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REGULARITY OF THE INVERSE OF A HOMEOMORPHISM OF A SOBOLEV–ORLICZ SPACE

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Abstract: Given a homeomorphism $\varphi \in W^1_M$, we determine the conditions that guarantee the belonging of the inverse of φ in some Sobolev–Orlicz space W^1_F. We also obtain necessary and sufficient conditions under which a homeomorphism of domains in a Euclidean space induces the bounded composition operator of Sobolev–Orlicz spaces defined by a special class of N-functions. Using these results, we establish requirements on a mapping under which the inverse homeomorphism also induces the bounded composition operator of another pair of Sobolev–Orlicz spaces which is defined by the first pair.

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Introduction

The article is devoted to the problem of describing the regularity properties of the inverse mapping to a homeomorphism of a Sobolev–Orlicz space W^1_M if the regularity properties of the homomorphism are known.

The main problem consists in finding the conditions for a homeomorphism $\varphi \in W^1_M$ that guarantee the regularity of the inverse of φ. Moreover, we determine necessary and sufficient conditions under which a homeomorphism $\varphi : D \to D'$, where D and D' are domains in $\mathbb{R}^n$, induces the bounded composition operator $\varphi^* : L^1_{M_1}(D') \to L^1_M(D)$ acting by the rule $\varphi^* f = f \circ \varphi$. Note that if the N-function M defining the Sobolev–Orlicz space W^1_M is a power function then the problem is reduced to the case of the Sobolev space W^1_p which is rather studied in [1]. The methods of [1] are applied to proving the assertions of this article.

Two directions can be distinguished in the study of the problem of determining the regularity properties of the inverse mapping from the known regularity properties of the direct mapping. The first direction is based on the methods of quasiconformal analysis, while the second, appeared in the study of the properties of the special classes of mappings that arose in the study of some questions of nonlinear elasticity. Let us list the main stages of steps in each direction.

Let us first consider the historically first direction, connected with quasiconformal analysis. The following main result is formulated in [2, 3]: *The inverse homeomorphism to a quasiconformal homeomorphism $\varphi : \Omega \to \Omega'$, $\Omega, \Omega' \subset \mathbb{R}^n$, $n \geq 2$, is quasiconformal as well.* This result can be represented as follows: *A homeomorphism $\varphi : \Omega \to \Omega'$, $\Omega, \Omega' \subset \mathbb{R}^n$, $n \geq 2$, of the Sobolev space $W^1_{n,\mathrm{loc}}(\Omega)$ satisfying the condition*

$$|D\varphi(x)|^n \leq K|\det D\varphi(x)|$$

for almost all $x \in \Omega$ with some constant K independent of $x \in \Omega$ has the inverse belonging to $W^1_{n,\mathrm{loc}}(\Omega')$ and such that

$$|D\varphi^{-1}(y)|^n \leq K'|\det D\varphi^{-1}(y)|$$

for almost all $y \in \Omega'$, where K' is a constant expressed through K.

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For studying the problem of describing the properties of the inverse mapping of importance are the results of the article [4] which relied on the basis of the methods of [5, 6]. In [4], there were firstly studied the properties of a mapping with finite distortion of $W^1_{n,\mathrm{loc}}(\Omega)$: $D\varphi(x) = 0$ almost everywhere on the zero set of the Jacobian for $\varphi \in W^1_{n,\mathrm{loc}}(\Omega)$. The name for this space of mappings was later proposed in [7, 8].

Using the new methods and methods of quasiconformal analysis of [9, 10], the authors of [11, 12] studied the analytical properties of a homeomorphism $\psi : \Omega' \to \Omega$ whose inverse φ induces the bounded operator $\varphi^* : L^1_p(\Omega') \to L^1_q(\Omega)$, $n \le q \le p < \infty$. The obtained properties of ψ guarantee the boundedness of $\psi^* : L^1_{q/(q-n+1)}(\Omega) \to L^1_{p/(p-n+1)}(\Omega')$. The final results were established in [1] by geometric measure theory. Note that, in [11, 12], there appears the class of mappings with bounded (p, q)-distortion (a mapping $f : \Omega \to \Omega'$ belongs to $W^1_{q,\mathrm{loc}}(\Omega)$, and its local p-distortion

$$K_p(x, f) = \inf\{k(x) : |Df(x)| \le k(x)J(x, f)^{1/p}\}$$

is integrable to the power κ, $1/\kappa = 1/q - 1/p$, $p \ge q \ge 1$). A detailed study of such mappings can be found in [13].

As was observed above, the most complete results on the problem under consideration in the case of Sobolev spaces were obtained in [1]. One if the main results of [1] is Theorem 1 of Section 4, whose peculiarity is that the initial mapping is not assumed to belong to a Sobolev space, which guarantees its wider applications. Let us cite one more result of [1], whose analog for Sobolev–Orlicz spaces we will prove in the present article (below $\operatorname{adj} A$ stands for the matrix adjoint to A, i.e., A is the transpose of the cofactor matrix).

Theorem 1 [1]. *Suppose that a homeomorphism $\varphi : D \to D'$ possesses the properties:*

(1) $\varphi \in W^1_{q,\mathrm{loc}}(D)$, $n - 1 \le q \le \infty$;

(2) *φ has finite codistortion* ($\operatorname{adj} D\varphi(x) = 0$ *almost everywhere on the set* $Z = \{x \in D : J(x, \varphi) = 0\}$);

(3) $\mathscr{K}_{\varphi,p}(x) = \frac{|\operatorname{adj} D\varphi(x)|}{|J(x,\varphi)|^{(n-1)/p}} \in L_\rho(D)$, *where* $1/\rho = (n-1)/q - (n-1)/p, n - 1 \le q \le p \le \infty$ ($\rho = \infty$ *for* $q = p$).

