(\alpha)$ is decreasing on $(0, \infty)$ and

$$\lambda(0) = D(0) \ge \lambda_+, \quad \lambda(a) = -A(0) = 0 < \lambda_+.$$

The cases $a < 0$ are similar. □

For $\lambda_+ > D(0)$ the set $[\beta_-, \beta_+]$ is empty. For $\lambda_+ < D(0)$ the closed interval $[\beta_-, \beta_+]$ is *forbidden* in the study of the normalized deviations α of the process $Z(t)$. In this domain the influence of the large horizontal jump $\tau_{\eta(t)}$ is too large, violating the regularity of the behavior of distributions under study. It is not difficult to see that if $[\beta_-, \beta_+] \subset [\alpha_-, \alpha_+]$ then (β_-, β_+) is the maximal interval on which $\lambda(\alpha) > \lambda_+$ and $\widehat{D}(\alpha) < D(\alpha)$, and that in the case $\lambda_+ = D(0)$ the closed interval $[\beta_-, \beta_+]$ degenerates to the point $\beta_- = \beta_+ = 0$.

As [6, § 4.10; 9] show, in general, meaning without the condition $\lambda_+ \ge D(0)$, the function

$$\widehat{D}(\alpha) := \inf_{0 \le t \le 1} \{D(t, \alpha) + \lambda_+(1 - t)\}$$

is the deviation function for the compound renewal process $Z(t)$. Lemma 2.2 and (2.31) imply that $\widehat{D}(\alpha) < D(\alpha)$ for $\alpha \in (\beta_-, \beta_+)$ and $\widehat{D}(\alpha) = D(\alpha)$ for $\alpha \notin (\beta_-, \beta_+)$. Therefore, the interval (β_-, β_+) coincides with the domain on which $\widehat{D}(\alpha) < D(\alpha)$.

Recall that the condition $\lambda_+ \ge D(0)$ is rather weak. It is met whenever at least one of the following holds:

(1) τ and ζ are independent;

(2) $\mathbf{E}\zeta = 0$;

(3) τ or ζ satisfies condition $[\mathbf{C}_0]$ for all $v > 0$.

We can expand condition 1 on the independence of τ and ζ to the condition that the domain $\mathscr{A}^{\le 0}$ is embedded into the half-plane $\{\lambda \le \lambda_+\}$:

$$\mathscr{A}^{\le 0} \subset \{\lambda \le \lambda_+\}; \tag{2.32}$$

for τ and ζ independent we have

$$\mathscr{A}^{\le 0} \subset \mathscr{A} \subset \{\lambda \le \lambda_+\} \times \{\mu_- \le \mu \le \mu_+\} \subset \{\lambda \le \lambda_+\},$$

and (2.32) is satisfied. Indeed, (2.11) and (2.32) yield

$$D(0) = \sup_{(\lambda,\mu)\in\mathscr{A}^{\le 0}} \lambda.$$

By (2.32) this means that $D(0) \le \lambda_+$.

It is not difficult to see that (2.32) is satisfied whenever $\zeta = \omega + g(\tau, \omega)$, where τ and ω are independent and $|g(t, y)| = o(t + |y|)$ as $t + |y| \to \infty$.

We can note also that $D(0) = \lambda_+$ for independent τ and ζ in the case

$$\mathbf{P}(\zeta > 0) = 1 \quad \text{or} \quad \mathbf{P}(\zeta < 0) = 1. \tag{2.33}$$

Indeed, in this case

$$D(0) = \sup\{\lambda : A^{(\tau)}(\lambda) + A^{(\zeta)}(\mu) \le 0\}. \tag{2.34}$$

In other words, for μ arbitrary we have

$$A^{(\tau)}(D(0) + 0) + A^{(\zeta)}(\mu) > 0 \tag{2.35}$$

and for $\varepsilon > 0$ arbitrary there is μ such that $A^{(\tau)}(D(0) - \varepsilon) + A^{(\zeta)}(\mu) \le 0$. Since $\inf A^{(\zeta)}(\mu) = -\infty$ in the case (2.33), it follows that $D(0) = \lambda_+$ is the unique value of $D(0)$ satisfying (2.34) and (2.35).

The condition $\lambda_+ < D(0)$ is related to the strong dependence between τ and ζ in the domain of large deviations; see [9].

2.4. Examples.

Example 2.1. Suppose that $\zeta = c\tau + \omega$, where τ and ω are independent. Then with natural notational conventions we have

$$\mathbf{A}(\lambda, \mu) = \ln \mathbf{E} e^{\lambda\tau + \mu\zeta} = \ln \mathbf{E} e^{\lambda\tau + \mu(c\tau + \omega)} = A^{(\tau)}(\lambda + c\mu) + A^{(\omega)}(\mu). \tag{2.36}$$

Therefore, the interior of the effective domain $\mathscr{A}$ of $\mathbf{A}(\lambda, \mu)$ is of the form

$$(\mathscr{A}) = \{(\lambda, \mu) : \lambda + c\mu < \lambda_+,\ \mu_-^{(\omega)} < \mu < \mu_+^{(\omega)}\},$$

where $(\mu_-^{(\omega)},\ \mu_+^{(\omega)})$ is the effective domain of the (analytic) function $A^{(\omega)}(\mu)$.

Since $A^{(\tau)}(\lambda)$ is monotone increasing, we have the inverse (generalized) function

$$(A^{(\tau)})^{(-1)}(v) = \sup\{\lambda : A^{(\tau)}(\lambda) \le v\}.$$

It is clear that

$$(A^{(\tau)})^{(-1)}(v) = \lambda_+ \quad \text{for } v \ge A^{(\tau)}(\lambda_+). \tag{2.37}$$

According to (2.12) and (2.36),

$$\begin{aligned} A(\mu) &= -\sup\{\lambda : \mathbf{A}(\lambda, \mu) \le 0\} = -\sup\{\lambda : A^{(\tau)}(\lambda + c\mu) \le -A^{(\omega)}(\mu)\} \\ &-(-c\mu + (A^{(\tau)})^{(-1)}(-A^{(\omega)})(\mu)) = c\mu - (A^{(\tau)})^{(-1)}(-A^{(\omega)}(\mu)). \end{aligned} \tag{2.38}$$

If $\mathbf{P}(\tau > t) = e^{-t}$ then

$$A^{(\tau)}(\lambda) = -\ln(1 - \lambda), \quad (A^{(\tau)})^{(-1)}(v) = 1 - e^{-v},$$

$$A(\mu) = c\mu + e^{A^{(\omega)}(\mu)} - 1,$$

$$D(0) = \sup_{\mu}\{-A(\mu)\} = 1 + \sup_{\mu}[-c\mu - (e^{A^{(\omega)}(\mu)} - 1)].$$

This means that $D(0)$ is the Legendre transform at $-c$ of the convex function $e^{A^{(\omega)}(\mu)}-1$. If $\mathbf{P}(\omega>0)>0$ and $\mathbf{P}(\omega<0)>0$ then $A^{(\omega)}(\mu)\to\infty$ as $|\mu|\to\infty$, and consequently $D(0)\to\infty$ as $|c|\to\infty$. Therefore, by choosing c we can always satisfy the inequality $\lambda_+<D(0)$, so the forbidden set $[\beta_-,\beta_+]$ is nonempty for sufficiently large $|c|$.

