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# DESCRIBING NEIGHBORHOODS OF 5-VERTICES IN A CLASS OF 3-POLYTOPES WITH MINIMUM DEGREE 5

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**Abstract:** Lebesgue proved in 1940 that each 3-polytope with minimum degree 5 contains a 5-vertex for which the set of degrees of its neighbors is majorized by one of the following sequences:

$$(6,6,7,7,7),\ (6,6,6,7,9),\ (6,6,6,6,11),$$

$$(5,6,7,7,8),\ (5,6,6,7,12),\ (5,6,6,8,10),\ (5,6,6,6,17),$$

$$(5,5,7,7,13),\ (5,5,7,8,10),\ (5,5,6,7,27),\ (5,5,6,6,\infty),\ (5,5,6,8,15),\ (5,5,6,9,11),$$

$$(5,5,5,7,41),\ (5,5,5,8,23),\ (5,5,5,9,17),\ (5,5,5,10,14),\ (5,5,5,11,13).$$

We prove that each 3-polytope with minimum degree 5 without vertices of degree from 7 to 10 contains a 5-vertex whose set of degrees of its neighbors is majorized by one of the following sequences: $(5,6,6,5,\infty)$, $(5,6,6,6,15)$, and $(6,6,6,6,6)$, where all parameters are tight.

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## 1. Introduction

By a 3-*polytope* we mean a finite 3-dimensional convex polytope. As proved by Steinitz [1], the 3-polytopes are in 1–1 correspondence with 3-connected plane graphs.

The *degree* $d(v)$ of a vertex $v$ ($r(f)$ of a face $f$) in a 3-polytope $P$ is the number of incident edges. Denote the maximum and minimum vertex degree of $P$ by $\Delta$ and $\delta$, respectively. A $k$-*vertex* ($k$-*face*) is a vertex (face) with degree $k$; a $k^+$-*vertex* has degree at least $k$, etc.

In 1904, Wernicke [2] proved that each 3-polytope $P$ with $\delta(P)=5$ has a 5-vertex adjacent to some $6^-$-vertex, which was strengthened by Franklin [3] in 1922 by proving that each 3-polytope $P$ with $\delta(P)=5$ has a 5-vertex adjacent to two $6^-$-vertices. The Franklin Theorem is unimprovable since the class $\mathbf{P}_5$ of 3-polytopes with minimum degree 5 has a 3-polytope with each 5-vertex completely surrounded by 6-vertices.

Recently, Borodin and Ivanova [4] proved an analog of Franklin's Theorem saying that each such 3-polytope has also a vertex of degree at most 6 adjacent to a 5-vertex and another vertex of degree at most 6, which is tight. Also [4] proves that there are no other tight descriptions of 3-vertex paths in $\mathbf{P}_5$ other than those obtained in [3, 4].

We say that $v$ is a *vertex of type* $(k_1,k_2,\dots)$ or simply a $(k_1,k_2,\dots)$-*vertex* if the set of degrees of the vertices adjacent to $v$ is majorized by the vector $(k_1,k_2,\dots)$. If the order of neighbors in the type is neglected, then we put a line over the corresponding degrees. The following description of the neighborhoods of 5-vertices in a 3-polytope $P$ with $\delta(P)=5$ was given by Lebesgue [5, p. 36] in 1940, which includes the results of Wernicke [2] and Franklin [3]:

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**Theorem 1** [5]. *Each 3-polytope with minimum degree 5 has a 5-vertex of one of the following types:*

$$(\overline{6,6,7,7,7}),\ (\overline{6,6,6,7,9}),\ (\overline{6,6,6,6,11}),$$
$$(\overline{5,6,7,7,8}),\ (5,6,\overline{6,7},11),\ (5,6,\overline{6,8},8),$$
$$(5,6,\overline{6,9},7),\ (5,7,6,6,12),\ (5,8,6,6,10),\ (5,6,6,6,17),$$
$$(5,5,\overline{7,7,8}),\ (5,13,5,7,7),\ (5,10,5,7,8),$$
$$(5,8,5,7,9),\ (5,7,5,7,10),\ (5,7,5,8,8),$$
$$(5,5,7,6,12),\ (5,5,8,6,10),\ (5,6,5,7,12),$$
$$(5,6,5,8,10),\ (5,17,5,6,7),\ (5,11,5,6,8),$$
$$(5,11,5,6,9),\ (5,7,5,6,13),\ (5,8,5,6,11),\ (5,9,5,6,10),\ (5,6,6,5,\infty),$$
$$(5,5,7,5,41),\ (5,5,8,5,23),\ (5,5,9,5,17),\ (5,5,10,5,14),\ (5,5,11,5,13).$$

Theorem 1, along with other ideas from [5], has many applications to plane graph coloring problems (first examples of such applications and a recent survey can be found in [6, 7]). Some parameters of Lebesgue's Theorem were improved for narrow classes of plane graphs. For example, in 1963, Kotzig [8] proved that every plane triangulation with minimum degree 5 satisfies $w \le 18$, where $w$ is the minimum weight (the sum of degrees of the boundary vertices) of $5^-$-faces, and conjectured that $w \le 17$. Only in 1989, Borodin [9] gave a confirmation of Kotzig's conjecture, which also allowed him to prove Grünbaum's [10] conjecture about the cyclic 11-connectedness of 5-connected 3-polytopes.

**Theorem 2** [9]. *Each 3-polytope with minimum degree 5 has a $(5,5,7)$-face or a $(5,6,6)$-face, where all parameters are tight.*

By a *minor $k$-star* we mean a star with $k$ rays centered at a $5^-$-vertex. Lebesgue's description [5, p. 36] of the neighborhoods of 5-vertices in class $\mathbf{P}_5$ shows that there is a 5-vertex having three $7^-$-neighbors. In 1996, Jendrol' and Madaras [11] gave a precise description of minor 3-stars in $\mathbf{P}_5$: there is a $(6,6,6)$-star or a $(5,6,7)$-star. Borodin and Ivanova [12], using the tight bound on the minimal weight (sum of vertex degrees) of minor 4-stars by Borodin and Woodall [13], obtained the tight description of minor 4-stars in $\mathbf{P}_5$.

The problem of tightly describing 5-stars in $\mathbf{P}_5$ is far from solution. It is even difficult to obtain tight upper bounds for the minimal weight and height (the maximum degree of neighbors) of minor 5-stars in bounded subclasses of $\mathbf{P}_5$. Some results in this direction can be found in [14–20]. In particular, we [19] proved that if 6-vertices are absent then there is a 5-star of height at most 17, where 17 is tight, which improves the bound 41 implied by Lebesgue's Theorem.

In [5], Lebesgue did not give a proof of Theorem 1 but provided only its idea. In 2013, Ivanova and Nikiforov [21] gave a full proof for Theorem 1 and corrected the following imprecisions in the statement:

(1) in the type $(5,11,5,6,8)$, there should be 15 instead of 11;
(2) in the type $(5,17,5,6,7)$, there should be 27 instead of 17;
(3) in the type $(\overline{6,6,6,6,11})$, the line is not needed;
(4) the type $(\overline{5,6,7,7,8})$ should be replaced by $(5,8,\overline{6,7,7})$ and $(5,7,6,8,7)$;
(5) the type $(5,6,\overline{6,9},7)$ is redundant;
(6) instead of $(5,5,\overline{7,7,8})$, it suffices to write $(5,5,7,\overline{7,8})$.

Later on, Ivanova and Nikiforov [22, 23] improved this corrected version of Theorem 1 by replacing 41 and 23 in the types $(5,5,7,5,41)$ and $(5,5,8,5,23)$ by 31 and 22, respectively.