Then the inverse homeomorphism φ^{-1} has the properties:

(4) $\varphi^{-1} \in W^1_{p',\mathrm{loc}}(D')$, *where* $p' = p/(p - n + 1)$, $p' = 1$ *for* $p = \infty$;

(5) *φ^{-1} has finite distortion;*

(6) $K_{\varphi^{-1},q'}(y) = \frac{|D\varphi^{-1}(y)|}{|J(y,\varphi^{-1})|^{1/q'}} \in L_\rho(D')$, *where* $q' = q/(q - n + 1)$, $q' = \infty$ *for* $q = n - 1$.

By the conditions on the initial mapping in Theorem 1, we can notice one more substantial peculiarity of the results of [1]. It is assumed in Theorem 1 that $\varphi : D \to D'$ possesses the finite codistortion property (i.e., $\operatorname{adj} D\varphi(x) = 0$ almost everywhere on the set $Z = \{x \in D : J(x, \varphi) = 0\}$), whereas in most of the previous articles on this problem, the finiteness of the distortion is required ($D\varphi(x) = 0$ almost everywhere on $Z = \{x \in D : J(x, \varphi) = 0\}$).

Now, consider the main stages of the second direction in studying the problem. A necessity for finding the properties of the inverse of a homeomorphism of some Sobolev spaces W^1_p appeared also in connection with the study of some problems of nonlinear elasticity. In [14, 15], J. M. Ball proved

Theorem 2 [15]. *Let $\Omega \in \mathbb{R}^n$ be a bounded strictly Lipschitz domain. If $u : \overline{\Omega} \to \mathbb{R}^n$ belongs to $W^1_q(\Omega)$, $q > n$, coincides on $\partial\Omega$ with some homeomorphism u_0 of Ω, while $u_0(\Omega)$ satisfies the cone condition and* $\det Du(x) > 0$ *almost everywhere in Ω, and*

$$\int\limits_{\Omega} |Du^{-1}(u(x))|^{p'} \det Du(x)\, dx < \infty$$

for some $p' > n$, then u is a homeomorphism from Ω into $u_0(\Omega)$ and the inverse u^{-1} belongs to $W^1_{p'}(u_o(\Omega))$.

The proof of the fact that $u : \Omega \to u_0(\Omega)$ is a homeomorphism is given in [15]. Show that the second claim of Theorem 2 can be obtained as a particular case of Theorem 1:

$$\int\limits_{\Omega} |Du^{-1}(u(x))|^{p'} \det Du(x)\, dx = \int\limits_{\Omega} \frac{|\operatorname{adj} Du(x)|^{p'}}{J(x,u)^{(p'-1)}}\, dx = \int\limits_{\Omega} \left(\frac{|\operatorname{adj} Du(x)|}{J(x,u)^{(n-1)/p}}\right)^{p'} dx < \infty,$$

where $p = p'(n-1)/(p'-1)$. Consequently, $p > n-1$, since, by hypothesis, $p' > n$. In this case, the condition $1/\rho = (n-1)/q - (n-1)/p$ of Theorem 1 looks as follows: $1/p' = (n-1)/q - (p'-1)/p'$, whence $q = n-1$. Since in [15] it is required that the initial mapping u belongs to W^1_q, $q > n$, and in Theorem 1, for getting the same result, it is required that u belong just to W^1_{n-1}, the above arguments imply that the Ball Theorem is a particular case of Theorem 1.

Alongside this result, in [14, 15], there were also defined the classes of mappings

$$A_{p,q} = \{f \in W^1_p(\Omega) : \operatorname{adj} Df \in L_q\},$$

where $p \geq n-1$ and $q \geq p/(p-1)$. The works given below are devoted to investigating various subspaces of $A_{p,q}$.

Some results in this direction were already mentioned above (mappings with finite distortion). In [16–18], there were defined some properties of mappings with finite distortion without the assumption of the belonging of such a mapping in W^1_n. Also in [16–18], the space of mappings with exponentially integrable distortion was considered and the following assertion was given: *Suppose that $f \in W^1_1(\Omega)$ satisfies the condition*

$$|Df(x)|^n \leq K(x)J(x,f)$$

almost everywhere in Ω, where $K \geq 1$ and $\exp(\lambda K)$ is integrable for some $\lambda > 0$. If $J(x,f)$ is integrable then f is either constant or open and discrete. Further study of mappings with exponentially integrable distortion is presented in [19], where the more general case is considered of $\exp(\Psi(K)) \in L_{1,\mathrm{loc}}(\Omega)$, where $\Psi(t)$ is a strictly increasing differentiable function satisfying some additional conditions.

We should also mention the articles about the conditions under which mappings with finite distortion satisfy Luzin's condition (N). The main methods for solving this problem were laid in establishing condition (N) for mappings with bounded distortion in [20]. For mappings with finite distortion, this question was consecutively treated in [6, 21–23] and [18]. In [24], this problem was studied for mappings with exponentially integrable distortion.

Achievements in the study of the above-mentioned various classes of mappings made it possible to solve the problem of finding the properties of the inverse mapping from the known properties of the direct mapping. This problem was first investigated for various homeomorphisms $f : \Omega \to f(\Omega)$, $\Omega, f(\Omega) \in \mathbb{R}^2$, and then a generalization was obtained for the spaces $\mathbb{R}^n$, $n > 2$.