Returning to (2.38), take $\mu\in(\mu_-^{(\omega)},\mu_+^{(\omega)})$ and $v=-A^{(\omega)}(\mu)<A^{(\tau)}(\lambda_+)$. For this μ the function $(A^{(\tau)})^{(-1)}(v)$ becomes the ordinary inverse function to $A^{(\tau)}(\lambda)$, i.e., $A^{(\tau)}((A^{(\tau)})^{(-1)}(v))\equiv v$, while $(A^{(\tau)})^{(-1)}(-A^{(\omega)}(\mu))$ is an analytic function of μ.

But if $-A^{(\omega)}(\mu)\ge A^{(\tau)}(\lambda_+)$, which is possible for $\mathbf{E}\omega\ne 0$ and $|\mu_\pm^{(\omega)}|>0$ because $\min_\mu A^{(\omega)}(\mu)<0$ in this case, then for μ we have $A(\mu)=c\mu-\lambda_+=A^\infty(\mu)$; see (2.37) and (2.38). For instance, if $\mathbf{E}\omega<0$ then $\min_{\mu>0}A^{(\omega)}(\mu)<0$ and

$$\mu_-=\mu^-=\mu_-^{(\omega)},\quad \mu_+<\mu^+=\mu_+^{(\omega)},\quad \mu_+>0.$$

If $c=0$ and $\tau\equiv 1$ then $A^{(\tau)}(\lambda)=\lambda$, $\lambda_+=\infty$, and $A(\mu)=A^{(\omega)}(\mu)=A^{(\zeta)}(\mu)$.

In the important particular case that τ and ζ are independent ($c=0$) we have the following statement, see Theorem 2.1 of [13]; we practically proved it while considering Example 2.1.

Lemma 2.3. *If τ and ζ are independent then*
(i)

$$\mu^-=\mu_-^{(\zeta)}:=\inf\{\mu : A^{(\zeta)}(\mu)<\infty\},\quad \mu^+=\mu_+^{(\zeta)}:=\sup\{\mu : A^{(\zeta)}(\mu)<\infty\},$$
$$A^\infty(\mu)=-\lambda_+\quad \text{for } \mu\in(\mu^-,\mu^+);$$

(ii)

$$\lambda_+\ge D(0);$$

(iii) *if*

$$-\inf_\mu A^{(\zeta)}(\mu)<A^{(\tau)}(\lambda_+),$$

which is always so for $\mathbf{E}\zeta=0$, then $\lambda_+>D(0)$, the interval $(\mu^-,\mu^+)=(\mu_-,\mu_+)$ is the main domain of analyticity of $A(\mu)$, and $A(\mu)=-(A^{(\tau)})^{(-1)}(-A^{(\zeta)}(\mu))$.

If

$$-\inf_\mu A^{(\zeta)}(\mu)\ge A^{(\tau)}(\lambda_+)$$

then the solutions $\mu'_-\le\mu'_+$ are defined to the equation $A^{(\zeta)}(\mu)=-A^{(\tau)}(\lambda_+)$. For $\mathbf{E}\zeta>0$ we have

$$\mu'_\pm<0,\quad \mu_-=\mu'_+,\quad \mu_+=\mu^+,\quad A(\mu)=A^\infty(\mu)\quad \text{for } \mu\in[\mu'_-,\mu'_+].$$

For $\mathbf{E}\zeta<0$ we have

$$\mu'_\pm>0,\quad \mu_-=\mu^-,\quad \mu_+=\mu'_-,\quad A(\mu)=A^\infty(\mu)\quad \text{for } \mu\in[\mu'_-,\mu'_+].$$

There are few examples in which we can find $\mathbf{D}(\theta,\alpha)$ and $A(\mu)$ explicitly. Let us present one of them.

Example 2.2. Suppose that τ and ζ are independent and Γ-distributed with parameters (λ_+,γ) and (μ_+,γ) respectively. Let us indicate the dependence of the characteristics in question on γ with the upper left index γ, so that in our case

$$^{\gamma}\mathbf{A}(\lambda,\mu)=-\gamma\ln\left(1-\frac{\lambda}{\lambda_+}\right)-\gamma\ln\left(1-\frac{\mu}{\mu_+}\right)\tag{2.39}$$

and

$$\mathbf{A}(l,\mu):={}^{1}\mathbf{A}(\lambda,\mu)=-\ln\left(1-\frac{\lambda}{\lambda_+}\right)\left(1-\frac{\mu}{\mu_+}\right),$$

and put

$$\Lambda(\theta,\alpha) := {}^1\Lambda(\theta,\alpha), \quad \mathbf{D}_\Lambda(\theta,\alpha) := {}^1\mathbf{D}_\Lambda(\theta,\alpha).$$

Then

$$ {}^\gamma\mathbf{A}(\lambda,\mu) = \gamma\mathbf{A}(\lambda,\mu),$$
$$ {}^\gamma\Lambda(\theta,\alpha) = \sup_{(\lambda,\mu)}\{\lambda\theta+\mu\alpha-\gamma\mathbf{A}(\lambda,\mu)\} = \gamma\Lambda\left(\frac{\theta}{\gamma},\frac{\alpha}{\gamma}\right),$$
$$ {}^\gamma\mathbf{D}_\Lambda(\theta,\alpha) = \inf_{r>0} r\gamma\Lambda\left(\frac{\theta}{r\gamma},\frac{\alpha}{r\gamma}\right) = \mathbf{D}_\Lambda(\theta,\alpha), \tag{2.40}$$

i.e., ${}^\gamma\mathbf{D}_\Lambda(\theta,\alpha)$ is independent of γ.

