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A SEMIFIELD PLANE OF ODD ORDER ADMITTING AN AUTOTOPISM SUBGROUP ISOMORPHIC TO A_5

© **O. V. Kravtsova and B. K. Durakov**
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Abstract: We develop an approach to constructing and classifying semifield projective planes with the use of a spread set. The famous conjecture is discussed on the solvability of the full collineation group of a finite semifield nondesarguesian plane. We construct a matrix representation of a spread set of a semifield plane of odd order admitting an autotopism subgroup isomorphic to the alternating group A_5 and find a series of semifield planes of odd order not admitting A_5.

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Introduction

Consider a finite projective plane coordinatized by a semifield and, as a consequence, possessing a large group of central collineations (automorphisms). There is a conjecture [1, VIII.6] on the solvability of the full collineation group of every semifield nondesarguesian plane of finite order; see also Question 11.76 in [2]. Presently this conjecture is confirmed only for some classes of semifield planes; see [3–5] for instance. As [1] shows, the conjecture on the solvability of the full automorphism group for a semifield nondesarguesian plane reduces to the solvability of the group of autotopisms which are collineations fixing a triangle. If an autotopism group is of odd order then it is solvable by the Feit–Thompson Theorem. Therefore, while discussing the question of solvability, we should consider only semifield planes admitting order 2 autotopisms.

On assuming that the full collineation group is not solvable, the simple composition factors must be isomorphic to the known simple groups. The straightforward brute-force search through the list of simple nonabelian groups leads to a rather large number of situations. We propose to test the existence of a subgroup of the full collineation group isomorphic to the alternating group A_5 which is a subgroup of many simple nonabelian groups.

On the other hand, it is of great interest to construct and classify the translation planes that admit some collineation subgroups; see [6, 7] for instance. Nowadays these planes are often constructed and studied with the use of computer algebra, combinatorial approach, and the methods of linear algebra.

A method for constructing semifield planes as well as other translation planes rests on a vector space of even dimension and a spread set, i.e., a family of linear transformations defining congruence partition. A matrix representation of the spread set determines the geometric properties of the plane and the algebraic properties of the coordinatizing semifield.

A matrix representation of the spread set of the semifield plane of arbitrary odd order p^N admitting an autotopism group isomorphic to the alternating group A_4 was obtained in [8]. The authors apply those results to construct a matrix representation of the spread set of the semifield plane of odd order p^N admitting an autotopism subgroup isomorphic to A_5. We find a series of semifield planes of odd order not admitting A_5.

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Theorem 1. *Consider a semifield plane π of odd order p^N, where $p > 2$ is a prime, whose autotopism group Λ includes a subgroup $H \simeq A_5$. Then $N = 4n$ and the plane π can be defined using the $8n$-dimensional vector space over $\mathbb{Z}_p$ so that the spread set $R \subset GL_{4n}(p) \cup \{0\}$ of the plane is formed by $4n \times 4n$-matrices of the form*

$$\theta(V_1, U_1, V_2, U_2) = \begin{pmatrix} \mu(U_2) & -\psi(V_2) & \psi(U_1) & \varphi(V_1) \\ \psi(V_2) & \mu(U_2) & -\psi(V_1) & \varphi(U_1) \\ -\psi(U_1) & \psi(V_1) & \mu(U_2) & \varphi(V_2) \\ V_1 & U_1 & V_2 & U_2 \end{pmatrix}, \quad (1)$$

where $V_1, U_1, V_2 \in Q_1$ and $U_2 \in Q_2$, while Q_1 and Q_2 are spread sets in $GL_n(p) \cup \{0\}$; ψ, μ, and φ are injective linear mappings from Q_1 and Q_2 respectively into $GL_n(p) \cup \{0\}$; furthermore,

$$\mu(E) = E, \quad \varphi(E) \neq E, \quad \psi(E) = -E.$$

Theorem 2. *The semifield plane of order p^N, where $p > 2$ is a prime and $p - 1$ is divisible by 4, does not admit a subgroup of autotopisms isomorphic to the alternating group A_5.*

Note that Theorem 2 holds for many primes, in particular, for 5, 13, 17, 29, and so on. The case $p = 5$ requires a separate consideration.

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1. The Main Definitions and Notation

Let us give some definitions and notation following [1, 9].

For the points and lines of a finite projective plane we can introduce a coordinate system using the elements of a certain coordinatizing set. The properties of the incidence relation in the projective plane enable us to introduce addition and multiplication on the coordinatizing set. Its algebraic properties are closely related to the geometric properties of the corresponding projective plane. In particular, every classical or *desarguesian* projective plane is coordinatized by a field and every translation plane by a quasifield. The coordinatizing set of each semifield plane is a division ring or a semifield.

A method for constructing a finite semifield plane, as well as an arbitrary translation plane, is known [1, VII.3]; it is based on a linear space and a special set of matrices called a spread set.

Take a translation plane π of order q^n, with $q = p^k$ for a prime p, and a linear space W of dimension n over the field $GF(q)$. Then we can express the affine points of the plane π as vectors (x, y) with $x, y \in W$ and the affine lines as cosets of the subgroups

$$V_i = \{(x, xT_i) \mid x \in W\}, \ i = 1, 2, \dots, q^n, \quad V_0 = \{(0, y) \mid y \in W\}.$$

Here T_i are $n \times n$ matrices with entries in $GF(q)$ constituting a *spread set* R of π; see [9].

Definition 1. A set $R = \{T_i \mid i = 1, 2, \dots, q^n\}$ consisting of q^n matrices of size $n \times n$ over the field $GF(q)$ is called a *spread set* whenever the following conditions are met:

(1) R contains the zero and identity matrices;

(2) $\det(T_i - T_j) \neq 0$ for all $i \neq j$.

Therefore, we can express a spread set as $R = \{\theta(w) \mid w \in W\}$, where $\theta : W \to GL(W) \cup \{0\}$ with $\theta(0) = 0$. Define on W the operation $*$ as $x * y = x \cdot \theta(y)$ for $x, y \in W$. Then $\langle W, +, * \rangle$ is a quasifield.

As [9] showed, if a spread set $R \subset GL(W) \cup \{0\}$ is closed under addition then $\langle W, +, * \rangle$ is a semifield. Since this article treats only semifields, we tacitly assume throughout that R is closed.

Definition 2. The subsets

$$N_r = \{x \in W \mid (ab)x = a(bx) \ \forall a, b \in W\},$$

$$N_m = \{x \in W \mid (ax)b = a(xb) \ \forall a, b \in W\},$$

$$N_l = \{x \in W \mid (xa)b = x(ab) \ \forall a, b \in W\}$$

are called respectively the *right, middle*, and *left nuclei* of the semifield W.

