Siberian Mathematical Journal, Vol. 59, *No.* 2, *pp.* 202–206, 2018
Original Russian Text Copyright © 2018 *Afanas′eva S.G.*

ALGEBRAIC SETS IN A DIVISIBLE 2-RIGID GROUP

© **S. G. Afanas′eva** UDC 512.5

Abstract: We answer the following question: Which finite unions of special irreducible algebraic sets in a divisible 2-rigid group are algebraic?

DOI: 10.1134/S0037446618020027

Keywords: divisible rigid group, special algebraic set

1. Introduction

For a normal abelian subgroup C of a group G the action $c \to c^g = g^{-1}cg$ of G on C by conjugation determines on C the structure of a right module over the group ring $\mathbb{Z}[G/C]$.

A group G is called *m-rigid* whenever there is a subnormal series

$$G = G_1 > G_2 > \dots > G_m > G_{m+1} = 1$$

with the abelian quotients G_i/G_{i+1} lacking modular torsion when regarded as right modules over $\mathbb{Z}[G/G_i]$. This series, if it exists, is uniquely determined by G and is called the *rigid series* of the group. The solvability length of every m-rigid group equals precisely m [1]. Every subgroup of a rigid group is rigid too, and its rigid series can be obtained by intersecting the subgroup with a rigid series of the group and removing repetitions. The free solvable groups are important examples of rigid groups. The definition of rigid group is due to Romanovskii whose joint articles with Myasnikov studied many aspects of algebraic geometry over rigid groups [1–6].

A rigid group G is called *divisible* whenever the elements of G_i/G_{i+1} are divisible by the nonzero elements of $\mathbb{Z}[G/G_i]$ or, in other words, whenever G_i/G_{i+1} is a vector space over the skew field $Q(G/G_i)$ of fractions of this ring. Finally, a rigid group G is called *split* whenever G splits into a sequence $A_1A_2\dots A_m$ of semidirect products of the abelian groups $A_i \cong G_i/G_{i+1}$, where A_i normalizes A_j for $i < j$. A split divisible rigid group is uniquely determined by the cardinalities α_i of the bases of the corresponding vector spaces A_i; we denote it by $M(\alpha_1, \dots, \alpha_m)$. For the definitions, see [2]. In fact, every divisible rigid group is split [3].

The coordinate groups of irreducible algebraic sets in an affine space over a divisible rigid group or, more exactly, special irreducible algebraic sets are described in [4]; we give the definition below. In essence, describing coordinate groups is equivalent to describing these sets themselves. Recall that in algebraic geometry over an equationally Noetherian group G the Zariski topology on the affine space G^n is Noetherian; hence, each closed set is a union of finitely many irreducible algebraic sets, but it is not necessarily algebraic itself. Therefore, once a description of irreducible algebraic sets is available, there arises a problem of understanding the conditions under which the union of finitely many irreducible algebraic sets is itself an algebraic set. In this regard, note the articles [7, 8]. The first of them described irreducible algebraic sets in the free length 2 solvable group and in the wreath product of two free abelian groups, i.e., in the corresponding affine space of dimension 1; the second article found conditions for the union of finitely many such sets to be algebraic. This article solves a similar problem for divisible 2-rigid groups: we determine when the union of special irreducible algebraic sets in the group itself is algebraic. Thus, we tackled only the length 2 solvable case and dimension 1, and the problem still remains open in general.

The author was supported by the Russian Foundation for Basic Research (Grant 15–01–01485).

Novosibirsk. Translated from *Sibirskiĭ Matematicheskiĭ Zhurnal*, Vol. 59, No. 2, pp. 257–263, March–April, 2018; DOI: 10.17377/smzh.2018.59.202. Original article submitted July 4, 2017.