The following question was studied in [25, 26]: Under what conditions does the inverse mapping for a homeomorphism $f : \Omega \to f(\Omega)$, $\Omega, f(\Omega) \subset \mathbb{R}^2$, $f \in W^1_{p,\mathrm{loc}}(\Omega)$, $p \geq 1$, belongs to $W^1_{1,\mathrm{loc}}(f(\Omega))$ or even to $W^1_{q,\mathrm{loc}}(f(\Omega))$ for some $q > 1$? The most important of such conditions in [25, 26] is the finiteness of distortion. The analogous problem was considered in [25, 27–29] for mappings with exponentially integrable distortion. Let us state one of the results of [29]: *Let Ω be a domain in $\mathbb{R}^2$ and let $f \in W^1_{1,\mathrm{loc}}(\Omega)$ be a homeomorphism with finite distortion. Suppose that the distortion function K_f satisfies the condition $\exp(\lambda K_f) \in L_{1,\mathrm{loc}}(\Omega)$ for some $\lambda > 0$. Then $K^p_{f^{-1}} \in L_{1,\mathrm{loc}}(f(\Omega))$ for all $p < \lambda$.* The article [30] deals with the invertibility of mappings with bounded variation. The following is proved for such mappings: *Let Ω be a domain in $\mathbb{R}^2$ and let $f : \Omega \to \Omega'$ be a homeomorphism. Then $f \in BV_{\mathrm{loc}}(\Omega)$ if and only if $f^{-1} \in BV_{\mathrm{loc}}(\Omega')$. Moreover, f and f^{-1} are differentiable almost everywhere.*

The study of the problem in $\mathbb{R}^n$, $n > 2$, is the contents of [30–32]. The results of these articles were generalized in [33]. The main assertion is as follows: *Suppose that Ω is a domain in $\mathbb{R}^n$ and a homeomorphism f belongs to $W^1_{n-1,\mathrm{loc}}(\Omega)$. Then $f^{-1} \in BV_{\mathrm{loc}}(f(\Omega))$. If f has finite distortion then f^{-1} belongs to $W^1_{1,\mathrm{loc}}(f(\Omega))$ and has finite distortion.*

Note that the results of the previous two paragraphs which concern mappings of finite distortion and mappings with bounded variation are also particular cases of the results of [1].

This paper is organized as follows: Section 1 contains the main information about Orlicz spaces and some theorems necessary for proving the assertions. In Section 2, in Theorems 6 and 7, we formulate assertions on the properties of the inverse mappings to a homeomorphism of Sobolev–Orlicz spaces. In Section 3, we generalized the previous result on the bounded composition operator of the spaces L^1_M. Using this generalization, Theorems 6 and 7, and a result from [1], we prove one more assertion which generalize Theorem 8 in a sense.

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1. Preliminaries

Recall the main definitions of the theory of Orlicz spaces (the definitions and assertions formulated below can be found in [34]).

Definition 1. A continuous convex function $M : \mathbb{R} \to \mathbb{R}^+$ is called an *N-function* if M is even and satisfies the conditions

$$\lim_{u\to 0} \frac{M(u)}{u} = 0, \quad \lim_{u\to\infty} \frac{M(u)}{u} = \infty.$$

For an N-function $M : \mathbb{R} \to \mathbb{R}^+$, the complementary function $M^* : \mathbb{R} \to \mathbb{R}^+$ is defined by

$$M^*(v) = \sup\{u|v| - M(u) : u \geq 0\}.$$

It is not hard to see that $M^*(v)$ is an N-function too.

Using the considerations usually applied in deriving Hölder's inequality, we can obtain Young's inequality which plays an important role in the study of N-functions:

$$uv \leq M(u) + M^*(v) \tag{1}$$

for all u and v. Inequality (1) implies that

$$u < M^{-1}(u)M^{*-1}(u) \leq 2u. \tag{2}$$

A convex function $Q(u)$ is called the *principal part of an N-function* $M(u)$ if $Q(u) = M(u)$ for the large values of u.

We write $M = M_1$ if the functions M and M_1 coincide; the notation $M < M_1$ is used if $M(u) < M_1(u)$ for large values of u.

A substantial role is played by the growth rate of an N-function $M(u)$ as $u \to \infty$. Therefore, it is convenient to consider the special classes of N-functions whose behavior satisfies some conditions.

An N-function M is said to *satisfies the Δ_2-condition* (globally) if there exists a constant $k > 0$ such that

$$M(2u) \leq kM(u) \quad \text{for all } u.$$

We can show that N-functions satisfying the Δ_2-condition grow not faster than power functions.

An N-function is said to *satisfy the Δ'-condition* (globally) if there exists a constant $c > 0$ such that

$$M(uv) \leq cM(u)M(v) \quad \text{for all } u \text{ and } v.$$

Recall the definition of Orlicz spaces. The *Orlicz class* $\widetilde{L}_M(D)$ defined by an N-function M is the class of all real measurable functions $u : D \to \mathbb{R}$ (D is a domain in $\mathbb{R}^n$) for which

$$\int_D M(u(x))\,dx < \infty.$$

Definition 2. The *Orlicz space* $L_M(D)$ is the set of measurable functions $u : D \to \mathbb{R}$ satisfying the condition

$$\int\limits_D u(x)v(x)\,dx < \infty$$

for all $v : D \to \mathbb{R}$, $v \in \widetilde{L}_{M^*}(D)$.

If a function $M : \mathbb{R} \to \mathbb{R}^+$ satisfies the Δ_2-condition then $\widetilde{L}_M(D)$ coincides with $L_M(D)$.

We will consider the Orlicz space with the Luxemburg norm:

$$\|u\|_M = \inf\left\{\lambda > 0 : \int\limits_D M\left(\frac{u(x)}{\lambda}\right)\,dx \le 1\right\}.$$

Note that if an N-function $M(u)$ satisfies the Δ_2-condition then

$$\frac{1}{k}\int\limits_D M(u)\,dx \le \|u\|_M^{\log k/\log 2} \le k\int\limits_D M(u)\,dx,$$

where k is the constant of the definition of Δ_2-condition.