REMARK 2.1. This circumstance corresponds to the property that the second deviation function $\mathbf{D}(\theta,\alpha)$ is invariant under enlargement of jumps. By the latter we understand the consideration of the jumps ${}^k\boldsymbol{\xi} =_d \mathbf{S}_k$ for fixed k instead of $\boldsymbol{\xi}$. Then it is natural to expect that the number of hits of the enlarged random walk $\{{}^k\mathbf{S}_n\}$, for $n = 0, 1, \dots$, in the receding set TB is about k times less than for the original walk $\{\mathbf{S}_n\}$. This means that the corresponding renewal measure ${}^kH$ satisfies

$$ {}^kH(TB) \approx \frac{1}{k}H(TB).$$

This in turn means that the asymptotics for $\ln {}^kH(TB)$ and $\ln H(TB)$ are the same. This elucidates the invariance of $\mathbf{D}(\theta,\alpha)$ with respect to enlargement.

However, if we mean by enlargement the change of $\boldsymbol{\xi}$ into ${}^{(b)}\boldsymbol{\xi} := b\boldsymbol{\xi}$, which is a change of scale, then with obvious notational conventions we have

$$ {}^{(b)}\mathbf{A}(\lambda,\mu) = \mathbf{A}(b\lambda,b\mu), \quad {}^{(b)}\Lambda(\theta,\alpha) = \sup_{(\lambda,\mu)}\{\lambda\theta+\mu\alpha-\mathbf{A}(b\lambda,b\mu)\} = \Lambda\left(\frac{\theta}{b},\frac{\alpha}{b}\right),$$
$$ {}^{(b)}\mathbf{D}_\Lambda(\theta,\alpha) = \inf_{r>0} r\Lambda\left(\frac{\theta}{rb},\frac{\alpha}{rb}\right) = \frac{1}{b}\mathbf{D}_\Lambda(\theta,\alpha) = \mathbf{D}_\Lambda\left(\frac{\theta}{b},\frac{\alpha}{b}\right).$$

Let us return to Example 2.2. The relation (2.40) means that, in order to find ${}^\gamma\mathbf{D}(\theta,\alpha) = \mathbf{D}(\theta,\alpha)$, it suffices to consider the case that $\gamma = 1$, i.e., τ and ζ are exponentially distributed, while $Z(t)$ is a Poisson renewal process. In this case

$$\Lambda(\theta,\alpha) = \Lambda^{(\tau)}(\theta) + \Lambda^{(\zeta)}(\alpha) = \lambda_+\theta + \mu_+\alpha - 2 - \ln\theta\alpha\lambda_+\mu_+,$$
$$\mathbf{D}_\Lambda(\theta,\alpha) = \lambda_+\theta + \mu_+\alpha + \inf_{r>0} r[2\ln r - 2 - \ln\theta\alpha\lambda_+\mu_+].$$

Differentiating with respect to r the function under the infimum, we obtain the equation

$$2\ln r - 2 - \ln\theta\alpha\lambda_+\mu_+ + 2 = 0$$

for the point

$$r(\theta,\alpha) = \sqrt{\theta\alpha\lambda_+\mu_+}$$

at which the infimum is attained. This implies that

$$\mathbf{D}(\theta,\alpha) = \mathbf{D}_\Lambda(\theta,\alpha) = \lambda_+\theta + \mu_+\alpha - 2\sqrt{\theta\alpha\lambda_+\mu_+} \tag{2.41}$$

for θ and α positive. But if we consider the boundary of the positive quadrant then, for instance, for $\theta = 0$ we have $\Lambda(0,\alpha) = \infty$ and $\mathbf{D}_\Lambda(0,\alpha) = \infty$. Similarly, $\mathbf{D}_\Lambda(\theta,0) = \infty$. The function $\mathbf{D}(\theta,\alpha)$ removes this discontinuity and $\mathbf{D}(\theta,\alpha)$ equals the right-hand side of (2.41) for all $\theta \geq 0$ and $\alpha \geq 0$. In particular,

$$D(\alpha) = \mathbf{D}(1,\alpha) = \mu_+\alpha - 2\sqrt{\alpha\lambda_+\mu_+} + \lambda_+, \tag{2.42}$$

$$D(0) = \lambda_+, \quad \mu(\alpha) = D'(\alpha) = \mu_+ - \sqrt{\frac{\lambda_+\mu_+}{\alpha}}. \tag{2.43}$$

Furthermore, the basic function ${}^{\gamma}A(\mu)$ is a solution to the equation ${}^{\gamma}\mathbf{A}(-\lambda,\mu)=0$ or, which is the same, see (2.39), to

$$\left(1+\frac{\lambda}{\lambda_+}\right)\left(1-\frac{\mu}{\mu_+}\right)=1,$$

and is also independent of γ,

$${}^{\gamma}A(\mu)=A(\mu)=\frac{\lambda_+\mu}{\mu_+-\mu},\quad \lambda(\alpha)=-A(\mu(\alpha))=\lambda_+-\sqrt{\alpha\lambda_+\mu_+}. \tag{2.44}$$

We can show that (2.42)–(2.44) imply the equality $D(\alpha)=\lambda(\alpha)+\mu(\alpha)\alpha$, and the fact that $A(\mu)$ is the Legendre transform of $D(\alpha)$.

3. The Main Statements

3.1. Integro-local theorem for the process $Z(t)$. Along with the moment and structure conditions of the beginning of Section 1, we assume that the normalized deviation $\alpha=\frac{x}{T}$ belongs to some compact set

$$K\subset(\alpha_-,\alpha_+)\setminus[\beta_-,\beta_+], \tag{3.1}$$

which is a closed interval including a neighborhood of the point $\alpha=a$. Consider the compact set

$$\mathscr{A}_K:=\{(\lambda(\alpha'),\mu(\alpha')):\alpha'\in K\},$$

which is an analytic segment of the curve $\partial\mathscr{A}^{\le 0}$ in $\mathbb{R}^2$ including the intersection of a neighborhood of $(0,0)$ with $\partial\mathscr{A}^{\le 0}$. Put $\zeta(t):=\zeta_{\eta(t)}$ and

$$\mathbf{w}=[w_1,w_2),\quad B(u,v,\mathbf{w}):=\{\gamma(T)\ge u,\ \chi(T)\ge v,\ \zeta(T)\in\mathbf{w}\},$$

hoping that the use of $\mathscr{A}_K$ next to $\mathscr{A}_1=\{(\lambda,\mu):\mathbf{A}_1(\lambda,\mu)<\infty\}$ is not confusing because, in contrast to 1, the symbol K stands for a set.