2. General Facts of Algebraic Geometry over Groups

Given a group G, denote by F the free product of G and the free group with base $\{x_1, \dots, x_n\}$. The set $S \subseteq G^n$ of solutions to a system of equations $\{v_i(x) = 1 \mid i \in I\}$ in $x = (x_1, \dots, x_n)$ with left-hand sides lying in F is called an *algebraic subset* of G^n. Denote by $I(S) = \{v(x) \in F \mid v(s) = 1,\ s \in S\}$ the *annihilator* of a nonempty algebraic set S and refer as the *coordinate group* of S to the quotient $\Gamma(S) = F/I(S)$. It is obvious that G embeds into this quotient group and $\Gamma(S)$ is generated as a G-group by the images of $x_1, \dots, x_n$.

We can view F as the group of equations on x with coefficients in G. In general, refer as a *group of equations* over G to an arbitrary group D generated by G and the set $\{x_1, \dots, x_n\}$ satisfying the condition that each mapping $x \to (g_1, \dots, g_n) \in G^n$ determines a G-epimorphism $D \to G$. Clearly, we can express D as the quotient group F/H. Among these groups D there exists a group with maximal H, equal to $I(G^n)$: it is $\Gamma(G^n)$.

For an algebraic subset S of G^n, allowing in the definition of D above only the mappings $x \to (g_1, \dots, g_n) \in S$, we obtain a more general definition of a group of equations over G provided that $x \in S$. This group covers $\Gamma(S)$.

Observe that the intersection of an arbitrary family of algebraic sets in G^n is an algebraic set, whereas the union of two algebraic sets need not be algebraic.

The set G^n is endowed with the Zariski topology: we should take algebraic sets as a prebase for the family of closed sets. Recall that a topology is called *Noetherian* whenever all decreasing chains of closed sets are finite. In this case we can uniquely express every closed set as a minimal union of finitely many irreducible closed sets. The Noetherian property of the Zariski topology on the affine space G^n is equivalent to the equationally Noetherian property of G. The latter means that for every n each system of equations in $x_1, \dots, x_n$ over G is equivalent to its finite subsystem. The fundamental result that every rigid group is equationally Noetherian is established in [5].

3. The Divisible Rigid Group $M(\alpha_1, \alpha_2)$, Special Variables, and Irreducible Algebraic Sets

Since this article discusses only metabelian groups, for simplicity we include the necessary definitions and statements from [4] only for them.

3.1. The abelian case. A divisible rigid abelian group $A = M(\alpha_1)$ is isomorphic to the direct sum of α_1 copies of the additive group $\mathbb{Q}$ of rationals. As the group of equations in $x_1, \dots, x_n$ over A, take $A[x_1, \dots, x_n] = A \times \langle x_1 \rangle \times \dots \times \langle x_n \rangle$, i.e., the direct product of A and the free abelian group with base $\{x_1, \dots, x_n\}$. Every algebraic set in A^n is irreducible. The annihilators of nonempty algebraic sets $S \subseteq A^n$ are precisely the isolated subgroups B of the group $A[x_1, \dots, x_n]$ trivially intersecting A. Accordingly, the coordinate groups of (irreducible) algebraic sets here are of the form $\Gamma(S) = A[x_1, \dots, x_n]/B$.

3.2. The length 2 solvable case. We can represent $G = M(\alpha_1, \alpha_2)$ as the group of matrices $\begin{pmatrix} A & 0 \\ T & 1 \end{pmatrix}$, where $A = M(\alpha_1)$ and T is the right vector space with basis $\{t_k \mid k \in K\}$ of cardinality α_2 over the fraction field $Q(A)$ of the group ring $\mathbb{Z}A$. This group splits: $G = A_1 A_2$, where $A_1 = \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}$ is isomorphic to A and $A_2 = \begin{pmatrix} 1 & 0 \\ T & 1 \end{pmatrix}$ is isomorphic to the additive group of T. Every splitting of G into a semidirect product of two abelian subgroups is of the form $A_1^g A_2^g$ with $g \in G$. Fix a splitting. It stands to reason to consider the set $X = \{x_{ij} \mid i = 1, \dots, n;\ j = 1, 2\}$ of special variables subject to the condition that x_{i1} take values in A_1, while x_{i2} do in A_2. Put $X' = \{x_{11}, \dots, x_{n1}\}$. We can express the usual variables x_i as $x_{i1}x_{i2}$, and they can take arbitrary values in G. Conversely, we can regard the special variables as the ordinary ones, but satisfying the additional equations $[x_{i1}, a_1] = 1$ and $[x_{i2}, a_2] = 1$, where a_1 and a_2 are nontrivial elements of A_1 and A_2 respectively.