We will need an analog of Hölder's inequality for functions of an Orlicz space [35]. It has the form

$$\|uv\|_M \le 2\|u\|_{M_1}\|v\|_{M_2}, \tag{3}$$

where M, M_1, and M_2 are N-functions such that

$$M_1(u) = M(2M_3(u)), \quad M_2(u) = M(2M_3^*(u)). \tag{4}$$

Let us now define the Sobolev–Orlicz spaces.

Definition 3 [36]. The *Sobolev–Orlicz space* $W^1_M(D)$ (with D a domain in $\mathbb{R}^n$) consists of the equivalence classes of functions in the Orlicz space $L_M(D)$ having first weak derivatives in $L_M(D)$.

The Sobolev–Orlicz space $W^1_M(D)$ is endowed with the norm $\|f \mid W^1_M\| = \|f \mid L_M\| + \|Df \mid L_M\|$.

Definition 4 [36]. The *Sobolev–Orlicz space* $L^1_M(D)$ (with D a domain in $\mathbb{R}^n$) consists of the equivalence classes of locally integrable functions with first weak derivatives in $L_M(D)$.

The space $L^1_M(D)$ is endowed with the seminorm $\left\|f \mid L^1_M\right\| = \|Df \mid L_M\|$. Denote by $\overset{\circ}{L}{}^1_M(D)$ the closure of the set of compactly supported smooth functions in $L^1_M(D)$.

We say that a homeomorphism $\varphi : D \to D'$, with D and D' domains in $\mathbb{R}^n$, *induces the bounded composition operator* $\varphi^* : L^1_{M_1}(D') \to L^1_M(D)$ by the rule $\varphi^* f = f \circ \varphi$ if there exists a constant $K < \infty$ such that $\left\|\varphi^* f \mid L^1_M(D)\right\| \le K\left\|f \mid L^1_{M_1}(D')\right\|$ for all $f \in L^1_{M_1}(D') \cap \mathrm{Lip}(D')$.

For proving the results formulated here, we will need a few theorems:

Theorem 3 [37]. *Let functions M and M_1 be such that the function M_2 of* (4) *satisfies the Δ'-condition. A homeomorphism $\varphi : D \to D'$ induces the bounded composition operator $\varphi^* : L^1_{M_1}(D') \cap \mathrm{Lip}(D') \to L^1_M(D)$ if and only if*

(1) $\varphi \in \mathrm{ACL}(D)$ *(i.e., φ is absolutely continuous on almost all straight lines parallel to each coordinate axis and having nonempty intersection with D);*

(2) *φ has finite distortion ($\varphi \in \mathrm{ACL}(D), D\varphi(x) = 0$ almost everywhere on the set $Z = \{x \in D : J(x,\varphi) = 0\}$);*

(3) $K = \left\|\frac{|D\varphi|}{M_1^{-1}(|J(x,\varphi)|)} \mid L_{M_2}\right\|$ ($K = \left\|\frac{|D\varphi|}{M_1^{-1}(|J(x,\varphi)|)} \mid L_\infty\right\|$ *for $M = M_1$) is finite.*

The norm of the operator $\varphi^* : L^1_{M_1}(D') \to L^1_M(D)$ is equivalent to K; namely, $\alpha K \le \|\varphi^*\| \le K$, where α is a positive constant.

In this article, we strengthen Theorem 3 and prove it for a wider class of N-functions (Theorem 9).

The proof of the main result, formulated in Theorem 6, is based on the owing assertion (details on approximate differentiability can be found, for example, in [1]):

Theorem 4 [1]. *Suppose that a homeomorphism $\varphi : D \to D'$ possesses the properties:*
(1) *φ is approximately differentiable almost everywhere;*
(2) $\operatorname{adj} D\varphi \in L_1$*;*
(3) *φ has finite distortion;*
(4) *φ satisfies Luzin's condition on hypersurfaces.*
Then the inverse $\varphi^{-1} : D' \to D$ has the properties:
(5) *φ^{-1} belongs to* ACL*;*
(6) *$D\varphi^{-1} \in L_1(D')$;*
(7) *φ^{-1} has finite distortion.*
On the other hand, if the inverse of a homeomorphism $\varphi : D \to D'$ satisfies (5)–(7) *and φ is approximately differentiable on Z then φ possesses properties* (1)–(4).

Note that property (4) is fulfilled for mappings of Sobolev–Orlicz spaces under the hypotheses of the following

Theorem 5 [38]. *Suppose that D is an open set in $\mathbb{R}^n$, $n \geq 3$, while $\varphi : D \to \mathbb{R}^n$ is an open continuous mapping in $W^1_{M,\mathrm{loc}}(D)$, where $M : (0, \infty) \to (0, \infty)$ is a nondecreasing function such that*

$$\int\limits_1^\infty \left(\frac{t}{M(t)}\right)^{\frac{1}{n-2}} dt < \infty.$$

Then φ has total differential almost everywhere in D. Moreover, φ satisfies condition (N) with respect to the $(n-1)$-dimensional Hausdorff measure on almost all hypersurfaces parallel to an arbitrary fixed hypersurface.

2. The Main Result

In the assertions of this section, we assume that N-functions M and M_1 are chosen so that the function M_2 defined from (4) satisfies the Δ'-condition.