In the subsequent propositions we emphasize the two most important particular cases: The processes Z and Y either are homogeneous or have stationary increment. Denote the latter by $Z^{(st)}$ and $Y^{(st)}$. It is known, see [9, § 3] for instance, that if (τ_1,ζ_1) has the Laplace transform

$$\psi_1(\lambda,\mu)=\psi^{(st)}(\lambda,\mu):=\frac{\psi(\lambda,\mu)-\psi(0,\mu)}{a_\tau\lambda}, \tag{3.2}$$

then $Z(t)$ is a compound renewal process *with stationary increment*; for it the distribution $(\chi(t),\zeta(t))$ is independent of t, as well as the distribution of the increment $Z(t+u)-Z(t)$,

$$\mathscr{A}^{(st)}:=\{(\lambda,\mu):\psi^{(st)}(\lambda,\mu)<\infty\}=\mathscr{A}\cap\mathscr{A}^{(\zeta)}\subset\mathscr{A},\quad \mathscr{A}^{(\zeta)}:=\{\mu:\psi^{(\zeta)}(\mu)<\infty\}.$$

Theorem 3.1. *Suppose that $\alpha\in K$ and the admissible inhomogeneity condition*

$$\mathscr{A}_K\subset(\mathscr{A}_1) \tag{3.3}$$

is satisfied. Then the following hold:

(i) *For some $\varepsilon>0$*

$$\begin{aligned}\mathbf{P}(Z(T)\in\Delta[x);\ B(u,v,\mathbf{w}))&=\mathbf{I}_{\{x\in(-\Delta,0]\}}\mathbf{P}(\tau_1\ge T+v,\ \zeta\in\mathbf{w})\\&+\frac{\Delta}{\sqrt{T}}\psi_1C(\alpha)e^{-TD(\alpha)}I_Z(\alpha,u,v,\mathbf{w})(1+o(1))+O(e^{-T(D(\alpha)+\varepsilon)})\end{aligned} \tag{3.4}$$

as $T\to\infty$, where $\psi_1=\psi_1(\lambda(\alpha),\mu(\alpha))$ and

$$I_Z(\alpha,u,v,\mathbf{w}):=\int_u^\infty e^{\lambda(\alpha)y}\mathbf{P}(\tau>y+v,\ \zeta\in\mathbf{w})\,dy,\quad C(\alpha):=C(1,\alpha),$$

while $C(\theta,\alpha)$ is a positive continuous function on the cone $\mathscr{D}$ defined in (4.5), $\Delta = \Delta_T$ *tends to* 0 *sufficiently slowly as $T\to\infty$, the remainders $o(1)$ and $O(\cdot)$ are uniform in $\alpha\in K$, $u\le u_0$, and all $v\ge 0$ and $\mathbf{w}$; here $u_0<\infty$ is an arbitrary constant.*

(ii) *If $Z(t)$ is a homogeneous compound renewal process then the first term on the right-hand side of* (3.4) *may be omitted* (*treated as remainder*).

(iii) *Suppose that $Z(t)=Z^{(st)}(t)$ is a process with stationary increment and the condition*

$$\mathscr{M}_K := \{\mu(\alpha') : \alpha'\in K\} \subset (\mu_-^{(\zeta)}, \mu_+^{(\zeta)}) \tag{3.5}$$

is met. Then $\mathscr{A}_1=\mathscr{A}^{(st)}$ satisfies the admissible inhomogeneity condition (3.3) *and the first term on the right-hand side of* (3.4) *may be omitted.*

Condition (3.5) *is met whenever the domain $(\mathscr{A})$ is rectangular:*

$$(\mathscr{A}) = (\lambda<\lambda_+)\times(\mu_-<\mu<\mu_+), \tag{3.6}$$

as in the case of τ and ζ independent.

Remark 3.1. It is clear that if

$$\mathbf{P}(\tau_1\ge T+v,\ \zeta_1\in\mathbf{w}) = o\left(\frac{1}{\sqrt{T}}e^{-TD(0)}I_Z(0,0,v,\mathbf{w})\right) \quad \text{as } T\to\infty \tag{3.7}$$

for arbitrary v and $\mathbf{w}$ then we may omit the first term on the right-hand side of (3.4).

If

$$\mathbf{P}(\tau_1\ge T) = o\left(\frac{1}{\sqrt{T}}e^{-TD(0)}\right) \quad \text{as } T\to\infty \tag{3.8}$$

then (3.7) is satisfied with $v=0$ and $\mathbf{w}=\mathbb{R}$. For the fulfillment of (3.8), it is sufficient to require that

$$\lambda_+^{(\tau_1)} > D(0). \tag{3.9}$$

Then the first term on the right-hand side of (3.4) is absorbed into the remainder $O(e^{-T(D(\alpha)+\varepsilon)})$.

Remark 3.2. Integrating $I_Z(\alpha,0,0,\mathbb{R})$ by parts yields

$$I_Z(\alpha,0,0,\mathbb{R}) = \frac{\psi^{(\tau)}(\lambda(\alpha))-1}{\lambda(\alpha)}, \quad \text{where } \psi^{(\tau)}(\lambda) := \mathbf{E}e^{\lambda\tau}. \tag{3.10}$$

From (3.10) and (3.2) we infer that

$$I_Z(\alpha,0,0,\mathbb{R}) = a_\tau \psi^{(st)}(\lambda(\alpha),0). \tag{3.11}$$

If $Z(t)$ is a compound renewal process with stationary increment, $\psi_1(\lambda,\mu)=\psi^{(st)}(\lambda,\mu)$, and the domain $(\mathscr{A})$ is rectangular (see (2.32)) then for the set $\mathscr{A}^{(st)}$ we have

$$(\mathscr{A}^{(st)}) = \{\lambda<\lambda_+\}\times\{\mu_-^{(\zeta)}<\mu<\mu_+^{(\zeta)}\} = (\mathscr{A}).$$

Since $\mathscr{A}_K\subset(\mathscr{A})=(\mathscr{A}^{(st)})$, the admissible inhomogeneity condition (3.3) in this case is always satisfied.