**Theorem 3** [21–23]. *Each 3-polytope with minimum degree 5 contains a 5-vertex of one of the following types:*

$$(\overline{6,6,7,7,7}),\ (\overline{6,6,6,7,9}),\ (6,6,6,6,11),$$
$$(5,8,\overline{6,7,7}),\ (5,7,6,8,7),\ (5,6,\overline{6,7},11),\ (5,6,\overline{6,8},8),$$

$$(5,7,6,6,12),\ (5,8,6,6,10),\ (5,6,6,6,17),$$
$$(5,5,7,\overline{7,8}),\ (5,13,5,7,7),\ (5,10,5,7,8),\ (5,8,5,7,9),$$
$$(5,7,5,7,10),\ (5,7,5,8,8),\ (5,5,7,6,12),\ (5,5,8,6,10),$$
$$(5,6,5,7,12),\ (5,6,5,8,10),\ (5,27,5,6,7),\ (5,15,5,6,8),$$
$$(5,11,5,6,9),\ (5,7,5,6,13),\ (5,8,5,6,11),\ (5,9,5,6,10),$$
$$(5,6,6,5,\infty),$$
$$(5,5,7,5,31),\ (5,5,8,5,22),\ (5,5,9,5,17),\ (5,5,10,5,14),\ (5,5,11,5,13).$$

If we ignore the cyclic order of neighbors of 5-vertices in Theorem 3, then we obtain this shorter stated fact.

**Corollary 4.** *Each 3-polytope with minimum degree 5 has a 5-vertex of one of the following types:*

$$(\overline{6,6,7,7,7}),\ (\overline{6,6,6,7,9}),\ (\overline{6,6,6,6,11}),$$
$$(\overline{5,6,7,7,8}),\ (\overline{5,6,6,7,12}),\ (\overline{5,6,6,8,10}),\ (\overline{5,6,6,6,17}),$$
$$(\overline{5,5,7,7,13}),\ (\overline{5,5,7,8,10}),\ (\overline{5,5,6,7,27}),$$
$$(\overline{5,5,6,6,\infty}),\ (\overline{5,5,6,8,15}),\ (\overline{5,5,6,9,11}),$$
$$(\overline{5,5,5,7,41}),\ (\overline{5,5,5,8,23}),\ (\overline{5,5,5,9,17}),\ (\overline{5,5,5,10,14}),\ (\overline{5,5,5,11,13}).$$

We can see already from Corollary 4 that if vertices of degree from 7 to 11 are forbidden, then there is a 5-vertex of one of the following types: $(\overline{5,5,6,6,\infty})$, $(\overline{5,6,6,6,17})$, and $(6,6,6,6,6)$. Borodin, Ivanova, and Kazak [24] obtained a precise description of 5-stars in this narrow subclass of $\mathbf{P}_5$.

**Theorem 5** [24]. *Each 3-polytope with minimum degree 5 and without vertices of degree from 7 to 11 has a 5-vertex of one of the following types:* $(\overline{5,5,6,6,\infty})$, $(\overline{5,6,6,6,15})$, *and* $(6,6,6,6,6))$, *where all parameters are tight.*

The purpose of this note is to prove the next theorem which generalizes Theorem 5, maximally extends the scope of its applicability, and reduces the number of involved types of 5-vertices to the minimum.

**Theorem 6.** *Each 3-polytope with minimum degree 5 and without vertices of degree from 7 to 10 has a 5-vertex of one of the following types:* $(5,6,6,5,\infty)$, $(5,6,6,6,15)$, *and* $(6,6,6,6,6)$, *where all parameters are tight.*

Note that the description in Theorem 6 does not hold anymore in the broader class of 3-polytopes with minimum degree 5 and without vertices of degree from 7 to 9 due to a construction by Borodin and Woodall [13], in which every 5-vertex has two $10^+$-neighbors.