Theorem 6. *Let a homeomorphism $\varphi : D \to D'$ have the properties:*
(1) *$\varphi \in W^1_{M,\mathrm{loc}}(D)$, where the N-function M satisfies the conditions of Theorem* 5*;*
(2) *φ has finite codistortion;*
(3) $\mathscr{K}_{\varphi,F_2} = \left\| \frac{|\operatorname{adj} D\varphi|}{(M_1^{-1}(|J(x,\varphi)|))^{n-1}} \mid L_{F_2} \right\| < \infty.$
Then the inverse φ^{-1} possesses the properties:
(4) *$\varphi^{-1} \in W^1_{F_1,\mathrm{loc}}(D')$;*
(5) *φ^{-1} has finite distortion;*
(6) $K_{\varphi^{-1},F} = \left\| \frac{|D\varphi^{-1}|}{F^{-1}(|J(x,\varphi^{-1})|)} \mid L_{F_2} \right\| < \infty.$
The N-functions F and F_2 are defined as

$$F^{-1}(u) = u(M^{-1}(1/u))^{n-1}, \quad F_2(u) = M_2(u^{\frac{1}{n-1}}),$$

and the function F_1 is defined by the equality

$$F(u) = F_1(2F_3(u)), \quad F_2(u) = F_1(2F_3^*(u)).$$

Proof. Note that conditions (1)–(3) of Theorem 4 stem from conditions (1) and (2) in Theorem 6. Condition (4) of Theorem 4 is fulfilled since the N-function M satisfies the hypothesis of Theorem 5. Therefore, the conclusions of Theorem 4 hold for the mapping $\varphi^{-1} : D' \to D$, and item (5) of the present theorem is established.

Pass to proving item (6). Denote by Z the zero set of the Jacobian of φ and designate as Z' the zero set of the Jacobian of φ^{-1}. It is shown in [1] that Z can be chosen so that $\varphi(Z) = \Sigma'$, where Σ' is the singularity set for φ^{-1} (see [1] concerning singularity sets). Analogously, the singularity set Σ for φ can be chosen so that $Z' = \varphi(\Sigma)$.

At the nondegeneracy points of the Jacobian matrix $D\varphi(x)$, we have (see [1]) $J(y, \varphi^{-1}) = J(x, \varphi)^{-1}$ and $|D\varphi^{-1}(y)| = |J(\varphi^{-1}(y), \varphi)|^{-1}|\operatorname{adj} D\varphi(\varphi^{-1}(y))|$. Using these and the above-mentioned corollary to Young's inequality (2), we obtain

$$\left\| \frac{|\operatorname{adj} D\varphi|}{(M_1^{-1}(|J(x,\varphi)|))^{n-1}} \mid L_{F_2} \right\|^{\log k_{F_2}/\log 2} \geq C \int\limits_{D\setminus Z} F_2\left(\frac{|\operatorname{adj} D\varphi(x)|}{\left(M_1^{-1}(|J(x,\varphi)|)\right)^{n-1}} \right) \frac{|J(x,\varphi)|}{|J(x,\varphi)|}\, dx$$

$$\geq C \int\limits_{D'\setminus(Z'\cup\Sigma')} F_2\left(\frac{|J(\varphi^{-1}(y),\varphi)|^{-1}|\operatorname{adj} D\varphi(\varphi^{-1}(y))|}{|J(\varphi^{-1}(y),\varphi)|^{-1}(M^{-1}(|J(\varphi^{-1}(y),\varphi)|))^{n-1}} \right) dy$$

$$= C \int\limits_{D'\setminus Z'} F_2\left(\frac{|D\varphi^{-1}(y)|}{F^{-1}(|J(y,\varphi^{-1})|)} \right) dy \geq C \left\| \frac{|D\varphi^{-1}|}{F^{-1}(|J(x,\varphi^{-1})|)} \mid L_{F_2} \right\|^{\log k_{F_2}/\log 2}.$$

Moreover, the definition of F_2 implies that F_2 satisfies the Δ'-condition.

Show that $\varphi^{-1} \in W^1_{F_1,\mathrm{loc}}(D')$. To this end, use Hölder's inequality (3):

$$\left\| D\varphi^{-1} \mid L_{F_1} \right\| \leq 2\|F^{-1}(|J(x,\varphi^{-1})|) \mid L_F\| \left\| \frac{|D\varphi^{-1}|}{F^{-1}(|J(x,\varphi^{-1})|)} \mid L_{F_2} \right\|,$$

where F and F_1 must satisfy the equalities of the hypothesis of the Theorem.

The above inequality implies the finiteness of $\|D\varphi^{-1} \mid L_{F_1}\|$, and item (4) of the Theorem is proved.

As an example, find the functions F, F_1, and F_2, if M is a power function (for instance, $M(u) = u^q$) and the principal part of M_1 has the form $Q_1(u) = u^p(\log u)^a$, $a \in \mathbb{R}$. Put $q' = q/(q-n+1)$ and $p' = p/(p-n+1)$.

First find the function M_2 from (4). Its principal part is $Q_2(u) = u^{\frac{pq}{p-q}}(\log u)^{\frac{-qa}{p-q}}$. Then the principal part of $F_2(u)$ looks as $G_2(u) = u^{\frac{p'q'}{q'-p'}}(\log u^{1/(n-1)})^{\frac{-qa}{p-q}}$. Obviously, $F(u) = u^{q'}$. Now we can define the principal part of $F_1(u)$ on using the relations that are given in the Theorem:

$$G_1(u) = u^{p'}(\log u^{1/(n-1)})^{a(p'-1)}.$$

Using Theorem 6, we prove

Theorem 7. *Suppose that a homeomorphism $\varphi : D \to D'$ possesses the properties:*
(1) $\varphi \in W^1_{M,\mathrm{loc}}(D)$, *where the N-function M satisfies the hypotheses of Theorem 5;*
(2) φ *has finite distortion;*
(3) $K_{\varphi,M_1} = \left\|\frac{|D\varphi|}{M_1^{-1}(|J(x,\varphi)|)} \mid L_{M_2}\right\| < \infty$.
Then the inverse of φ has the properties:
(4) $\varphi^{-1} \in W^1_{F_1,\mathrm{loc}}(D')$;
(5) φ^{-1} *has finite distortion;*
(6) $K_{\varphi^{-1},F} = \left\|\frac{|D\varphi^{-1}|}{F^{-1}(|J(x,\varphi^{-1})|)} \mid L_{F_2}\right\| < \infty$.