These sets are subfields of W, and it is known [1, VIII.2] that we can regard a semifield plane as a linear space over either of the nuclei of the semifield. Usually the left nucleus N_l is convenient. Moreover, a representation of a semifield plane and, accordingly, its spread set over the prime subfield of W is also convenient, and we use precisely this representation.

Take the translation line $[\infty]$ and the translation point (∞) of a plane π. The subgroup Λ formed by the collineations fixing the triangle with vertices P_1, $P_2 = (\infty)$, $P_3 \in [\infty]$ and sides l_1, $l_2 = [\infty]$, $l_3 \ni (\infty)$ is called the *autotopism group*. Since the semifield plane is $((\infty),(\infty))$-transitive and $([\infty],[\infty])$-transitive, without loss of generality we may assume that $P_1 = (0,0)$, $P_3 = (0)$, $l_1 = [0,0]$, and $l_3 = [0]$; our notation agrees with [1].

As we indicated in the Introduction, the conjecture on the solvability of the full collineation group of every semifield nondesarguesian plane reduces to the solvability of the autotopism group. Moreover, by the Feit–Thompson Theorem, while discussing solvability questions, we should consider only the semifield planes admitting order 2 autotopisms.

2. An Autotopism Subgroup Isomorphic to the Alternating Group A_4

We use the main results of [8].

Consider a semifield plane π of odd order p^{4n} whose autotopism group includes a subgroup isomorphic to A_4. Then we may assume that π is determined by an $8n$-dimensional linear space over $\mathbb{Z}_p$ with a spread set in $GL_{4n}(p) \cup \{0\}$.

Take a subgroup $H < \Lambda$ of the autotopism group isomorphic to A_4. Note that $H = \langle \tau, \sigma \rangle \leftthreetimes \langle \gamma \rangle$, where $\sigma, \gamma \in \Lambda$ with $|\tau| = |\sigma| = 2$, $|\gamma| = 3$, $\sigma\tau = \tau\sigma$, and $\tau^\gamma = \sigma$.

To represent involutions in the autotopism group, put

$$\tau = \begin{pmatrix} -E & 0 & 0 & 0 \\ 0 & E & 0 & 0 \\ 0 & 0 & -E & 0 \\ 0 & 0 & 0 & E \end{pmatrix}, \tag{2}$$

$$\sigma = \begin{pmatrix} L & 0 & 0 & 0 \\ 0 & L & 0 & 0 \\ 0 & 0 & L & 0 \\ 0 & 0 & 0 & L \end{pmatrix}, \tag{3}$$

where $L = \begin{pmatrix} -E & 0 \\ 0 & E \end{pmatrix}$ and E is the identity matrix; the matrix entries here and henceforth are square blocks of equal dimensions. The collineation γ is of order 3 and can be represented by the matrix with blocks of lower dimension:

$$\gamma = \begin{pmatrix} 0 & 0 & E & 0 & 0 & 0 & 0 & 0 \\ E & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & E & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & J & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & E & 0 \\ 0 & 0 & 0 & 0 & E & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & E & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & J \end{pmatrix}. \tag{4}$$

Here the matrix J satisfies $J^3 = E$. The matrices (2)–(4) are $8n \times 8n$, while the size of blocks is indicated in the text when necessary.

Let us recall the main result of [8].

Theorem 3. *Consider a semifield plane π of odd order p^N for a prime $p > 2$ whose autotopism group Λ includes a subgroup $H \simeq A_4$. Then $N = 4n$ and the plane π can be prescribed by an $8n$-dimensional vector space over $\mathbb{Z}_p$ such that a spread set $R \subset GL_{4n}(p) \cup \{0\}$ is formed by the $4n \times 4n$*

matrices of the form

$$\theta(V_1, U_1, V_2, U_2) = \begin{pmatrix} \mu(J^{-1}U_2J) & \nu(J^{-1}V_2) & \psi(J^{-1}U_1) & \varphi(J^{-1}V_1)J^{-1} \\ \psi(JV_2) & \mu(JU_2J^{-1}) & \nu(JV_1) & \varphi(JU_1)J^{-1} \\ \nu(U_1) & \psi(V_1) & \mu(U_2) & \varphi(V_2) \\ V_1 & U_1 & V_2 & U_2 \end{pmatrix}, \tag{5}$$

where $J^3 = E$; $\{V_1\} = Q_1$, $\{U_1\} = K_1$, $\{V_2\} = Q_2$, and $\{U_2\} = K_2$ are spread sets in $GL_n(p) \cup \{0\}$; $J^{-1}K_2J = K_2$, $JK_1 = Q_2$, $JQ_1 = K_1$, $JQ_2 = Q_1$; ν, ψ, μ, and φ are injective linear mappings from K_1, Q_1, K_2, and Q_2 respectively to $GL_n(p) \cup \{0\}$; furthermore,

$$\mu(E) = E, \quad \nu(E) = E, \quad \varphi(E) \neq E, \quad \psi(E) \neq E.$$

3. An Autotopism Subgroup Isomorphic to the Alternating Group A_5

Let us pass to the semifield planes admitting a subgroup isomorphic to A_5, accounting for the copresentation of A_5 appearing in [10, 6.3] for instance. Note that we present calculations with substantial omissions, but mention their main points, which could be used to recover complete calculations.

Using a permutation representation of the subgroup H and putting

$$\tau \leftrightarrow (12)(34)(5), \quad \sigma \leftrightarrow (13)(24)(5), \quad \gamma \leftrightarrow (132)(4)(5),$$

take the permutation $(125)(3)(4)$ and denote by α the corresponding autotopism. Then

$$|\alpha| = 3, \quad (\tau\alpha)^2 = \varepsilon, \quad \alpha\gamma^2 = \gamma\alpha^2, \quad |\alpha\sigma| = 5,$$

and $G = \langle \tau, \gamma, \alpha \rangle \simeq A_5$, where ε is the identity mapping. Find a matrix representation of α basing on the listed conditions. Put $\alpha = \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}$ and write down firstly the conditions satisfied by the block matrix $D = \begin{pmatrix} D_1 & D_2 \\ D_3 & D_4 \end{pmatrix}$. For brevity, denote the blocks of the matrix γ as

$$\begin{pmatrix} 0 & 0 & E & 0 \\ E & 0 & 0 & 0 \\ 0 & E & 0 & 0 \\ 0 & 0 & 0 & J \end{pmatrix} = \begin{pmatrix} E_3 & E_1 \\ E_2 & J_4 \end{pmatrix}.$$