Remark 3.3. The form of the integral $I_Z(\alpha,0,0,\mathbb{R})$ to some extent elucidates the necessity of the presence of the forbidden set $[\beta_-,\beta_+]$ in (3.1): for $\alpha\in(\beta_-,\beta_+)$ or, which is the same, for $\lambda(\alpha)>\lambda_+$ the integral $I_Z(\alpha,0,0,\mathbb{R})$ diverges and the asymptotics for $\mathbf{P}(Z(T)\in\Delta[x))$ is different; cf. Theorem 1.1 of [9]. The presence of the factor $\psi_1(\lambda(\alpha),\mu(\alpha))$ on the right-hand side of (3.4) elucidates the necessity of the admissible inhomogeneity condition (3.3).

Finally, in the example of integro-local theorems for random walks (see [5, Chapter 9] for instance) it is clear that outside Cramér's domain (α_-,α_+) the asymptotics for $\mathbf{P}(Z(T)\in\Delta[x))$ is different. The above means that the hypotheses of Theorem 3.1 are nearly minimal.

Remark 3.4. Henceforth we bear in mind that the choice of the compact set $K\subset\mathbb{R}$ to be as large as possible to satisfy the admissible inhomogeneity condition (3.3), and if $(\mathscr{A})\subset(\mathscr{A}_1)$ then (3.3) ceases to be restrictive. Since $\mathscr{A}_1$ always includes a neighborhood of zero, while $a\in(\alpha_-,\alpha_+)$ and $a\notin[\beta_-,\beta_+]$, we always assume that the compact set K includes a neighborhood of the point a, while the compact set $\mathscr{A}_K$ includes the intersection of the curve $\{(\lambda(\alpha),\mu(\alpha)):\alpha\in\mathbb{R}\}$ with a neighborhood of the point $(0,0)$.

Corollary 3.1. *Under the hypotheses of Theorem 3.1, if Δ is fixed then for some $\varepsilon > 0$ we have*

$$\mathbf{P}(Z(T) \in \Delta[x];\ B(u, v, \mathbf{w})) \\ = \frac{1 - e^{-\mu(\alpha)\Delta}}{\mu(\alpha)\sqrt{T}} \psi_1 C(\alpha) e^{-TD(\alpha)} I_Z(\alpha, u, v, \mathbf{w})(1 + o(1)) + O(e^{-T(D(\alpha)+\varepsilon)}), \tag{3.12}$$

where the remainder is uniform in $\alpha \in K$, $u \le u_0$, v, and $\mathbf{w}$.

Furthermore, if $\alpha \to a$ and $\Delta \ge \Delta_0 > 0$ so that $|\alpha - a|\Delta \to 0$ as $T \to \infty$ then we can replace the factor $\frac{1-e^{-\mu(\alpha)\Delta}}{\mu(\alpha)}$ with Δ.

Proof of Corollary 3.1. Fix $\Delta > 0$. Let $N \to \infty$ sufficiently slowly as $T \to \infty$ and put

$$\Delta_N := \frac{\Delta}{N}, \quad x_k = x + k\Delta_N,\ k = 0, 1, \dots, N-1; \quad \Delta_N[x_k) = [x_k, x_k + \Delta_N).$$

Then for $\alpha_k := \alpha + \frac{k\Delta_N}{T}$ we have

$$TD(\alpha_k) = TD(\alpha) + k\Delta_N \mu(\alpha) + O\left(\frac{(k\Delta_N)^2}{T}\right)$$

as $T \to \infty$. Since

$$\Delta[x) = \bigcup_{k=0}^{N-1} \Delta_N[x_k) \quad \text{as } \Delta_N \to 0$$

and

$$\sum_{k=0}^{N-1} \Delta_N e^{-TD(\alpha_k)} = e^{-TD(\alpha)} \int_0^{\Delta} e^{-\mu(\alpha)y}\, dy(1 + o(1)) = e^{-TD(\alpha)} \frac{1 - e^{-\mu(\alpha)\Delta}}{\mu(\alpha)}(1 + o(1));$$

Theorem 3.1 implies that for the left-hand side of (3.12), equal to

$$\sum_{k=0}^{N-1} \mathbf{P}(Z(T) \in \Delta_N[x_k),\ B(u, v, \mathbf{w})),$$

we have (3.12).

The second claim of the corollary is obvious because $\mu(\alpha)\Delta$ and $(\alpha - a)\Delta$ are of the same order as $\alpha \to a$. The proof of Corollary 3.1 is complete. $\square$

Similarly, Theorem 3.1 and the large deviation principle for $Z(T)$ imply integro-local limit theorems for arbitrary Δ, including integral theorems. For instance, for $\alpha \ge \alpha_0 > a$ we have $\mu(\alpha) \ge \mu_0 > 0$, and the claim in (3.12) remains valid for Δ arbitrary; in particular, we can put $\Delta = \infty$.

3.2. Integro-local theorem for $Y(t)$. The analog of Theorem 3.1 for $Y(t)$ has a somewhat different form. Put

$$B(u, v) := \{\gamma(T) \ge u,\ \chi(T) \ge v\} = B(u, v, \mathbb{R}),$$

$$I_Y(\alpha, u, v) := \int_u^{\infty} e^{\lambda(\alpha)y} \mathbf{E}(e^{\mu(\alpha)\zeta};\ \tau \ge y + v)\, dy,$$

$$\mu_-^{(\zeta)} := \inf\{\mu : \mathbf{E}e^{\mu\zeta} < \infty\}, \quad \mu_+^{(\zeta)} := \sup\{\mu : \mathbf{E}e^{\mu\zeta} < \infty\}.$$

Below in Theorem 3.2, as in Theorem 3.1, we consider the normalized deviation $\alpha = \frac{x}{T} \in K$, where $K \subset (\alpha_-, \alpha_+) \setminus [\beta_-, \beta_+]$ is an arbitrary fixed compact set containing the point a; see (3.1).