Proof. Prove that conditions (2) and (3) of Theorem 6 are fulfilled if conditions (2) and (3) of Theorem 7 hold. The fact that $\operatorname{adj} D\varphi(x) = 0$ almost everywhere on Z is straightforward from (2). Further, using the relation (see [1]) $|\operatorname{adj} D\varphi(x)| \le |D\varphi(x)|^{n-1}$, we infer

$$\left\| \frac{|D\varphi|}{M_1^{-1}(|J(x,\varphi)|)} \mid L_{M_2} \right\|^{\log k_{M_2}/\log 2} \ge C \int\limits_D M_2\left(\frac{|D\varphi(x)|}{M_1^{-1}(|J(x,\varphi)|)} \right) dx$$

$$\ge C \int\limits_D M_2\left(\frac{|\operatorname{adj} D\varphi(x)|^{\frac{1}{n-1}}}{M_1^{-1}(|J(x,\varphi)|)} \right) dx = \int\limits_D F_2\left(\frac{|\operatorname{adj} D\varphi(x)|}{\big(M_1^{-1}(|J(x,\varphi)|)\big)^{n-1}} \right) dx$$

$$\ge C \left\| \frac{|\operatorname{adj} D\varphi^{-1}|}{\big(M_1^{-1}(|J(x,\varphi)|)\big)^{n-1}} \mid L_{F_2} \right\|^{\log k_{F_2}/\log 2}.$$

Consequently, the hypotheses of Theorem 6 are fulfilled, and its conclusion holds.

Imposing additional conditions on M and M_1 and using Theorems 3 and 7, we obtain the following

Theorem 8. *Let functions M and M_1 satisfy the Δ'-condition. Assume, moreover, that the N-function M satisfies the conditions of Theorem 5. If a homeomorphism $\varphi : D \to D'$ induces the bounded composition operator $\varphi^* : L^1_{M_1}(D') \to L^1_M(D)$ and has finite codistortion then the inverse $\varphi^{-1} : D' \to D$ induces the bounded composition operator ${\varphi^{-1}}^* : L^1_F(D) \to L^1_{F_1}(D')$ and has bounded distortion.*

Proof. Since by hypothesis the N-functions M, M_1, and M_2 satisfy the Δ'-condition, we can use Theorem 3 for the homeomorphism $\varphi : D \to D'$. Then for this mapping we have the following assertions:

(1) $\varphi \in \mathrm{ACL}(D)$;

(2) φ has finite distortion;

(3) $K_{\varphi,M_1} = \big\| \frac{|D\varphi|}{M_1^{-1}(|J(x,\varphi)|)} \mid L_{M_2} \big\|$ ($K = \big\| \frac{|D\varphi|}{M_1^{-1}(|J(x,\varphi)|)} \mid L_\infty \big\|$ for $M = M_1$) is finite.

As in Theorem 6, using an analog of Hölder's inequality (3), we conclude that $\varphi \in W^1_{M,\mathrm{loc}}(D)$. Note that all conditions of Theorem 7 are fulfilled and $F(u)$ satisfies the Δ'-condition if $M(u)$ does. In result, we infer that $\varphi^{-1} : D' \to D$ possesses the properties:

(4) $\varphi^{-1} \in \mathrm{ACL}(D')$;

(5) φ^{-1} has finite distortion;

(6) $K_{\varphi^{-1},F} = \big\| \frac{|D\varphi^{-1}|}{F^{-1}(|J(x,\varphi^{-1})|)} \mid L_{F_2} \big\| < \infty$.

By Theorem 3, for φ^{-1} to induce the bounded composition operator ${\varphi^{-1}}^* : L^1_F(D) \to L^1_{F_1}(D')$, it suffices that conditions (4)–(6) be fulfilled and the N-functions defining the spaces L^1_F and $L^1_{F_1}$ satisfy the Δ'-condition. But in Theorem 9 given below, it is shown that, in proving this fact, a constraint must be imposed only on the N-function F (it must be satisfied the Δ'-condition). Therefore, we infer that $\varphi^{-1} : D' \to D$ induces the bounded composition operator ${\varphi^{-1}}^* : L^1_F(D) \to L^1_{F_1}(D')$ and has finite distortion. $\square$

3. Generalization of the Theorem on the Bounded Composition Operator

Consider the second problem mentioned in the Introduction. The theorem proved below is a generalization of Theorem 3 of Section 1.

Introduce the notations: $\alpha = \log C_M/\log 2$, $\beta = \log C_{M_1}/\log 2$, and $\gamma = \alpha\beta/(\beta - \alpha)$, where C_M and C_{M_1} are the constants in the Δ_2-condition for M and M_1 respectively.

Before formulating the main result, let us prove the following

Lemma 1. *Suppose that a homeomorphism $\varphi : D \to D'$ induces the bounded composition operator $\varphi^* : L^1_{M_1}(D') \to L^1_M(D)$, while M and M_1 satisfy the Δ_2-condition. Then*

$$\Phi(A') = \sup_{f \in L^1_{M_1}(A')} \left(C \frac{\big\| \varphi^* f \mid L^1_M(\varphi^{-1}(A')) \big\|}{\big\| f \mid L^1_{M_1}(A') \big\|} \right)^\gamma,$$

with C a constant, is a bounded quasiadditive function defined on open subsets of D'.