(1) The equalities $(\tau\alpha)^2 = \varepsilon$ and $\alpha^3 = \varepsilon$ yield

$$\left(\begin{pmatrix} -E & 0 \\ 0 & E \end{pmatrix} \begin{pmatrix} D_1 & D_2 \\ D_3 & D_4 \end{pmatrix} \right)^2 = \begin{pmatrix} -D_1 & -D_2 \\ D_3 & D_4 \end{pmatrix}^2 = \begin{pmatrix} E & 0 \\ 0 & E \end{pmatrix},$$

$$\begin{pmatrix} -D_1 & -D_2 \\ D_3 & D_4 \end{pmatrix} \begin{pmatrix} -E & 0 \\ 0 & E \end{pmatrix} = \begin{pmatrix} D_1 & D_2 \\ D_3 & D_4 \end{pmatrix}^2.$$

Expanding, we arrive at

$$\begin{cases} D_1^2 + D_2D_3 = D_1, \\ D_1^2 - D_2D_3 = E, \\ D_1D_2 + D_2D_4 = -D_2, \\ D_1D_2 - D_2D_4 = 0, \\ D_3D_1 + D_4D_3 = -D_3, \\ -D_3D_1 + D_4D_3 = 0, \\ D_4^2 + D_3D_2 = D_4, \\ D_4^2 - D_3D_2 = E. \end{cases}$$

Rearrange this system, adding and subtracting the equations pairwise:

$$\begin{cases} 2D_1^2 - D_1 - E = 0, \\ 2D_4^2 - D_4 - E = 0, \\ 2D_2D_3 = D_1 - E, \\ 2D_3D_2 = D_4 - E, \\ (2D_1 + E)D_2 = 0, \\ D_2(2D_4 + E) = 0, \\ (2D_4 + E)D_3 = 0, \\ D_3(2D_1 + E) = 0. \end{cases} \tag{6}$$

It is obvious that the eigenvalues of D_1 and D_4 can be equal only to 1 and $-\frac{1}{2}$; hence, D_1 and D_4 are nondegenerate matrices.

If D_2 or D_3 are nondegenerate then $D_1 = D_4 = -\frac{1}{2}E$. In this case $2D_2D_3 = -\frac{3}{2}E$ and $D_3 = -\frac{3}{4}D_2^{-1}$.

Suppose that $|D_2| \neq 0$ and $|D_3| \neq 0$. Then

$$D = \begin{pmatrix} -\frac{1}{2}E & D_2 \\ -\frac{3}{4}D_2^{-1} & -\frac{1}{2}E \end{pmatrix}, \quad D^2 = D^{-1} = \begin{pmatrix} -\frac{1}{2}E & -D_2 \\ \frac{3}{4}D_2^{-1} & -\frac{1}{2}E \end{pmatrix}.$$

Under the condition $\alpha\gamma^2 = \gamma\alpha^2$ the matrix D satisfies

$$\begin{pmatrix} -\frac{1}{2}E_2 + D_2E_1 & -\frac{1}{2}E_3 + D_2J_4^2 \\ -\frac{3}{4}D_2^{-1}E_2 - \frac{1}{2}E_1 & -\frac{3}{4}D_2^{-1}E_3 - \frac{1}{2}J_4^2 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2}E_3 + \frac{3}{4}E_1D_2^{-1} & -E_3D_2 - \frac{1}{2}E_1 \\ -\frac{1}{2}E_2 + \frac{3}{4}J_4D_2^{-1} & -E_2D_2 - \frac{1}{2}J_4 \end{pmatrix}.$$

Comparing the entries in position 21, we obtain $-\frac{1}{2}E_3 + D_2J_4^2 = -E_3D_2 - \frac{1}{2}E_1$. Inserting $D_2 = \begin{pmatrix} D_{21} & D_{22} \\ D_{23} & D_{24} \end{pmatrix}$ yields

$$\begin{pmatrix} 0 & D_{22}J^2 \\ -\frac{1}{2}E & D_{24}J^2 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2}E & 0 \\ -D_{21} & -D_{22} \end{pmatrix},$$

which is impossible. Thus, $|D_2| = |D_3| = 0$.

Suppose that $D_2 = 0$. Then $D_1 = D_4 = E$ and $D_3 = 0$; hence, D is the identity matrix, and the condition $\alpha\gamma^2 = \gamma\alpha^2$ is violated. For $D_3 = 0$ we obtain a similar result. Suppose that $D_1 = -\frac{1}{2}E$ or $D_4 = -\frac{1}{2}E$. Then $2D_2D_3 = -\frac{3}{2}E$ and the matrices D_2 and D_3 are nondegenerate. We obtain the same result in the case of arbitrary scalar matrices D_1 and D_4; let us state the argument as a lemma.

Lemma 1. *D_1 and D_4 are nondegenerate nonscalar matrices, while D_2 and D_3 are degenerate nonzero matrices.*

(2) Consider the equality $\alpha\gamma^2 = \gamma\alpha^2$:

$$\begin{pmatrix} D_1 & D_2 \\ D_3 & D_4 \end{pmatrix}\begin{pmatrix} E_2 & E_3 \\ E_1 & J_4^2 \end{pmatrix} = \begin{pmatrix} E_3 & E_1 \\ E_2 & J_4 \end{pmatrix}\begin{pmatrix} D_1 & -D_2 \\ -D_3 & D_4 \end{pmatrix},$$

$$\begin{cases} D_1E_2 + D_2E_1 = E_3D_1 - E_1D_3, \\ D_1E_3 + D_2J_4^2 = -E_3D_2 + E_1D_4, \\ D_3E_2 + D_1E_1 = E_2D_1 - J_4D_3, \\ D_3E_3 + D_4J_4^2 = -E_2D_2 + J_4D_4. \end{cases} \tag{7}$$

Replace each of the blocks D_i for $i = 1, 2, 3, 4$ with $\begin{pmatrix} D_{i1} & D_{i2} \\ D_{i3} & D_{i4} \end{pmatrix}$. Then (7) turns into 16 equalities:

$$\begin{cases} D_{21} = -D_{31}, \\ D_{11} = -D_{32}, \\ D_{23} = D_{11}, \\ D_{13} = D_{12}; \end{cases} \quad \begin{cases} D_{12} = D_{41}, \\ D_{22}J^2 = D_{42}, \\ D_{14} = -D_{21}, \\ D_{24}J^2 = -D_{22}; \end{cases} \quad \begin{cases} D_{41} = D_{13}, \\ D_{31} = D_{14}, \\ D_{43} = -JD_{33}, \\ D_{33} = -JD_{34}; \end{cases} \quad \begin{cases} D_{32} = -D_{23}, \\ D_{42}J^2 = -D_{24}, \\ D_{34} = JD_{43}, \\ D_{44}J^2 = JD_{44}. \end{cases}$$