## 2. Proving Theorem 6

All parameters in Theorem 6 are best possible. Indeed, the following construction confirming the tightness of the type $(5,6,6,5,\infty)$ appears in [14]. Take three concentric $n$-cycles $C^i = v_1^i \dots v_n^i$, where $n$ is not limited and $1 \le i \le 3$, and join $C^2$ to $C^1$ by edges $v_j^2 v_j^1$ and $v_j^2 v_{j+1}^1$, where $1 \le j \le n$ (addition modulo $n$). Then do the same with $C^2$ and $C^3$. Finally, join all vertices of $C^1$ to a new $n$-vertex, and do the same for $C^3$.

The tightness of $(6,6,6,6,6)$ is confirmed by putting a 5-vertex in each face of the dodecahedron. For the tightness of $(5,6,6,6,15)$, see [24].

Suppose now that a 3-polytope $P'$ is a counterexample to Theorem 6. Let $P$ be a counterexample on the same vertices as $P'$ having the most edges.

Remark 7. In $P$, each $4^+$-face $f = v_1 \dots v_{d(f)}$ with $d(v_1) \ne 6$ satisfies $d(v_i) \ne 6$ whenever $3 \le i \le d(f) - 1$. Otherwise, we could put a diagonal $v_1 v_i$ which contradicts the maximality of $P$.

**Corollary 8.** *In $P$, each $4^+$-face has at most two vertices with degree $\neq 6$. Moreover, if there are precisely two such vertices, then they are adjacent to each other.*

**2.1. Discharging.** The sets of vertices, edges, and faces of $P$ are denoted by $V$, $E$, and $F$, respectively. Euler's formula $|V| - |E| + |F| = 2$ for $P$ implies

$$\sum_{v \in V} (d(v) - 6) + \sum_{f \in F} (2r(f) - 6) = -12. \tag{1}$$

We assign the *initial charge* $\mu(v) = d(v) - 6$ to every vertex $v$ and $\mu(f) = 2d(f) - 6$ to every face $f$, so that only $5^-$-vertices have negative charge. Using the properties of $P$ as a counterexample, we define a local redistribution of charges, such that the *new charge* $\mu'(x)$ is nonnegative whenever $x \in V \cup F$. This will contradict the fact that the sum of the new charges is, by (1), equal to $-12$.

By $v_1, \dots, v_{d(v)}$ denote the neighbors of a vertex $v$ in a cyclic order round $v$, and let $f_1, \dots, f_{d(v)}$ be the faces incident with $v$ in the same order. The vertex $v$ is *simplicial* if $v$ is completely surrounded by 3-faces. The simplicial 5-vertex $v$ is *bad* if $v$ is adjacent to an 11-vertex $v_1$, 6-vertices $v_2$ and $v_5$, and $6^-$-vertices $v_3$ and $v_4$.

We use the following rules of discharging (see Fig. 1).

**R1.** Every $4^+$-face gives 1 to each incident 5-vertex.

**R2.** Every 11-vertex $v$ gives $\frac{1}{4}$ to each simplicial 5-vertex $v_1$ through each incident face with the following exception.

(e) If $v_1$ is bad, then $v$ gives $\frac{1}{2}$ to $v_1$ through each incident face.

**R3.** Suppose that a simplicial 5-vertex $v$ is adjacent to an 11-vertex $v_1$, 5-vertices $v_2$ and $v_5$, and $11^+$-vertices $v_3$ and $v_4$. Then $v$ gives $\frac{1}{2}$ to $v_1$.

**R4.** Every $12^+$-vertex $v$ gives a simplicial 5-vertex $v_2$ the following charge through a face $f = v_2 v v_3$:

(a) $\frac{1}{4}$ if $d(v_3) = 5$,

(b) $\frac{1}{2}$ if $d(v_3) \geq 6$,

with the following exception.

(e) If $d(v) \geq 16$, $d(v_1) = 5$, $d(v_3) = 6$, and $v_2$ has four $6^-$-neighbors, then $v$ gives $\frac{2}{3}$ to $v_2$ through the face $v_2 v v_3$ and $\frac{1}{3}$ through the face $v_1 v v_2$.