The group $G[X]$ of equations in the special variables is defined as follows: Firstly, take the abelian group $A[X'] = A \times \langle x_{11} \rangle \times \cdots \times \langle x_{n1} \rangle$, the right $\mathbb{Z}A[X']$-module $T \otimes_{\mathbb{Z}A} \mathbb{Z}A[X']$, and the direct sum

$$T \otimes_{\mathbb{Z}A} \mathbb{Z}A[X'] + x_{12} \cdot \mathbb{Z}A[X'] + \cdots + x_{n2} \cdot \mathbb{Z}A[X']$$

of $T \otimes_{\mathbb{Z}A} \mathbb{Z}A[X']$ and the free right $\mathbb{Z}A[X']$-module with basis $\{x_{12}, \ldots, x_{n2}\}$. Put

$$G[X] = \begin{pmatrix} A[X'] & 0 \\ T \otimes_{\mathbb{Z}A} \mathbb{Z}A[X'] + x_{12} \cdot \mathbb{Z}A[X'] + \cdots + x_{n2} \cdot \mathbb{Z}A[X'] & 1 \end{pmatrix}$$

and identify x_{i1} with the matrix $\begin{pmatrix} x_{i1} & 0 \\ 0 & 1 \end{pmatrix}$ and x_{i2} with the matrix $\begin{pmatrix} 1 & 0 \\ x_{i2} & 1 \end{pmatrix}$. It is clear that $G[X]$ is generated by the subgroup G and the set X. The solutions to the special equations lie in $A_1^n \times A_2^n$. Since we can express each element of G as a product of elements of A_1 and A_2, we can formally identify $A_1 \times A_2$ with G. In terms of special equations we can define special algebraic sets in G^n and all other concepts of algebraic geometry.

Let us indicate, in accordance with Theorem 3 of [4], how the coordinate groups of special irreducible algebraic sets in the affine space G^n look. Firstly, take an arbitrary quotient $A[x_1, \ldots, x_n]/B = C$, where B is an isolated subgroup of $A[x_1, \ldots, x_n]$ with $A \cap B = 1$. Then consider the right $\mathbb{Z}C$-module

$$T \otimes_{\mathbb{Z}A} \mathbb{Z}C + x_{12} \cdot \mathbb{Z}C + \cdots + x_{n2} \cdot \mathbb{Z}C$$

and an isolated submodule H of it trivially intersecting $T \otimes_{\mathbb{Z}A} \mathbb{Z}C$. Denote by W the corresponding quotient module. Then the coordinate groups of special irreducible algebraic sets in G^n are precisely the groups of the form $\begin{pmatrix} C & 0 \\ W & 1 \end{pmatrix}$.

4. Algebraic Sets in a Divisible 2-Rigid Group

In the notation of Section 3 we have $G = M(\alpha_1, \alpha_2) = \begin{pmatrix} A & 0 \\ T & 1 \end{pmatrix}$. Since we study algebraic sets in G itself, i.e., in dimension 1, only two special variables x_{11} and x_{12} appear. For convenience, put

$$x = x_{11},\ y = x_{12},\ A_x = A \times \langle x \rangle,\ T_{xy} = T \otimes_{\mathbb{Z}A} \mathbb{Z}A_x + y \cdot \mathbb{Z}A_x,\ G_{xy} = \begin{pmatrix} A_x & 0 \\ T_{xy} & 1 \end{pmatrix}.$$