With these, express the matrix D as

$$D = \begin{pmatrix} D_{11} & D_{12} & -D_{14} & D_{22} \\ D_{12} & D_{14} & D_{11} & -D_{22}J \\ D_{14} & -D_{11} & D_{12} & D_{22}J^2 \\ D_{33} & -J^2D_{33} & -JD_{33} & D_{44} \end{pmatrix} \tag{8}$$

with the condition

$$JD_{44}J = D_{44}. \tag{9}$$

Lemma 2. *The matrix D, or equivalently A, defining the autotopism α is of the form* (8)*; moreover, conditions* (6) *and* (9) *are met.*

Inserting now the blocks D_i for $i = 1, 2, 3, 4$ into (6), we obtain 32 new matrix equalities, which we list in the order most convenient for further calculations:

$$\begin{cases} 2D_{11}^2 + 2D_{12}^2 - D_{11} - E = 0, \\ 2D_{12}^2 + 2D_{14}^2 - D_{14} - E = 0, \\ 2D_{14}^2 + 2D_{11}^2 + D_{12} - E = 0, \\ 2D_{11}^2 - 2D_{22}D_{33} + D_{14} - E = 0, \\ 2D_{12}^2 - 2D_{22}D_{33} - D_{12} - E = 0, \\ 2D_{14}^2 - 2D_{22}D_{33} + D_{11} - E = 0; \end{cases} \tag{10}$$

$$\begin{cases} 2D_{11}D_{12} + 2D_{12}D_{14} - D_{12} = 0, \\ 2D_{12}D_{14} - 2D_{14}D_{11} - D_{11} = 0, \\ 2D_{14}D_{11} - 2D_{11}D_{12} + D_{14} = 0, \\ 2D_{11}D_{12} + 2D_{22}J^2D_{33} + D_{11} = 0, \\ 2D_{12}D_{14} + 2D_{22}J^2D_{33} + D_{14} = 0, \\ 2D_{14}D_{11} - 2D_{22}J^2D_{33} - D_{12} = 0; \end{cases} \tag{11}$$

$$\begin{cases} 2D_{12}D_{11} + 2D_{14}D_{12} - D_{12} = 0, \\ 2D_{14}D_{12} - 2D_{11}D_{14} - D_{11} = 0, \\ 2D_{11}D_{14} - 2D_{12}D_{11} + D_{14} = 0, \\ 2D_{12}D_{11} + 2D_{22}JD_{33} + D_{11} = 0, \\ 2D_{14}D_{12} + 2D_{22}JD_{33} + D_{14} = 0, \\ 2D_{11}D_{14} - 2D_{22}JD_{33} - D_{12} = 0; \end{cases} \tag{12}$$

$$\begin{cases} 2D_{33}D_{11} - 2D_{44}D_{33} - JD_{33} = 0, \\ 2D_{33}D_{12} + 2JD_{44}D_{33} - D_{33} = 0, \\ 2D_{33}D_{14} - 2J^2D_{44}D_{33} - J^2D_{33} = 0, \\ 2D_{33}D_{11} - 2J^2D_{33}D_{12} + D_{33} = 0, \\ 2D_{33}D_{12} - 2J^2D_{33}D_{14} - J^2D_{33} = 0, \\ 2D_{33}D_{14} + 2J^2D_{33}D_{11} - JD_{33} = 0; \end{cases} \tag{13}$$

$$\begin{cases} 2D_{11}D_{22} - 2D_{22}D_{44} - D_{22}J^2 = 0, \\ 2D_{12}D_{22} + 2D_{22}JD_{44} - D_{22} = 0, \\ 2D_{14}D_{22} - 2D_{22}D_{44}J - D_{22}J = 0, \\ 2D_{11}D_{22} - 2D_{12}D_{22}J + D_{22} = 0, \\ 2D_{12}D_{22} - 2D_{14}D_{22}J - D_{22}J = 0, \\ 2D_{14}D_{22} + 2D_{11}D_{22}J - D_{22}J^2 = 0; \end{cases} \tag{14}$$

$$\begin{cases} 2JD_{33}D_{22}J^2 - 2D_{44}^2 + D_{44} + E = 0, \\ 2D_{33}D_{22} + 2J^2D_{33}D_{22}J - D_{44} + E = 0. \end{cases} \tag{15}$$

Solving all systems in turns for the products and squares of the matrices D_{ij}, we obtain

$$\left\{\begin{array}{l}
D_{11}^2 = \frac{1}{4}(D_{11} - D_{12} - D_{14} + E), \\
D_{12}^2 = \frac{1}{4}(D_{11} + D_{12} + D_{14} + E), \\
D_{14}^2 = \frac{1}{4}(-D_{11} - D_{12} + D_{14} + E), \\
D_{44}^2 = \frac{1}{4}D_{44}(E + J + J^2) + \frac{1}{4}E, \\
D_{22}D_{33} = \frac{1}{4}(D_{11} - D_{12} + D_{14} - E), \\
D_{33}D_{22} = \frac{1}{4}D_{44}(E + J - J^2) - \frac{1}{4}E, \\
D_{11}D_{12} = D_{12}D_{14} = \frac{1}{4}(-D_{11} + D_{12} + D_{14}), \\
D_{12}D_{14} = D_{14}D_{12} = \frac{1}{4}(D_{11} + D_{12} - D_{14}), \\
D_{14}D_{11} = D_{11}D_{14} = \frac{1}{4}(-D_{11} + D_{12} - D_{14}), \\
D_{22}JD_{33} = D_{22}J^2D_{33} = \frac{1}{4}(-D_{11} - D_{12} - D_{14}), \\
D_{33}D_{11} = D_{33}D_{14} = \frac{1}{4}(-E + J + J^2)D_{33}, \\
D_{33}D_{12} = \frac{1}{4}(E + J + J^2)D_{33}, \\
D_{44}D_{33} = \frac{1}{4}(-E - J + J^2)D_{33}, \\
D_{11}D_{22} = D_{14}D_{22} = \frac{1}{4}D_{22}(-E + J + J^2), \\
D_{12}D_{22} = \frac{1}{4}D_{22}(E + J + J^2), \\
D_{22}D_{44} = \frac{1}{4}D_{22}(-E + J - J^2).
\end{array}\right. \tag{16}$$