Theorem 3.2. *Suppose that $\alpha \in K$ and condition* (3.5) *is met. Then the following hold:*
(i) *If $\lambda(\alpha) \neq 0$ then*

$$I_Y(\alpha, 0, 0) = \frac{1 - \psi^{(\zeta)}(\mu(\alpha))}{l(\alpha)} < \infty, \tag{3.13}$$

where $\psi^{(\zeta)}(\mu) := \psi(0, \mu)$; cf. (3.10).
If $\lambda(\alpha) = 0$ then

$$I_Y(\alpha, 0, 0) = \psi'_{(1)}(0, \mu(\alpha)) < \infty \quad (= \mathbf{E}\tau \text{ if } \alpha = a). \tag{3.14}$$

(ii) *Suppose that the admissible inhomogeneity condition* (3.3) *is met. Then for arbitrary fixed $\Delta > 0$ we have*

$$\mathbf{P}(Y(T) \in \Delta[x);\ B(u, v)) = P_1(v, \Delta[x)) + \frac{e^{\mu(\alpha)\Delta} - 1}{\mu(\alpha)\sqrt{T}} \psi_1 C(\alpha) e^{-TD(\alpha)} [I_Y(\alpha, u, v) + o(1)] \quad \text{as } T \to \infty, \tag{3.15}$$

where

$$P_1(v, \Delta[x)) := \mathbf{P}(\tau_1 > T + v,\ \zeta_1 \in \Delta[x)),$$

while the remainder $o(1)$ is uniform in $\alpha \in K$, $v \geq 0$, and $u \leq u_0$.
(iii) *If the process $Y(t)$ is homogeneous then* (3.15) *holds, and we may omit the term $P_1(v, \Delta[x))$ on the right-hand side of* (3.15).
(iv) *If $Y(t) = Y^{(st)}(t)$ is a process with stationary increment then the admissible inhomogeneity condition $(\mathscr{A}_K) \subset (\mathscr{A}^{(st)})$ and* (3.15) *are satisfied, where we may omit the term $P_1(v, \Delta[x))$.*

It is clear that if $\Delta = \Delta_T \to 0$ sufficiently slowly as $T \to \infty$ then we can replace the factor $\frac{e^{\mu(\alpha)\Delta} - 1}{\mu(\alpha)}$ in (3.15) with Δ. Remarks 3.3 and 3.4 to Theorem 3.1 fully apply here.

As we already noted in Theorem 3.1, condition (3.5) is satisfied for rectangular domains $\mathscr{A}$; see (3.6).

REMARK 3.5. It is clear from (3.15) that in order to neglect the term $P_1(v, \Delta[x))$ in (3.15) in general, we should impose the relation

$$\mathbf{P}(\tau_1 > T,\ \zeta_1 \in \Delta[x)) = o\left(\frac{e^{-TD(\alpha)}}{\sqrt{T}}\right) \quad \text{as } T \to \infty. \tag{3.16}$$

Thus, in order for the initial jump (τ_1, ζ_1) to have no effect on the asymptotics in integro-local theorems for compound renewal processes $Z(t)$ and $Y(t)$, in addition to the main admissible inhomogeneity condition (3.3), we must also impose conditions (3.7) and (3.16) in Theorems 3.1 and 3.2 respectively.

For (3.16) to hold, it suffices that

$$\Lambda_1(T, \alpha T) \geq \Lambda(T, \alpha T) + o(T) \quad \text{as } T \to \infty, \tag{3.17}$$

where $\Lambda_1(T, \alpha T)$ is the deviation function corresponding to (τ_1, ζ_1); in the homogeneous case this condition is always satisfied.

Indeed, suppose for simplicity that the rectangle $[T, \infty) \times \Delta[x)$ is tangent to the level surface of the deviation function Λ_1 at (T, x). Then

$$\mathbf{P}(\tau_1 \geq T,\ \zeta_1 \in \Delta[x)) \leq e^{-\Lambda_1(T, \alpha T)}. \tag{3.18}$$

Lemma 4.1 justified below implies that for some $\varepsilon > 0$ and all sufficiently large T we have

$$\frac{1}{T}\Lambda(T, \alpha T) \geq \mathbf{D}(1, \alpha) + \varepsilon = D(\alpha) + \varepsilon;$$

hence, (3.17) implies that

$$\Lambda_1(T,\alpha T) \ge \Lambda(T,\alpha T) + o(T) \ge TD(\alpha) + \varepsilon T + o(T) \ge TD(\alpha) + \frac{\varepsilon}{2}T$$

as $T \to \infty$. Combined with (3.18), this yields (3.16).

Another condition,

$$(0,\mu(\alpha)) \in (\mathscr{A}_1) \quad \text{for all } \alpha \in K,$$

sufficient for the fulfillment of (3.16), is presented in Lemma 6.1.

REMARK 3.6. Observe that

$$I_Y(\alpha,0,0) = \frac{\psi(\lambda(\alpha),\mu(\alpha)) - \psi(0,\mu(\alpha))}{\lambda(\alpha)} = a_\tau \psi^{(st)}(\lambda(\alpha),\mu(\alpha))$$

by analogy with (3.11).

For compound renewal processes with stationary increment and for rectangular domains $(\mathscr{A})$ the admissible inhomogeneity condition (3.3) is always satisfied.

3.3. Integro-local theorem for finite-dimensional distributions of $Z(t)$. If the conditions

$$\lambda_+ > D(0), \quad \mu_\pm = \mu^\pm, \quad A'(\mu_\pm \mp 0) = \pm\infty \tag{3.19}$$

are met then the forbidden set $[\beta_1, \beta_+]$ is empty, $\alpha_\pm = \pm\infty$, and so we can take as K the compact set $[-N, N]$ with $N > 0$ arbitrary. Assume that N at least ensures that $a \in K$ and $(0,0) \in \mathscr{A}_K$. As we already noted, $\lambda(\alpha) < \lambda_+$ for $\alpha \in K$. Therefore, for each K there exists $\delta_K > 0$ with

$$\lambda(\alpha) < \lambda_+ - \delta_K \quad \text{for } \alpha \in K.$$

We need the following condition:

$[\widehat{\mathbf{P}}_K]$ *There exist a distribution $\widehat{P}$ in $\mathbb{R}^2$ and a constant $c < \infty$ such that*

$$e^{-\delta_K t}\mathbf{P}(\tau > t+v,\ \zeta \in \mathbf{w} \mid \tau > t) \le c\widehat{P}((v,\infty),\ \mathbf{w}) \tag{3.20}$$

for all $v > 0$, $\mathbf{w}$, and all sufficiently large t. Furthermore,

$$\widehat{\psi}(\lambda,\mu) < \infty \quad \textit{for } (\lambda,\mu) \in \mathscr{A}_K, \tag{3.21}$$

where $\widehat{\psi}$ is the Laplace transform of $\widehat{P}$.

If τ and ζ are independent then it is not difficult to verify that condition $[\widehat{\mathbf{P}}_K]$ is met whenever τ satisfies the large deviation principle:

$$\ln \mathbf{P}(\tau > t) \ge -\Lambda^{(\tau)}(t) + o(t) \tag{3.22}$$

or

$$\ln \mathbf{P}(\tau > t) \sim -\lambda_+ t \text{ as } t \to \infty,$$

which is the same for $\lambda_+ < \infty$.