Proof. Let A_i', $i \in N$, be open pairwise disjoint subsets of D', $A_0' = \bigcup_{i=1}^\infty A_i'$, $A_i = \varphi^{-1}(A_i')$, $i = 0, 1, \dots$. Consider $f_i \in L^1_{M_1}(A_i')$ such that

$$\|\varphi^* f_i \mid L^1_M(A_i)\| \geq \Big(\Phi(A_i')\Big(1 - \frac{\varepsilon}{2^i}\Big)\Big)^{1/\gamma} \|f_i \mid L^1_{M_1}(A_i')\|,$$

$$\big(\|f_i \mid L^1_{M_1}(A_i')\|\big)^\beta = \Phi(A_i')\Big(1 - \frac{\varepsilon}{2^i}\Big), \quad \varepsilon \in (0, 1).$$

Such a function exists because the norm in L_M is homogeneous; this implies that multiplication by a positive constant does not change the sign of the above inequality and, choosing the constant appropriately, we can always achieve the fulfillment of the last inequality. Putting $f_N = \sum_{i=1}^N f_i$ and applying Hölder's inequality, we obtain

$$\Big\|\varphi^* f_N \mid L^1_M\Big(\bigcup_{i=1}^N A_i\Big)\Big\| \geq \Big(\frac{1}{C_M^2}\Big)^{1/\alpha} \Big(\sum_{i=1}^N \|\varphi^* f_i \mid L^1_M(A_i)\|^\alpha\Big)^{1/\alpha}$$
$$\geq \Big(\frac{1}{C_M^2}\Big)^{1/\alpha} \Big(\sum_{i=1}^N \Big(\Phi(A_i')\Big(1 - \frac{\varepsilon}{2^i}\Big)\Big)^{\alpha/\gamma} \|f_i \mid L^1_{M_1}(A_i')\|^\alpha\Big)^{1/\alpha}$$
$$\geq \Big(\frac{1}{C_M^2}\Big)^{1/\alpha} \Big(\sum_{i=1}^N \Phi(A_i')\Big(1 - \frac{\varepsilon}{2^i}\Big)\Big)^{1/\gamma} \Big(\sum_{i=1}^N \|f_i \mid L^1_{M_1}(A_i')\|^\beta\Big)^{1/\beta}$$
$$\geq \Big(\frac{1}{C_M^2}\Big)^{1/\alpha} \Big(\frac{1}{C_{M_1}^2}\Big)^{1/\beta} \Big(\sum_{i=1}^N \Phi(A_i') - \varepsilon\Phi(A_0')\Big)^{1/\gamma} \Big\|f_N \mid L^1_{M_1}\Big(\bigcup_{i=1}^N A_i'\Big)\Big\|.$$

Note that $C = \big(\frac{1}{C_M^2}\big)^{1/\alpha}\big(\frac{1}{C_{M_1}^2}\big)^{1/\beta}$ is a constant depending only on the particular form of the N-functions M and M_1. This implies that

$$(\Phi(A_0'))^{1/\gamma} \geq \sup C \frac{\|\varphi^* f_N \mid L^1_M(\bigcup_{i=1}^N A_i)\|}{\|f_N \mid L^1_{M_1}(\bigcup_{i=1}^N A_i')\|} \geq \Big(\sum_{i=1}^N \Phi(A_i') - \varepsilon\Phi(A_0')\Big)^{1/\gamma},$$

where the supremum is taken over all $f_N \in L^1_{M_1}\big(\bigcup_{i=1}^N A_i'\big)$ of the form indicated above. Since N and ε are arbitrary, the quasiadditivity of Φ is proved.

Using Lemma 1, we can find the following assertion:

Theorem 9. *A homeomorphism $\varphi : D \to D'$ induces the bounded composition operator $\varphi^* : L^1_{M_1}(D') \cap \mathrm{Lip}(D') \to L^1_M(D)$ if M_1 satisfies the Δ'-condition and the following conditions:*
(1) $\varphi \in \mathrm{ACL}(D)$;
(2) φ *has finite distortion;*
(3) $K_{\varphi,M_1} = \big\|\frac{|D\varphi|}{M_1^{-1}(|J(x,\varphi)|)} \mid L_{M_2}\big\|$ $\big(K_{\varphi,M_1} = \big\|\frac{|D\varphi|}{M_1^{-1}(|J(x,\varphi)|)} \mid L_\infty\big\|$ *for* $M = M_1\big)$ *is finite.*

If a homeomorphism $\varphi : D \to D'$ induces the bounded composition operator $\varphi^* : L^1_{M_1}(D') \cap \mathrm{Lip}(D') \to L^1_M(D)$, while M and M_1 satisfy the Δ_2-condition; then φ satisfies conditions (1) *and* (2) *as well as the condition:*
(4) $K_{\varphi,\beta} = \big\|\frac{(M(|D\varphi|))^{1/\alpha}}{|J(x,\varphi)|^{1/\beta}} \mid L_\gamma\big\|$ $\big(K_{\varphi,\beta} = \big\|\frac{M(|D\varphi|)}{|J(x,\varphi)|} \mid L_\infty\big\|$ *for* $M = M_1\big)$ *is finite.*

Proof. The article [39] contains a proof of the assertion that the mapping φ inducing the bounded composition operator $\varphi^* : L^1_p(D') \to L^1_q(D)$, $1 \leq q \leq p \leq \infty$, by the rule $\varphi^*(f) = f \circ \varphi$, $f \in L^1_p$, belongs to $\mathrm{ACL}(D)$. The method of proving the ACL-property is applicable also to Sobolev–Orlicz spaces.