(3) Consider the condition $|\alpha\sigma| = 5$ and raise the matrix

$$\overline{D} = \begin{pmatrix} -D_{11} & D_{12} & D_{14} & D_{22} \\ -D_{12} & D_{14} & -D_{11} & -D_{22}J \\ -D_{14} & -D_{11} & -D_{12} & D_{22}J^2 \\ -D_{33} & -J^2D_{33} & JD_{33} & D_{44} \end{pmatrix}$$

to power 5. Replace the appearing products of the blocks D_{ij} using (16). Write down

$$\overline{D}^2 = \begin{pmatrix} -D_{14} & D_{11} & -D_{12} & -D_{22}J^2 \\ -D_{11} & -D_{12} & D_{14} & -D_{22} \\ D_{12} & D_{14} & D_{11} & -D_{22}J \\ JD_{33} & -D_{33} & -J^2D_{33} & D_{44}J^2 \end{pmatrix},$$

$$\overline{D}^3 = \begin{pmatrix} -D_{14} & -D_{11} & D_{12} & -D_{22}J^2 \\ D_{11} & -D_{12} & D_{14} & D_{22} \\ -D_{12} & D_{14} & D_{11} & D_{22}J \\ JD_{33} & D_{33} & J^2D_{33} & D_{44}J^2 \end{pmatrix},$$

$$\overline{D}^5 = \begin{pmatrix} E & 0 & 0 & 0 \\ 0 & E & 0 & 0 \\ 0 & 0 & E & 0 \\ 0 & 0 & 0 & D_{44} \end{pmatrix}.$$

Since $D_{44} = E$, we have $J^2 = E = J^3$ by (9), and so $J = E$. Rearrange (16) as

$$\begin{cases} D_{11}^2 = \frac{1}{4}(D_{11} - D_{12} - D_{14} + E), \\ D_{12}^2 = \frac{1}{4}(D_{11} + D_{12} + D_{14} + E), \\ D_{14}^2 = \frac{1}{4}(-D_{11} - D_{12} + D_{14} + E), \\ D_{22}D_{33} = \frac{1}{4}(D_{11} - D_{12} + D_{14} - E), \\ D_{33}D_{22} = 0, \\ D_{11}D_{12} = D_{12}D_{14} = \frac{1}{4}(-D_{11} + D_{12} + D_{14}), \\ D_{12}D_{14} = D_{14}D_{12} = \frac{1}{4}(D_{11} + D_{12} - D_{14}), \\ D_{14}D_{11} = D_{11}D_{14} = \frac{1}{4}(-D_{11} + D_{12} - D_{14}), \\ D_{22}D_{33} = \frac{1}{4}(-D_{11} - D_{12} - D_{14}), \\ D_{33}D_{11} = D_{33}D_{14} = \frac{1}{4}D_{33}, \\ D_{33}D_{12} = \frac{3}{4}D_{33}, \\ D_{33} = -\frac{1}{4}D_{33}, \\ D_{11}D_{22} = D_{14}D_{22} = \frac{1}{4}D_{22}, \\ D_{12}D_{22} = \frac{3}{4}D_{22}, \\ D_{22} = -\frac{1}{4}D_{22}. \end{cases} \tag{17}$$

If the characteristic p of the field is not 5 then $D_{22} = D_{33} = 0$. Then $D_{12} = -\frac{1}{2}E$, $D_{14} = -D_{11} + \frac{1}{2}E$, and $D_{11}^2 - \frac{1}{2}D_{11} - \frac{1}{4}E = 0$. This justifies the following lemma.

Lemma 3. *If $p \neq 5$ then the matrix D, or equivalently A, defining the autotopism α is of the form*

$$D = \begin{pmatrix} D_{11} & -\frac{1}{2}E & -D11 - \frac{1}{2}E & 0 \\ -\frac{1}{2}E & -D_{11} + \frac{1}{2}E & D_{11} & 0 \\ -D_{11} + \frac{1}{2}E & -D_{11} & -\frac{1}{2}E & 0 \\ 0 & 0 & 0 & E \end{pmatrix} \tag{18}$$

with the condition that

$$D_{11}^2 - \frac{1}{2}D_{11} - \frac{1}{4}E = 0. \tag{19}$$

The case $p = 5$ is treated below.

Let us determine the form of the matrices in the spread set of the semifield plane for $p \neq 5$ admitting a subgroup of autotopisms isomorphic to A_5. Use the form (5) obtained in [8] and impose the restrictions following from the property that α is a collineation. For every matrix $\theta(x)$ in the spread set the product $A^{-1}\theta(x)D$ must also belong to the spread set. In particular, for $\theta(x) = E$ we obtain $A^{-1}D \in R$, where the matrices A and D are of the form (18). The multiplication yields

$$A^{-1}D = \begin{pmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ 0 & 0 & 0 & E \end{pmatrix} = E,$$

and so $A = D$. For brevity, put $D_{11} = Y$, $4Y^2 - 2Y - E = 0$, and $\alpha = \begin{pmatrix} D & 0 \\ 0 & D \end{pmatrix}$, with

$$D = \begin{pmatrix} Y & -\frac{1}{2}E & -Y - \frac{1}{2}E & 0 \\ -\frac{1}{2}E & -Y + \frac{1}{2}E & Y & 0 \\ -Y + \frac{1}{2}E & -Y & -\frac{1}{2}E & 0 \\ 0 & 0 & 0 & E \end{pmatrix}. \tag{20}$$

By Theorem 3, for $J = E$ we obtain

$$\theta(V_1, U_1, V_2, U_2) = \begin{pmatrix} \mu(U_2) & \nu(V_2) & \psi(U_1) & \varphi(V_1) \\ \psi(V_2) & \mu(U_2) & \nu(V_1) & \varphi(U_1) \\ \nu(U_1) & \psi(V_1) & \mu(U_2) & \varphi(V_2) \\ V_1 & U_1 & V_2 & U_2 \end{pmatrix}, \tag{21}$$

the matrices V_1, U_1, and V_2 lie in the same set Q_1, while $U_2 \in Q_2$, and

$$\mu(E) = \nu(E) = E, \quad \varphi(E) \neq E, \quad \psi(E) \neq E.$$

Impose the condition $D^{-1}\theta(V_1, U_1, V_2, U_2)D \in R$ for all $V_1, U_1, V_2 \in Q_1$ and $U_2 \in Q_2$ and specify the form of the spread set.