**R5.** Let $v$ with $16 \leq d(v) \leq 17$ has a path of neighbors $v_1, \dots, v_5$ such that $d(v_1) = 6$, $d(v_5) \geq 6$, and $v_2$, $v_3$, $v_4$ are simplicial 5-vertices, while $v_2$ has four $6^-$-neighbors. Then $v_4$ gives $\frac{1}{4}$ to $v$.

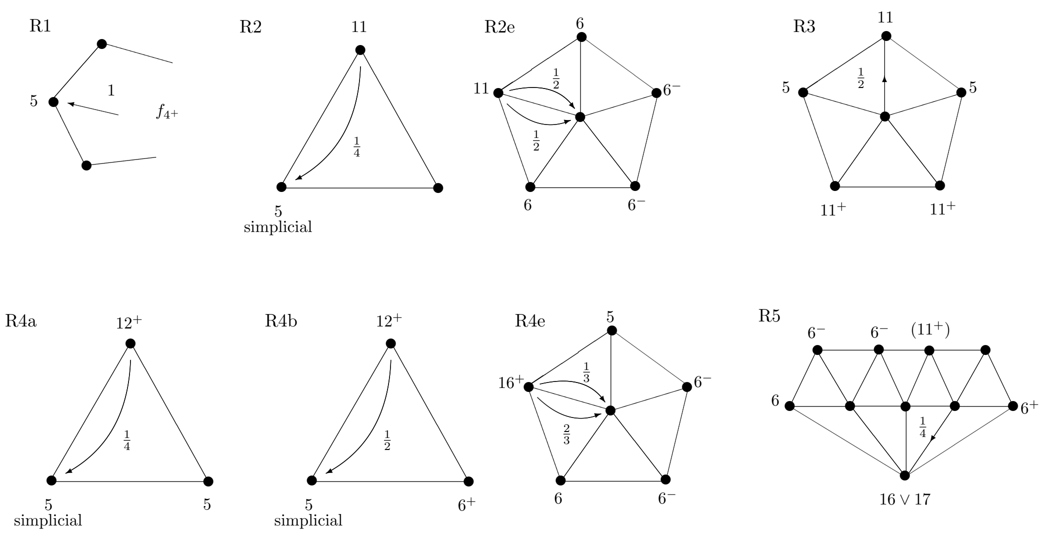


Fig. 1. Rules of discharging

**2.2. Proving $\mu'(x) \geq 0$ whenever $x \in V \cup F$.** First, consider a face $f$ in $P$. If $d(f) = 3$, then $f$ does not participate in discharging, and so $\mu'(v) = \mu(f) = 2\times 3 - 6 = 0$. Note that every $4^+$-face is incident with at most two 5-vertices due to Corollary 8, which implies that $\mu'(v) = 2d(f) - 6 - 2 \times 1 \geq 0$ by R1.

Now let $v$ be a vertex in $P$.

Case 1: $d(v) = 5$. If $v$ is incident with a $4^+$-face, then $\mu'(v) \geq 5 - 6 + 1 = 0$ due to R1. In what follows, we can assume that $v$ is simplicial.

Suppose that $v$ gives $\frac{1}{4}$ by R5 to its $17^-$-neighbor $v_1$; then $d(v_2) = 5$ and $d(v_5) \geq 6$, while $d(v_3) \geq 11$ due to the absence of vertices of type $(5, 6, 6, 5, \infty)$ in our counterexample.

If R3 is not applicable to $v$, then $\mu'(v) \geq -1 + 2 \times \frac{1}{4} + \frac{1}{2} + \frac{1}{4} - \frac{1}{4} = 0$ by R2 and R4.