Indeed, the fulfillment of (3.22) yields

$$e^{-\delta_K t}\mathbf{P}(\tau > t+v)/\mathbf{P}(\tau > t) \le \exp\{-\delta_K t - \Lambda^{(\tau)}(t+v) + \Lambda^{(\tau)}(t) + o(t)\}. \tag{3.23}$$

Since

$$\Lambda^{(\tau)}(t+v) - \Lambda^{(\tau)}(t) \ge \lambda^{(\tau)}(t)v, \quad \lambda^{(\tau)}(t) := (\Lambda^{(\tau)})'(t) \to \lambda_+ \quad \text{as } t \to \infty,$$

it follows that for $\lambda_+ < \infty$, arbitrary $\varepsilon > 0$, and all sufficiently large t the left-hand side of (3.23) is at most $\exp\{-(\lambda_+ - \varepsilon)v\}$. Therefore, if we put

$$\widehat{P}((v,\infty),\ \mathbf{w}) = e^{-(\lambda_+ - \varepsilon)v}\mathbf{P}(\zeta \in \mathbf{w})$$

then for sufficiently small $\varepsilon > 0$ the compact set $\mathscr{A}_K$ is embedded into the domain of convergence of $\widehat{\psi}(\lambda, \mu)$, and (3.21) is satisfied. For $\lambda_+ = \infty$ the argument is even simpler.

Along with Theorem 3.1, we have the following integro-local theorem for the finite-dimensional distributions of the process $Z(t)$. Given some tuples

$$0 = u_0 < u_1 < \cdots < u_M = 1, \quad \alpha_1, \ldots, \alpha_M, \quad \alpha_0 = 0$$

of numbers, put

$$\gamma_j := \frac{\alpha_j - \alpha_{j-1}}{u_j - u_{j-1}}, \ j = 1, \ldots, M, \quad I_Z(\gamma) := I_Z(\gamma, 0, 0, \mathbb{R}) = \int_0^\infty e^{\lambda(\gamma)y} \mathbf{P}(\tau > y)\, dy,$$

where the function $I_Z(\cdot, \cdot, \cdot, \cdot)$ is taken from Theorem 3.1. Suppose further that $P_\gamma((v, \infty), \mathbf{w})$ is the measure

$$P_\gamma((v, \infty), \mathbf{w}) := \int_0^\infty e^{\lambda(\gamma)y} \mathbf{P}(\tau > y + v,\ \zeta \in \mathbf{w})\, dy$$

and $\psi^{(\gamma)}(\lambda, \mu)$ is the Laplace transform of P_γ:

$$\psi^{(\gamma)}(\lambda, \mu) := \int_0^\infty \int_{-\infty}^\infty e^{\lambda v + \mu z} P_\gamma(dv, dz).$$

The value of γ_0 is undefined, and it is convenient to put

$$\psi^{(\gamma_0)}(\lambda, \mu) := \int_0^\infty \int_{-\infty}^\infty e^{\lambda v + \mu z} \mathbf{P}(\tau_1 \in dv, \zeta_1 \in dz).$$

With this notation, the following theorem holds.

Theorem 3.3. *Assume condition* (3.19), *condition* $[\widehat{\mathbf{P}}_K]$, *the admissible inhomogeneity condition* $\mathscr{A}_K \subset (\mathscr{A}_1)$, *and condition* (3.7) *if* $\alpha_1 = 0$. *Then for* $x_k = \alpha_k T$ *we have*

$$\mathbf{P}\biggl(\bigcap_{k=1}^M \{Z(u_k T) \in \Delta[x_k)\}\biggr)$$
$$= \frac{\Delta^M}{T^{M/2}} e^{-TJ(f)} \prod_{j=1}^M \frac{\psi^{(\gamma_{j-1})}(\lambda(\gamma_j), \mu(\gamma_j))}{\sqrt{u_j - u_{j-1}}} C(\gamma_j) I_Z(\gamma_j)(1 + o(1)), \tag{3.24}$$

where $f = f(t)$, *for* $0 \le t \le 1$, *is a continuous broken line with nodes at the points* (u_j, α_j) *for* $0 \le j \le M$, *while*

$$J(f) := \int_0^1 D(f'(t))\, dt,$$

and $\Delta = \Delta_T \to 0$ *sufficiently slowly as* $T \to \infty$. *The remainder* $o(1)$ *in* (3.24) *is uniform in* $(\gamma_1, \ldots, \gamma_M) \in K^M = \prod_{j=1}^M \{|\gamma_j| \le N\}$ *for* $N > 0$ *arbitrary.*

Along with condition $[\widehat{\mathbf{P}}_K]$ we can point out another condition, possibly simpler and more lucid, which also guarantees the fulfillment of an integro-local theorem for finite-dimensional distributions of the process $Z(t)$.

[**h**] *There exists a function $h(v) = o(v)$ as $v \to \infty$ such that for all $\varepsilon > 0$ and $\lambda < \lambda_+ - \varepsilon$ we have*

$$\int_v^\infty e^{\lambda t} \mathbf{P}(\tau > t,\ \zeta \in \mathbf{w})\, dt \le e^{\lambda v + h(v)} \mathbf{P}(\tau > v,\ \zeta \in \mathbf{w}).$$

For the fulfillment of this condition it suffices that there exist t_0 and $q < 1$, depending on λ, such that the measure

$$p(dt, \mathbf{w}) := e^{(\lambda_+ - \varepsilon)t} \mathbf{P}(\tau > t,\ \zeta \in \mathbf{w})\, dt$$

has the property

$$p(d(t + t_0), \mathbf{w}) \le q p(dt, \mathbf{w})$$

for all $\mathbf{w}$ and all sufficiently large t. Then obviously

$$\int_v^\infty e^{\lambda t} \mathbf{P}(\tau > t,\ \zeta \in \mathbf{w})\, dt \le \frac{t_0}{1 - q} e^{\lambda v} \mathbf{P}(\tau > v, \zeta \in \mathbf{w}).$$

If τ and ζ are independent then condition [**h**] is satisfied as soon as it is for $\mathbf{w} = \mathbb{R}$.