(1) Take $V_1 = U_1 = V_2 = 0$. Then

$$D^{-1}\theta(0,0,0,U_2)D = \begin{pmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ 0 & 0 & 0 & E \end{pmatrix} \begin{pmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ 0 & 0 & 0 & U_2 \end{pmatrix} \begin{pmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ 0 & 0 & 0 & E \end{pmatrix}$$
$$= \begin{pmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ 0 & 0 & 0 & U_2 \end{pmatrix} = \theta(0,0,0,U_2).$$

From $\theta(0,0,0,U_2)D = D\theta(0,0,0,U_2)$ it follows that

$$\mu(U_2)Y = Y\mu(U_2) \quad \forall U_2 \in Q_2. \tag{22}$$

(2) Take $V_1 = U_1 = U_2 = 0$ and denote by M the matrix $D^{-1}\theta(0,0,V_1,0)D = \theta(\overline{V_1}, \overline{U_2}, \overline{V_2}, \overline{U_2})$, where

$$\overline{V_1} = m_{41} = V_2\Big(-Y + \frac{1}{2}E\Big), \quad \overline{U_1} = m_{42} = -V_2Y,$$

$$\overline{V_2} = m_{43} = -\frac{1}{2}V_2, \quad \overline{U_2} = m_{44} = 0.$$

The condition $-V_2Y \in Q_1$ for all $V_2 \in Q_1$ implies that the matrix $Y \in Q_1$ belongs to the right nucleus of the semifield with the spread set Q_1.

Comparing the entries of M with the entries of the matrix in the spread set R, we obtain

$$m_{11} = \mu(\overline{U_2}) \Rightarrow -\frac{1}{2}\psi(V_2)Y - \frac{1}{2}Y\nu(V_2) = 0 \Rightarrow \nu(V_2) = -Y^{-1}\psi(V_2)Y;$$

$$m_{13} = \psi(\overline{U_1}) \Rightarrow -\frac{1}{2}\psi(V_2)\Big(Y - \frac{1}{2}E\Big) + Y\nu(V_2)Y = \psi(-V_2Y) \Rightarrow \psi(V_2Y) = \psi(V_2)Y;$$

$$m_{14} = \varphi(\overline{V_1}) \Rightarrow \Big(-Y + \frac{1}{2}E\Big)\varphi(V_2) = \varphi\Big(V_2\Big(-Y + \frac{1}{2}E\Big)\Big) \Rightarrow \varphi(V_2Y) = Y\varphi(V_2);$$

$$m_{33} = \mu(\overline{U_2}) \Rightarrow Y\psi(V_2)\Big(Y - \frac{1}{2}E\Big) + \Big(Y - \frac{1}{2}E\Big)\nu(V_2)Y = 0 \Rightarrow \begin{cases} \psi(V_2)Y = Y\psi(V_2), \\ \nu(V_2) = -\psi(V_2). \end{cases}$$

The cases $\theta(0, U_1, 0, 0)$ and $\theta(V_1, 0, 0, 0)$ yield no new restrictions on the functions μ, ν, φ, and ψ and the matrix Y. Finally, we represent the matrix $\theta(V_1, U_1, V_2, U_2)$ in the form (1) and, in addition to Theorem 1 for the case $p \neq 5$, state two lemmas on the properties of functions.

Lemma 4. *In the hypotheses of Theorem 1, the matrix Y lies in the right nucleus of Q_1 and*

$$\mu(U_2)Y = Y\mu(U_2) \quad \forall U_2 \in Q_2,$$
$$\psi(V_2)Y = Y\psi(V_2) = \psi(V_2Y) \quad \forall V_2 \in Q_1,$$
$$\varphi(V_2Y) = Y\varphi(V_2) \quad \forall V_2 \in Q_1.$$

Lemma 5. *If the spread set of the semifield plane consists of matrices of the form (1) then -1 is not a square in $\mathbb{Z}_p$ and $\varphi(E) \neq k^2E$ for $k \in \mathbb{Z}_p$.*

Proof. Consider the matrix of the form (1) with $V_1 = U_1 = 0$, $V_2 = E$, and $U_2 = kE$, for $k \in \mathbb{Z}_p$:

$$\theta(0,0,E,kE) = \begin{pmatrix} k\mu(E) & -\psi(E) & 0 & 0 \\ \psi(E) & k\mu(E) & 0 & 0 \\ 0 & 0 & k\mu(E) & \varphi(E) \\ 0 & 0 & E & kE \end{pmatrix} = \begin{pmatrix} kE & E & 0 & 0 \\ -E & kE & 0 & 0 \\ 0 & 0 & kE & \varphi(E) \\ 0 & 0 & E & kE \end{pmatrix}.$$

Its determinant equals $\pm|(k^2+1)E| \cdot |\varphi(E) - k^2E|$; consequently, -1 is not a square, and $\varphi(E)$ is not the square of a scalar matrix.

If $p-1$ is divisible by 4 then the multiplicative group of the field $\mathbb{Z}_p$ contains an order 4 element whose square equals -1. Thus, in the case $p \neq 5$ Theorems 1 and 2 are justified.

4. The Case $p = 5$

In the case $p = 5$ rearrange (17) as

$$\begin{cases} D_{11}^2 = -D_{11} + D_{12} + D_{14} - E, \\ D_{12}^2 = -D_{11} - D_{12} - D_{14} - E, \\ D_{14}^2 = D_{11} + D_{12} - D_{14} - E, \\ D_{22}D_{33} = -D_{11} + D_{12} - D_{14} + E, \\ D_{33}D_{22} = 0, \\ D_{11}D_{12} = D_{12}D_{14} = D_{11} - D_{12} - D_{14}, \\ D_{12}D_{14} = D_{14}D_{12} = -D_{11} - D_{12} + D_{14}, \\ D_{14}D_{11} = D_{11}D_{14} = D_{11} - D_{12} + D_{14}, \\ D_{22}D_{33} = D_{11} + D_{12} + D_{14}, \\ D_{33}D_{11} = D_{33}D_{14} = -D_{33}, \\ D_{33}D_{12} = 2D_{33}, \\ D_{11}D_{22} = D_{14}D_{22} = -D_{22}, \\ D_{12}D_{22} = 2D_{22}. \end{cases} \tag{23}$$

The fourth and ninth equalities yield $D_{14} = -D_{11} - 2E$. Inserting this expression into the remaining equalities, we obtain

$$\begin{cases} (D_{11}+E)^2 = D_{12} - E, \\ (D_{12} - 2E)^2 = 0, \\ D_{22}D_{33} = D_{12} - 2E, \\ D_{33}D_{22} = 0, \\ (D_{11}+E)(D_{12}-2E) = (D_{12}-2E)(D_{11}+E) = 0, \\ D_{33}(D_{11}+E) = 0, \\ D_{33}(D_{12}-2E) = 0, \\ (D_{11}+E)D_{22} = 0, \\ (D_{12}-2E)D_{22} = 0. \end{cases} \tag{24}$$

In order to simplify the expressions below, put $D_{11}+E = X$, $D_{12}-2E = Y$, $D_{22} = Z$, and $D_{33} = T$.