An equation over the abelian group A is of the form $mx = a$. If $m = 0$ and $a \neq 0$ then the equation lacks solutions. If $m = 0$ and $a = 0$ then the solution set is the whole group A. If $m \neq 0$ then the equation has a unique solution. Therefore, (irreducible) algebraic sets in the abelian group A are the whole group and all singletons $\{a\}$, while their coordinate groups are respectively A_x and A. According to Section 3, the coordinate group of an irreducible algebraic set S in G is of the form $\begin{pmatrix} C & 0 \\ W & 1 \end{pmatrix}$. Therefore, for C there are two possibilities: A_x and A.

Assume that $C = A$. Then W amounts to the quotient of the module $T + y \cdot \mathbb{Z}A$ over an isolated submodule trivially intersecting T. Suppose that this submodule is nonzero, i.e., contains a nontrivial element $t + yu$, where $t \in T$ and $0 \neq u \in \mathbb{Z}A$. Then S satisfies the equation $t + yu = 0$, which has a unique solution; consequently, S consists of one point $g = \begin{pmatrix} a & 0 \\ -tu^{-1} & 1 \end{pmatrix}$. In the case that the submodule is zero, we obtain $S = \begin{pmatrix} a & 0 \\ T & 1 \end{pmatrix}$.

Assume that $C = A_x$. Then W amounts to the quotient of the module T_{xy} over an isolated submodule trivially intersecting $T \otimes_{\mathbb{Z}A} \mathbb{Z}A_x$. If this submodule is zero then $S = G$. Otherwise we obtain a proper subset of G.

Thus, the special irreducible algebraic sets in the group G are precisely the following:

(1) the whole group G;

(2) the singleton $\left\{\begin{pmatrix} a & 0 \\ t & 1 \end{pmatrix}\right\}$ determined by the system of equations $x = a$ and $y = t$;

(3) the set of the form $\begin{pmatrix} a & 0 \\ T & 1 \end{pmatrix}$ determined by the equation $x = a$;

(4) the set determined by the system of equations $f(x, y) = 0$, where $f(x, y) \in T_{xy}$ constitute a proper isolated submodule over $\mathbb{Z}A_x$ in T_{xy} trivially intersecting $T \otimes_{\mathbb{Z}A} \mathbb{Z}A_x$.

Theorem. *The special algebraic sets in* $G = M(\alpha_1, \alpha_2) = \begin{pmatrix} A & 0 \\ T & 1 \end{pmatrix}$ *are precisely the following* (*irreducible*) *unions of special irreducible algebraic sets:*

(1) *the whole group G;*

(2) *the union of finitely many sets of the form* (2) *and* (3)*:*

$$\left\{\begin{pmatrix} a_1 & 0 \\ e_1 & 1 \end{pmatrix}, \dots, \begin{pmatrix} a_m & 0 \\ e_m & 1 \end{pmatrix}\right\} \cup \begin{pmatrix} a_{m+1} & 0 \\ T & 1 \end{pmatrix} \cup \dots \cup \begin{pmatrix} a_n & 0 \\ T & 1 \end{pmatrix},$$

where $a_1, \dots, a_m, a_{m+1}, \dots, a_n$ are distinct elements of A and $e_i \in T$, although the sets of one of the forms, (2) *or* (3), *may be absent;*

(3) *the union of one set of the form* (4) *and finitely many distinct sets of the form* (3), *although the latter may be absent.*

Proof. Verify firstly that the listed sets are algebraic.