Now if R3 is applicable to $v$, then $d(v_5) \geq 11$, and so $\mu'(v) \geq -1 + 2 \times \frac{1}{4} + 2 \times \frac{1}{4} + \frac{1}{2} + \frac{1}{4} - \frac{1}{2} - \frac{1}{4} = 0$ by R2, R4, and R3.

So in what follows we can assume that neither R3 nor R5 is applicable to our $v$. Hence it suffices to show that $v$ receives at least 1 from our neighbor in total. Note that $v$ has at most one $11^+$-neighbor due to the absence of $(6, 6, 6, 6, 6)$-vertices in $P$.

Subcase 1.1: $v$ has at least two $11^+$-neighbors. Here $\mu'(v) \geq -1 + 4 \times \frac{1}{4} = 0$ by R2 and R4.

Subcase 1.2: $v$ has precisely one $11^+$-neighbor, $v_2$. Note that the case $d(v_1) = d(v_3) = 5$ is impossible due to the absence of $(5, 6, 6, 5, \infty)$-stars in $P$. Thus in what follows we can assume that $d(v_3) = 6$.

If $d(v_1) = 5$, then $d(v_2) \geq 16$ due to the absence of $(5, 6, 6, 6, 15)$-stars in $P$, which implies that $\mu'(v) \geq -1 + \frac{1}{3} + \frac{2}{3} = 0$ by R4e. If $d(v_1) = 6$, then $\mu'(v) \geq -1 + 2 \times \frac{1}{2} = 0$ by R2e or R4b.

Case 2: $d(v) = 6$. Since $v$ does not participate in discharging, we have $\mu'(v) = \mu'(v) = 6 - 6 = 0$.

Case 3: $d(v) = 11$. Recall that $v$ gives at most $\frac{1}{2}$ through each face by R2.

If $v$ is incident with at least one $4^+$-face, then $\mu'(v) \geq 11 - 6 - 10 \times \frac{1}{2} = 0$ by R1, R2. So we can assume that our $v$ is simplicial.

If $v$ is adjacent to an $11^+$-vertex $v_2$, then each of the $vv_2v_1$ and $vv_2v_3$ takes away at most $\frac{1}{4}$ from $v$, and hence $\mu'(v) \geq 5 - 9 \times \frac{1}{2} - 2 \times \frac{1}{4} = 0$.

If $v$ is adjacent to a nonsimplicial 5-vertex $v_2$, then again each of the $vv_2v_1$ and $vv_2v_3$ takes away at most $\frac{1}{4}$ from $v$, so again $\mu'(v) \geq 5 - 9 \times \frac{1}{2} - 2 \times \frac{1}{4} = 0$.

The two previous paragraphs imply that $v$ can be assumed to have only 6-neighbors and simplicial 5-neighbors.

If $v$ is surrounded by 5-vertices, then each of them has an $11^+$-neighbor other than $v$. Owing to the oddness of 11, there are two adjacent $11^+$-vertices among these $11^+$-neighbors, and so $v$ receives $\frac{1}{2}$ by R3, which implies $\mu'(v) \geq 5 - 11 \times \frac{1}{2} + \frac{1}{2} = 0$.

So, we can assume that $v$ has at least one 6-neighbor.

If $v$ has two consecutive $6^+$-neighbors, then a face incident with both of them and $v$ takes away nothing from $v$, which implies that $\mu'(v) \geq 5 - 10 \times \frac{1}{2} = 0$.

Further, we assume that $v$ has no consecutive 6-neighbors. Owing to the oddness of 11, there is a path $v_1, \ldots, v_k$ with $d(v_1) = d(v_k) = 6$ and $d(v_i) = 5$, where $2 \leq i \leq k-1$ and $k \geq 4$. Note that it is not excluded that $v_1 = v_k$, which takes place if $v$ has a unique 6-neighbor. Thus, $v_2$ and $v_{k-1}$ are not bad, and hence each of the faces $vv_1v_2$ and $vv_{k-1}v_k$ takes away at most $\frac{1}{4}$ from $v$ by R2. So again $\mu'(v) \geq 5 - 9 \times \frac{1}{2} - 2 \times \frac{1}{4} = 0$, as desired.