Lemma 6. *If $p = 5$ then the matrix D, or equivalently A, defining the autotopism α is of the form*

$$D = \begin{pmatrix} X - E & Y + 2E & X + E & Z \\ Y + 2E & -X - E & X - E & -Z \\ -X - E & -X + E & Y + 2E & Z \\ T & -T & -T & E \end{pmatrix} \tag{25}$$

with the following conditions on the blocks:

$$\begin{cases} X^2 = Y, \\ Y^2 = 0, \\ ZT = Y, \\ TZ = 0, \\ XY = YX = 0, \\ TX = TY = 0, \\ XZ = YZ = 0. \end{cases} \tag{26}$$

In general the matrices A and D are different, which complicates calculations. To narrow down their form further, we establish an auxiliary lemma.

Lemma 7. *Take a spread set R in $GL_n(p) \cup \{0\}$ and two $n \times n$ matrices T and Z over $\mathbb{Z}_p$. If $TUZ = 0$ for all $U \in R$ then either $T = 0$ or $Z = 0$.*

Proof. Put $T \neq 0$ and $Z \neq 0$. Suppose that T contains a nonzero row $t = (t_{i1}, \dots, t_{in})$, while Z contains a nonzero column $z = \begin{pmatrix} z_{1j} \\ \vdots \\ z_{nj} \end{pmatrix}$. Then all elements of the set $M = \{tU \mid U \in R\}$ are distinct. Indeed, suppose that $tU_1 = tU_2$ for some $U_1, U_2 \in R$ with $U_1 \neq U_2$. Then $t(U_1 - U_2) = 0$, but $\det(U_1 - U_2) \neq 0$, and so $t = 0$, which contradicts the assumption.

Consider the linear equation $z_{1j}x_1 + z_{2j}x_2 + \dots + z_{nj}x_n = 0$. Its solutions constitute an $(n-1)$-dimensional linear subspace of $\mathbb{Z}_p^n$, but condition $TUZ = 0$ implies that all p^n elements of M satisfy this equation. The resulting contradiction yields the claim.

Lemma 8. *If a semifield plane of order 5^{4n} admits an autotopism $\begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}$, where A and D are of the form* (25) *then $X_A = Y_A = X_D = Y_D = 0$.*

Proof. Write down the matrices A and D,

$$A = \begin{pmatrix} X_A - E & Y_A + 2E & X_A + E & Z_A \\ Y_A + 2E & -X_A - E & X_A - E & -Z_A \\ -X_A - E & -X_A + E & Y_A + 2E & Z_A \\ T_A & -T_A & -T_A & E \end{pmatrix},$$

$$D = \begin{pmatrix} X_D - E & Y_D + 2E & X_D + E & Z_D \\ Y_D + 2E & -X_D - E & X_D - E & -Z_D \\ -X_D - E & -X_D + E & Y_D + 2E & Z_D \\ T_D & -T_D & -T_D & E \end{pmatrix},$$

on assuming that X_A, Y_A, Z_A, T_A, X_D, Y_D, Z_D, and T_D satisfy (26).

Since α is an autotopism, for all matrices $\theta(V_1, U_1, V_2, U_2)$ in the spread set the product $A^{-1}\theta(V_1, U_1, V_2, U_2)D$ also belongs to the spread set. Consider the matrix $C = A^{-1}\theta(0, 0, 0, U_2)D$ and look at some of its entries:

$$C_{11} = (X_A - E)\mu(U_2)(X_D - E) + (Y_A + 2E)\mu(U_2)(Y_D + 2E) \\ +(-X_A - E)\mu(U_2)(-X_D - E) - Z_A U_2 T_D,$$

$$C_{12} = (X_A - E)\mu(U_2)(Y_D + 2E) + (Y_A + 2E)\mu(U_2)(-X_D - E) \\ +(-X_A - E)\mu(U_2)(-X_D + E) - Z_A U_2 T_D,$$

$$C_{13} = (X_A - E)\mu(U_2)(X_D + E) + (Y_A + 2E)\mu(U_2)(X_D - E) \\ +(-X_A - E)\mu(U_2)(Y_D + 2E) + Z_A U_2 T_D,$$

$$C_{14} = (X_A - E)\mu(U_2)Z_D - (Y_A + 2E)\mu(U_2)Z_D + (-X_A - E)\mu(U_2)Z_D - Z_A U_2,$$

$$C_{41} = -T_A\mu(U_2)(X_D - E) + T_A\mu(U_2)(Y_D + 2E) + T_A\mu(U_2)(X_D + E) + U_2 T_D,$$

$$C_{42} = -T_A\mu(U_2)(Y_D + 2E) - T_A\mu(U_2)(X_D + E) - T_A\mu(U_2)(-X_D + E) - U_2 T_D,$$

$$C_{43} = -T_A\mu(U_2)(X_D + E) + T_A\mu(U_2)(X_D - E) - T_A\mu(U_2)(Y_D + 2E) - U_2 T_D,$$

$$C_{44} = -3T_A\mu(U_2)Z_D + U_2.$$

Here $C_{44} \in Q_2$; hence, $C_{44} - U_2 \in Q_2$ and $T_A\mu(U_2)Z_D \in Q_2$ for all $U_2 \in Q_2$.

By (26), if $|T_A| \neq 0$ then $X_A = Y_A = Z_A = 0$ and if $|Z_D| \neq 0$ then $X_D = Y_D = T_D = 0$.

Suppose that $|T_A| = |Z_D| = 0$. Then $T_A\mu(U_2)Z_D = 0$ for all $U_2 \in Q_2$. Since $\{\mu(U_2) \mid U_2 \in Q_2\}$ is a spread set, Lemma 6 implies that $T_A = 0$ or $Z_D = 0$.