(1) It is straightforward to check that the set

$$\left\{\begin{pmatrix} a_1 & 0 \\ e_1 & 1 \end{pmatrix}, \dots, \begin{pmatrix} a_m & 0 \\ e_m & 1 \end{pmatrix}\right\} \cup \begin{pmatrix} a_{m+1} & 0 \\ T & 1 \end{pmatrix} \cup \dots \cup \begin{pmatrix} a_n & 0 \\ T & 1 \end{pmatrix},$$

with $a_1, \dots, a_m, a_{m+1}, \dots, a_n$ distinct, is determined by the system of equations

$$\begin{cases} t(x - a_1)(x - a_2)\dots(x - a_m)(x - a_{m+1})\dots(x - a_n) = 0, \\ (y - e_1)(x - a_2)\dots(x - a_m)(x - a_{m+1})\dots(x - a_n) = 0, \\ (y - e_2)(x - a_1)(x - a_3)\dots(x - a_m)(x - a_{m+1})\dots(x - a_n) = 0, \\ \dots \\ (y - e_m)(x - a_1)(x - a_2)\dots(x - a_{m-1})(x - a_{m+1})\dots(x - a_n) = 0, \end{cases}$$

where t is an arbitrary nontrivial element of T.

(2) If a set of the form (4) is determined by the system of equations $f(x, y) = 0$, where $f(x, y) \in T_{xy}$, while sets of the form (3) by the equations $x = a_j$ for $j = 1, \dots, n$, then their union is determined by the system of equations $f(x, y)(x - a_1)\dots(x - a_n) = 0$.

Let us explain the lack of other algebraic sets. Suppose that $S_1 \cup S_2 \cup \dots \cup S_r$ is the decomposition of an algebraic set into irreducible components. Then the algebraic closure of $S_1 \cup S_2$ must at least have S_1 and S_2 as two irreducible components. For instance, S_1 cannot lie in a large irreducible set in this algebraic closure.

(3) Take two distinct irreducible algebraic sets S_1 and S_2 of the form (4). We can identify their annihilators $I(S_1)$ and $I(S_2)$ with isolated submodules of the module T_{xy}. Since these submodules are distinct and their ranks are equal to 1, they intersect trivially. The algebraic closure S of $S_1 \cup S_2$ is determined by equations in $I(S_1) \cap I(S_2) = 0$. It coincides with the whole group G and is an irreducible algebraic set. Thus, the decomposition of an algebraic set in G into irreducible components cannot involve two sets of the form (4).

(4) Suppose that $S_1 = \left\{\begin{pmatrix} a & 0 \\ t & 1 \end{pmatrix}\right\}$ is a point and S_2 is an irreducible algebraic set of the form (4) whose equations in G_{xy} constitute an isolated submodule over $\mathbb{Z}A_x$ in T_{xy} of rank 1. If $\begin{pmatrix} a & 0 \\ t & 1 \end{pmatrix} \notin S_2$

then the algebraic closure S of $S_1 \cup S_2$ is determined by equations in $I(S) = I(S_1) \cap I(S_2)$, which constitute a $\mathbb{Z}A_x$-submodule in T_{xy} included into $I(S_2)$. Suppose that $f(x, y) \in I(S_2)$ and furthermore $f(a, t) \neq 0$ and $h(x, y) \in I(S)$. We can assert that A has a free abelian subgroup A' of finite rank and a finite subset $\{t_1, \dots, t_n\}$ of free generators of the module T such that $f(x, y)$ and $h(x, y)$ lie in $T' = t_1 \cdot \mathbb{Z}\langle A', x\rangle + \dots + t_n \cdot \mathbb{Z}\langle A', x\rangle + y \cdot \mathbb{Z}\langle A', x\rangle$.