Theorem 3.4. *Assume the hypotheses of Theorem* 3.3 *with condition* [$\widehat{\mathbf{P}}_K$] *replaced by condition* [**h**]. *Then* (3.24) *is preserved.*

From (3.24) we see that even on assuming (3.19), [$\widehat{\mathbf{P}}_K$], and [**h**] the increments of the process $Z(t)$ on large intervals are not asymptotically independent in the domain of large deviations in the sense of the integro-local theorem. (For random walks this independence holds.) This is clear because of the presence of the factors $\psi^{(\gamma_{j-1})}(\lambda(\gamma_j), \mu(\gamma_j))$ depending on two adjacent normalized increments γ_{j-1} and γ_j. In the domain of normal and moderately large deviations this dependence disappears.

3.4. Normal and moderately large deviations. If

$$\alpha = \frac{x}{T} \to a \quad \text{as } T \to \infty$$

then $(\lambda(\alpha), \mu(\alpha)) \to (0, 0)$ as $T \to \infty$,

$$\psi_1(l(\alpha), \mu(\alpha)) \to 1, \tag{3.25}$$

$$I_Z(\alpha, u, v, \mathbf{w}) \to \int_u^\infty \mathbf{P}(\tau \ge y + v,\ \zeta \in \mathbf{w})\, dy,$$

$$I_Y(\alpha, u, v) \to \int_u^\infty \mathbf{P}(\tau \ge y + v)\, dy = \mathbf{E}\tau \mathbf{P}(\chi \ge u + v), \tag{3.26}$$

where χ is the magnitude of the overshoot of the walk $\{T_n\}$ across the infinitely distant barrier.

Furthermore, since the function $D(\alpha)$ is analytic on the interval (α_-, α_+) containing the point $\alpha = a$, we have

$$e^{-TD(\alpha)} = e^{-\frac{(x - Ta)^2}{2\sigma^2 T}(1 + o(1))} \quad \text{as } T \to \infty,$$

where σ^2 is defined in (2.19). Finally, in this case the admissible inhomogeneity conditions in Theorems 3.1 and 3.2 are satisfied. All that enables us to state the integro-local theorem in the domain of normal and moderately large deviations in the following form:

Corollary 3.2. *Suppose that $\alpha = \frac{x}{T} \to a$ as $T \to \infty$. Then*

$$C(\alpha) \sim C(a) = \frac{1}{\mathbf{E}\tau\sigma\sqrt{2\pi}} \quad \text{as } T \to \infty, \tag{3.27}$$

and for every fixed $\Delta > 0$ and some $\varepsilon > 0$ the following hold.

(I)

$$\mathbf{P}(Z(T) \in \Delta[x);\ B(u, v, \mathbb{R})) = \frac{\Delta}{\sqrt{2\pi T}\sigma} e^{-TD(\alpha)} \mathbf{P}(\chi \geq u + v)(1 + o(1)) + O(e^{-T\varepsilon}), \tag{3.28}$$

where the remainders $o(1)$ and $O(e^{-T\varepsilon})$ are uniform in $u \leq u_0$ and $v \geq 0$.

If $y := x - aT = o(T^{\frac{2}{3}})$ as $T \to \infty$ then we can replace the factor $e^{-TD(\alpha)}$ on the right-hand side of (3.28) with $e^{-\frac{y^2}{2T\sigma^2}}$.

(II)

$$\mathbf{P}(Y(T) \in \Delta[x); \quad B(u, v)) = \frac{\Delta}{\sqrt{2\pi T}\sigma} e^{-TD(\alpha)} (\mathbf{P}(\chi \geq u + v) + o(1)), \tag{3.29}$$

where the remainder $o(1)$ is uniform in $u \leq u_0$ and $v \geq 0$.

Proof of Corollary 3.2. (I). In view of Corollary 3.1, as well as (3.25) and (3.26), Theorem 3.1 implies that

$$\begin{aligned} &\mathbf{P}(Z(T) \in \Delta[x);\ B(u, v, \mathbb{R})) = \mathbf{I}_{\{x \in (-\Delta, 0]\}} \mathbf{P}(\tau_1 > T + v) \\ &+ \frac{\Delta}{\sqrt{T}} e^{-TD(\alpha)} C(a) \mathbf{E}\tau \mathbf{P}(\chi \geq u + v)(1 + o(1)) + O(e^{-T\varepsilon}) \end{aligned} \tag{3.30}$$

for every fixed $\Delta > 0$, where the remainder is uniform in $u \leq u_0$ and $v \geq 0$. Since $\mathbf{P}(\tau_1 > T + v) = o(e^{-T\varepsilon})$ as $T \to \infty$ for sufficiently small $\varepsilon > 0$, we can omit the first term on the right-hand side of (3.30). In order to establish (3.28), it remains to justify (3.27).

From (3.30) we infer that the "predensities" $\frac{\mathbf{P}(Z(T) \in \Delta[x))}{\Delta}$ for Δ tending to 0 sufficiently slowly as $T \to \infty$ get closer to the function

$$\frac{C(\alpha)\mathbf{E}\tau}{\sqrt{T}} e^{-\frac{y^2}{2T\sigma^2}} \tag{3.31}$$

in the sense of relative convergence.

On the other hand, the distribution $\frac{Z(T) - aT}{\sigma\sqrt{T}}$ converges as $T \to \infty$ to the standard normal distribution; see [5, Theorem 10.6.2] for instance. Comparing the values of $\mathbf{P}(Z(T) - aT \in [-b\sqrt{T},\ b\sqrt{T}])$ for $b > 0$ obtained using (3.31) and the normal approximation, we find that necessarily

$$C(a)\mathbf{E}\tau = \frac{1}{\sqrt{2\pi}\sigma}.$$

This justifies (3.27).

(II) Claim (3.29) is proved similarly on using Theorem 3.2. □

As we already noted, the Integral Theorem for $Z(T)$ in the domain of normal deviations is well-known; see [5, Theorem 10.6.2] for instance, where instead of Cramér's condition $[\mathbf{C}_0]$ we need only the finiteness of the second moment $\mathbf{E}|\boldsymbol{\xi}|^2 < \infty$. In the particular case that the coordinates τ and ζ of the vector $\boldsymbol{\xi}$ are independent, [5] also obtained an integro-local theorem in the domain of normal deviations for the homogeneous processes $Z(T)$ and $Y(T)$ with finite second moment (Theorem 10.6.3 of [5]) and uniform remainder $\left(\frac{1}{\sqrt{T}}\right)$. In the general case an integro-local theorem in the domain of normal deviations is established in [8], but under the additional assumption $\mathbf{E}|\boldsymbol{\xi}|^{2+\delta} < \infty$ for $\delta \geq \sqrt{2} - 1$ and Cramér's condition on the characteristic function.

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