Suppose that $T_A = 0$. Then $C_{41} = U_2 T_D$. If $|T_D| \neq 0$ then $X_D = Y_D = 0$. For $|T_D| = 0$ we have $U_2 T_D = 0$ and $T_D = 0$. Then $A^{-1}\theta(0, 0, 0, U_2)D = \theta(0, 0, 0, U_2)$ for all $U_2 \in Q_2$. For $U_2 = E$ we obtain $A = D$ and $X_A = X_D$; then $\mu(U_2)X_A = X_A\mu(U_2)$ for all $U_2 \in Q_2$. Multiply this equality on the left by X_A:

$$X_A\mu(U_2)X_A = X_A^2\mu(U_2).$$

If $X_A^2 = 0$ then $X_A = 0$ by Lemma 6. If $X_A^2 \neq 0$ then another multiplication yields

$$X_A^2\mu(U_2)X_A = X_A^3\mu(U_2) = 0$$

and $X_A^2 = 0$ or $X_A = 0$, which is a contradiction. Thus, $X_A = X_D = 0$ for $T_A = 0$.

Suppose that $Z_D = 0$. Then $C_{14} = -Z_A U_2 = \varphi(C_{41})$. If $|Z_A| \neq 0$ then $X_A = Y_A = 0$. For $|Z_A| = 0$ we obtain $Z_A U_2 = 0$, and so $Z_A = 0$ and $C_{41} = 0$. Similarly, $C_{42} = C_{43} = 0$; then $A^{-1}\theta(0, 0, 0, U_2)D = \theta(0, 0, 0, U_2)$ and $X_A = X_D = 0$ as in the previous case.

The following cases require a more detailed consideration:

(1) $|T_A| \neq 0$ and $X_A = Y_A = Z_A = 0$;

(2) $|Z_A| \neq 0$ and $X_A = Y_A = Z_A = 0$;

(3) $|T_D| \neq 0$ and $X_D = Y_D = Z_D = 0$;

(4) $|Z_D| \neq 0$ and $X_D = Y_D = Z_D = 0$.

The arguments in all four cases are similar. For instance, in the first case we calculate $A^{-1}D = C$ and obtain $C_{11} = 2Y_D + E$. Since $Y_D^2 = 0$ and $Y_D = \mu(U)$ for some $U \in Q_2$, it follows that $Y_D = 0$. Then $C_{12} = -X_D$, and so $X_D = 0$ because it is degenerate. Doing calculations in the cases 2–4, we arrive at the final conclusion that $X_A = X_D = 0$, which proves the lemma.

Lemma 9. *Suppose that a semifield plane of order 5^{4n} admits an autotopism $\begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}$, where A and D are of the form* (25). *Then the matrix of the spread set of the plane is of the form* (1).

PROOF. Calculate $A^{-1}D$ for $X_A = Y_A = X_D = Y_D = 0$. Then $A^{-1}D = A^2 D$ is an element of the spread set:

$$A^{-1}D = \begin{pmatrix} E - Z_A T_D & Z_A T_D & Z_A T_D & Z_D - Z_A \\ Z_A T_D & E - Z_A T_D & -Z_A T_D & -Z_D + Z_A \\ Z_A T_D & -Z_A T_D & E - Z_A T_D & -Z_D + Z_A \\ -T_A + T_D & T_A - T_D & T_A - T_D & -3T_A Z_D + E \end{pmatrix}.$$

Since $A^{-1}D$ and E belong to the spread set, the degenerate matrices $Z_A T_D$ and $T_A Z_D$ must be equal to zero. Then

$$A^{-1}D - E = \begin{pmatrix} 0 & 0 & 0 & Z_D - Z_A \\ 0 & 0 & 0 & -Z_D + Z_A \\ 0 & 0 & 0 & -Z_D + Z_A \\ -T_A + T_D & T_A - T_D & T_A - T_D & 0 \end{pmatrix}$$

is the zero matrix, $Z_A = Z_D$, and $T_A = T_D$; thus, $A = D$.

Continue the consideration of this case and obtain restrictions on the matrices in the spread set using the condition $D^2\theta(V_1, U_1, V_2, U_2)D \in R$ for all possible V_1, U_1, V_2, and U_2. In particular, for $V_1 = U_1 = U_2 = 0$ put $D^2\theta(0, 0, V_2, 0)D = C$ and look at some blocks C_{ij}:

$$C_{41} = -T\psi(V_2) - 2T\nu(V_2) - V_2 - T\varphi(V_2)T,$$

$$C_{42} = 2T\psi(V_2) + T\nu(V_2) + V_2 + T\varphi(V_2)T,$$

$$C_{43} = T\psi(V_2) + T\nu(V_2) + 2V_2 + T\varphi(V_2)T,$$

$$C_{44} = T\varphi(V_2)Z + T\nu(V_2)Z + V_2 Z - T\varphi(V_2).$$

Subtracting $\theta(-V_2, V_2, 2V_2, 0)$ from the resulting matrix, in the fourth row we see degenerate matrices, which are therefore zero:

$$\begin{cases} C_{41} + V_2 = -T\psi(V_2) - 2T\nu(V_2) - T\varphi(V_2)T = 0, \\ C_{42} - V_2 = 2T\psi(V_2) + T\nu(V_2) + T\varphi(V_2)T = 0, \\ C_{43} - 2V_2 = T\psi(V_2) + T\nu(V_2) + T\varphi(V_2)T = 0. \end{cases}$$

The last system yields $T\nu(V_2) = T\psi(V_2) = 0$ for all V_2, and so $T = 0$. Then $C_{44} = V_2 Z$ is a degenerate matrix, and so $Z = 0$. Inserting $T = Z = 0$ into C, we arrive at

$$\begin{pmatrix} -2\psi(V_2) - 2\nu(V_2) & -\psi(V_2) + \nu(V_2) & 2\psi(V_2) + \nu(V_2) & -\varphi(V_2) \\ \psi(V_2) - \nu(V_2) & -2\psi(V_2) - 2\nu(V_2) & -\psi(V_2) - 2\nu(V_2) & \varphi(V_2) \\ \psi(V_2) + 2\nu(V_2) & -2\psi(V_2) - \nu(V_2) & -\psi(V_2) - \nu(V_2) & 2\varphi(V_2) \\ -V_2 & V_2 & 2V_2 & 0 \end{pmatrix}.$$

Since $C = \theta(-V_2, V_2, 2V_2, 0)$, it follows that $\nu(V_2) = -\psi(V_2)$, which leads to a spread set of the form (1). The proof of Lemma 9 is complete.

Since -1 is a square in the field $\mathbb{Z}_5$, we exclude the case under consideration: there exists no semifield planes of order 5^{4n} admitting a subgroup of automorphisms isomorphic to the alternating group A_5.

The proofs of Theorems 1 and 2 are complete.

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O. V. Kravtsova; B. K. Durakov
Siberian Federal University, Krasnoyarsk, Russia
E-mail address: ol71@bk.ru; bkdurakov@gmail.com