Observe that the ring $R' = \mathbb{Z}\langle A', x\rangle$ is a unique factorization domain. Represent $f(x, y)$ as $g(x, y) \cdot u(x)$, where $g(x, y) \in T'$ is not divisible by any prime divisor in R' and $u(x) \in R'$. By assumption, there are coprime $v(x), w(x) \in R'$ with $h(x, y) \cdot v(x) = g(x, y) \cdot w(x)$. Since $v(x)$ divides the right-hand side, $v(x)$ divides $w(x)$ by the conditions on $g(x, y)$. Hence, we may assume that $v(x) = 1$. Therefore, $h(x, y) = g(x, y) \cdot w(x)$. Since $h(a, t) = 0$ and $g(a, t) \neq 0$, it follows that $w(a) = 0$. Thus, $h(x, y)$ belongs to the annihilator of $\begin{pmatrix} a & 0 \\ T & 1 \end{pmatrix} \cup S_2$. Hence, $S \supseteq \begin{pmatrix} a & 0 \\ T & 1 \end{pmatrix} \cup S_2$. Since $\begin{pmatrix} a & 0 \\ T & 1 \end{pmatrix} \cup S_2$ is an algebraic set, $S = \begin{pmatrix} a & 0 \\ T & 1 \end{pmatrix} \cup S_2$. By the remark above, this implies that the decomposition of an algebraic set in G into irreducible components cannot involve a set of the form (2) and a set of the form (4) simultaneously.

(5) Suppose that $S_1 = \left\{\begin{pmatrix} a & 0 \\ t' & 1 \end{pmatrix}\right\}$ and $S_2 = \left\{\begin{pmatrix} a & 0 \\ t'' & 1 \end{pmatrix}\right\}$ are distinct singletons. The algebraic closure S of these sets lies in the irreducible algebraic set $\begin{pmatrix} a & 0 \\ T & 1 \end{pmatrix}$; therefore, here we can consider the coordinate group of the last set, which equals $\begin{pmatrix} A & 0 \\ T + y \cdot \mathbb{Z}A & 1 \end{pmatrix}$, as the group of equations. The annihilator of S_1 in this group of equations is identified with the submodule of the $\mathbb{Z}A$-module $T + y \cdot \mathbb{Z}A$ generated by $y - t'$, while the annihilator of S_2, with the submodule generated by $y - t''$. It is clear that $I(S_1) \cap I(S_2) = 0$. Hence, the set S, which is defined by equations in $I(S_1) \cap I(S_2)$, coincides with $\begin{pmatrix} a & 0 \\ T & 1 \end{pmatrix}$. We infer that the decomposition of an algebraic set in G into irreducible components cannot involve any pair of distinct sets $\left\{\begin{pmatrix} a & 0 \\ t' & 1 \end{pmatrix}\right\}$ and $\left\{\begin{pmatrix} a & 0 \\ t'' & 1 \end{pmatrix}\right\}$. This completes the proof of our theorem.

The author is grateful to Professor N. S. Romanovskii for useful discussions.

References

1. Myasnikov A. and Romanovskiy N., "Krull dimension of solvable groups," J. Algebra, vol. 324, no. 10, 2814–2831 (2010).
2. Romanovskii N. S., "Divisible rigid groups," Algebra and Logic, vol. 47, no. 6, 426–434 (2008).
3. Myasnikov A. G. and Romanovskii N. S., "Logical aspects of the theory of divisible rigid groups," Dokl. Math., vol. 90, no. 3, 697–698 (2014).
4. Romanovskii N. S., "Irreducible algebraic sets over divisible decomposed rigid groups," Algebra and Logic, vol. 48, no. 6, 449–464 (2009).
5. Romanovskii N. S., "Equational Noetherianness of rigid soluble groups," Algebra and Logic, vol. 48, no. 2, 147–160 (2009).
6. Romanovskii N. S., "Coproducts of rigid groups," Algebra and Logic, vol. 49, no. 6, 539–550 (2010).
7. Remeslennikov V. N. and Romanovskii N. S., "Irreducible algebraic sets in metabelian groups," Algebra and Logic, vol. 44, no. 5, 336–347 (2005).
8. Romanovskii N. S., "Algebraic sets in metabelian groups," Algebra and Logic, vol. 46, no. 4, 274–280 (2007).

S. G. Afanas′eva
Sobolev Institute of Mathematics, Novosibirsk State Technical University
Specialized Educational Scientific Center of Novosibirsk State University
Novosibirsk, Russia
E-mail address: `melesheva@gmail.com`