Case 4: $12 \leq d(v) \leq 15$. Now R4e is not applicable to $v$, so $v$ sends at most $\frac{1}{2}$ through each face by R4a,b, which implies that $\mu'(v) \geq d(v) - 6 - d(v) \times \frac{1}{2} = \frac{d(v)-12}{2} \geq 0$.

Case 5: $16 \leq d(v) \leq 17$. Note that $v$ gives at most $\frac{2}{3}$ through each 3-face, and only to a simplicial 5-vertex. If $v$ gives nothing through at least one incident face, then $\mu'(v) \geq d(v) - 6 - (d(v) - 1) \times \frac{2}{3} = \frac{d(v)-16}{3} = 0$ by R1 and R4.

If $v$ is adjacent to a nonsimplicial 5-vertex $v_2$, then $v_2$ receives nothing from $v$, while each of the vertices $v_1$ and $v_3$ receives at most $\frac{1}{3}$ from $v$ by R4a or R4e, which implies that $\mu'(v) \geq d(v) - 6 - 2 \times \frac{1}{3} - (d(v) - 2) \times \frac{2}{3} = \frac{d(v)-16}{3} \geq 0$.

From now on, we can assume that $v$ is simplicial and each face takes away some positive charge from $v$, which implies that each face at $v$ is incident with a 5-vertex, and all 5-vertices adjacent to $v$ are simplicial.

Thus, $\mu'(v) \geq d(v) - 6 - d(v) \times \frac{2}{3} = \frac{d(v)-18}{3}$, and we have the deficiency of $\frac{1}{3}$ for a 17-vertex and of $\frac{2}{3}$ for a 16-vertex with respect to donating $\frac{2}{3}$ per face.

Suppose that $S_k = v_1, \ldots, v_k$ is a sequence of neighbors of $v$ with $d(v_1) \geq 6$ and $d(v_k) \geq 6$, while $d(v_i) = 5$ whenever $2 \leq i \leq k-1$ and $k \geq 3$, and $f_1, \ldots, f_{k-1}$ are the corresponding faces. (It is not excluded that $S_k = S_{d(v)}$, which happens when $v$ has precisely one $6^+$-neighbor.) We say that the sequence of the faces $f_1, \ldots, f_{k-1}$ *saves* $\varepsilon$ with respect to the level of $\frac{2}{3}$ if these faces take away the total of $k \times \frac{2}{3} - \varepsilon$ from $v$.

Remark 9. Only $v_2$ and $v_{k-1}$ in $S_k$ can receive the charge $\frac{2}{3}$ from $v$ by R4e, while each of the other 5-vertices $v_i$ receives precisely $\frac{1}{4}$ from $v$ through each incident face. So, if $k \geq 5$, then $v_2$ receives at most 1, and $v_3$ receives $\frac{1}{2}$ from $v$ through the incident faces.

Remark 10. If $v$ is completely surrounded by 5-vertices, then $\mu'(v) \geq d(v) - 6 - \frac{d(v)}{2} = \frac{d(v)-12}{2} > 0$ by R4a, and so we can assume from now on that the neighborhood of $v$ is partitioned into paths of type $S_k$s.

**(P1)** If $k = 3$, then $\varepsilon = \frac{1}{3}$.

Indeed, here $v_2$ receives $\frac{1}{2}$ through each of the faces $v_1vv_2$ and $v_2vv_3$ by R4b, whence $\varepsilon = 2 \times \frac{2}{3} - 2 \times \frac{1}{2} = \frac{1}{3}$.

**(P2)** If $k = 4$, then $\varepsilon = 0$.

Now each of the $v_2$ and $v_3$ receives at most 1 from $v$ by Remark 9, and so $\varepsilon = 3 \times \frac{2}{3} - 2 = 0$.

**(P3)** If $k = 5$, then $\varepsilon = \frac{2}{3}$.

Suppose that $w_1, \ldots, w_4$ are the neighbors of $v_1, \ldots, v_5$ such that there are faces $v_iw_iv_{i+1}$, where $1 \leq i \leq 4$.

If $v_2$ receives 1 by R4e, then $d(w_1) \leq 6$ and $d(w_2) \leq 6$. Hence, $d(w_3) \geq 12$ due to the absence of a $(5, 6, 6, 5, \infty)$-vertex in $P$, which implies that $v_4$ is adjacent to two $11^+$-vertices. Therefore, it receives $\frac{1}{2}$ from $v$ through $f_4$ and $\frac{1}{4}$ through $f_3$ by R4a and R4b. Furthermore, $v_4$ gives $\frac{1}{4}$ to $v$ by R5. So, $\varepsilon = 4 \times \frac{2}{3} - 1 - \frac{1}{2} - \frac{3}{4} + \frac{1}{4} = \frac{2}{3}$.

If R4e is not applicable to $v$, then $\varepsilon = 4 \times \frac{2}{3} - 4 \times \frac{1}{2} = \frac{2}{3}$ by R4a and R4b.

**(P4)** If $k = 6$ then $\varepsilon = \frac{1}{3}$.

Here each of the $v_2$ and $v_5$ receives at most 1, while each of the $v_3$ and $v_4$ receives $\frac{1}{2}$ from $v$ by Remark 9, and so $\varepsilon = 5 \times \frac{2}{3} - 2 \times 1 - 2 \times \frac{1}{2} = \frac{1}{3}$.

**(P5)** If $k = 7$ then $\varepsilon = \frac{1}{2}$.

Now we have $\varepsilon = 6 \times \frac{2}{3} - 2 \times 1 - 3 \times \frac{1}{2} = \frac{1}{2}$ according to Remark 9.

**(P6)** If $k \geq 8$ then $\varepsilon \geq \frac{2}{3}$.

Now we have $\varepsilon = (k-1) \times \frac{2}{3} - 2 \times 1 - (k-4) \times \frac{1}{2} = \frac{k-4}{6} \geq \frac{2}{3}$.

If $d(v) = 17$, then it suffices to assume that the neighborhood of $v$ consists of pairs of 5-vertices separated from each other by $6^+$-vertices due to (P1)–(P6) (since otherwise we already pay off the deficiency of $\frac{1}{3}$), which is impossible due to the fact that 17 is not divisible by 3.

Suppose that $d(v) = 16$ and $\mu'(v) < 0$. As follows from (P1)–(P6), the neighborhood of $v$ can have at most one of the paths $S_{t+2}$ of $t$ vertices of degree 5, where $t \in \{1, 4, 5\}$, while all other vertices are partitioned into pairs of 5-vertices separated from each other by $6^+$-vertices. Indeed, if there are either two paths with $t \in \{1, 4, 5\}$, or at least one path with $t = 3$ or $t \geq 6$, then we can pay off the deficiency of $\frac{2}{3}$; a contradiction. But none of these cases is possible due to the division by 3. Indeed, if $t = 1$ then we either have $16 - 2 = 14$ faces to be divided into triplets of faces with a sequence $S_4$ of neighbors of $v$ as in (P2), or $16 - 5 = 11$ and $16 - 6 = 10$ faces for $t = 4$ and $t = 5$, respectively, but all 16 faces also cannot be partitioned into triplets of faces with a sequence $S_4$; a contradiction.

CASE 6: $d(v) \geq 18$. Now $\mu'(v) \geq d(v) - 6 - d(v) \times \frac{2}{3} = \frac{d(v)-18}{3} \geq 0$ by R4.

Thus we have proved $\mu'(x) \geq 0$ for every $x \in V \cup F$, which contradicts (1) and completes the proof of Theorem 6.

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