NECESSITY. By Lemma 1, for every $f \in L^1_{M_1}(A) \cap \mathrm{Lip}(A)$, we have

$$\|\varphi^* f \mid L^1_M(\varphi^{-1}(A))\| \leq C(\Phi(A))^{1/\gamma} \|f \mid L^1_{M_1}(A)\|,$$

where $A \subset D'$ is an open subset (for $M = M_1$, we put $(\Phi(A))^{1/\gamma} = \|\varphi^*\|$). Fix some truncation $\eta \in C_0^\infty(\mathbb{R}^n)$ equal to unity in $B(0,1)$ and to zero outside $B(0,2)$. Inserting the functions $f_i(y) = (y_i - y_{0,i})\eta\big(\frac{y-y_0}{r}\big)$ in this inequality, we infer

$$\bigg(\int\limits_{\varphi^{-1}(B(y_0,r))} M(|D\varphi|)\,dx\bigg)^{1/\alpha} \leq C(\Phi(B(y_0,2r)))^{1/\gamma}(r^n)^{1/\beta}. \tag{5}$$

If φ does not satisfy condition (N) then, by the change-of-variable formula of [40], there exists a Borel set E of measure zero such that

$$\int\limits_{D\setminus E} (g \circ \varphi)|J(x,\varphi)|\,dx = \int\limits_{D'} g(y)\,dy. \tag{6}$$

Demonstrate that $\varphi : D \to D'$ has finite distortion. To this end, show that

$$\int\limits_{Z} M(|D\varphi|)\,dx = 0, \tag{7}$$

where $Z = \{x \in D \setminus E \mid J(x,\varphi) = 0\}$.

By (6), $|\varphi(Z \setminus E)| = 0$. Fix $\varepsilon > 0$ and an open set $U \supset \varphi(Z \setminus E)$, $|U| < \varepsilon$. There exists a covering of U of finite multiplicity by balls $\{B(x_i, r_i)\}$ such that $\{B(x_i, 2r_i)\}$ constitute a covering of U of finite multiplicity too and $\sum r_i^n < N\varepsilon$ (the multiplicity N of the covering does not depend on U). Inequality (5) yields

$$\int\limits_{\varphi^{-1}(Z)} M(|D\varphi|)\,dx \leq \sum_{i=1}^{\infty} \int\limits_{\varphi^{-1}(B(y_i,r_i))} M(|D\varphi|)\,dx \leq C\|\varphi^*\|^\alpha \sum_{i=1}^{\infty} r_i^n \quad \text{for } M = M_1;$$

$$\sum_{i=1}^{\infty} \int\limits_{\varphi^{-1}(B(y_i,r_i))} M(|D\varphi|)\,dx \leq C\sum_{i=1}^{\infty} (\Phi(B(y_i,2r_i)))^{\alpha/\gamma}\big(r_i^n\big)^{\alpha/\beta}$$

$$\leq C(\Phi(D'))^{\alpha/\gamma}\bigg(\sum_{i=1}^{\infty} r_i^n\bigg)^{\alpha/\beta} < C(\Phi(D'))^{\alpha/\gamma}(N\varepsilon)^{\alpha/\beta} \quad \text{for } M < M_1.$$

Since $\Phi(D') < \infty$ and $\varepsilon > 0$ is an arbitrary number, (7) is proved, and so $|D\varphi| = 0$ almost everywhere on Z.

Pass to proving item (4) of the Theorem. Consider the case of $M = M_1$. Apply (6) to the left-hand side of (5). Lebesgue's Differentiation Theorem (see, for example, [41]) implies that

$$\frac{M(|D\varphi|)(\varphi^{-1}(y))}{|J(\varphi^{-1}(y),\varphi)|} \leq C\|\varphi^*\|^\alpha \quad \text{almost everywhere in } D' \setminus \varphi(E \cup Z).$$

Let $S \subset D' \setminus \varphi(E \cup Z)$ be the set where the last inequality fails. Then, by (6), $|J(x,\varphi)| = 0$ almost everywhere on $\varphi^{-1}(S)$. Therefore, $\varphi^{-1}(S) \subset Z$ and $M(|D\varphi|) \leq C\|\varphi^*\|^\alpha |J(x,\varphi)|$ almost everywhere in D.

For $M < M_1$, from inequality (5) we deduce that

$$\int\limits_{\varphi^{-1}(B(y_0,r))} M(|D\varphi|)\,dx \leq C\bigg(\frac{\Phi(B(y_0,2r))}{|B(y_0,2r)|}\bigg)^{\alpha/\gamma} r^n.$$

Apply the change-of-variable formula (6) to the left-hand side:

$$\int\limits_{\varphi^{-1}(B(y_0,r))} M(|D\varphi|)\,dx = \int\limits_{\varphi^{-1}(B(y_0,r))\setminus Z} M(|D\varphi|)\,dx$$
$$= \int\limits_{B(y_0,r)} \frac{M(|D\varphi|)(\varphi^{-1}(y))}{|J(\varphi^{-1}(y),\varphi)|}\,dy \le C\left(\frac{\Phi(B(y_0,2r))}{|B(y_0,2r)|}\right)^{\alpha/\gamma} r^n.$$

Lebesgue's Differentiation Theorem and the properties of the derivative of a countably additive set function (see, for example, [41]) imply that

$$\left(\frac{M(|D\varphi|)(\varphi^{-1}(y))}{|J(\varphi^{-1}(y),\varphi)|}\right)^{\gamma/\alpha} \le C\Phi'(y) \quad \text{almost everywhere in } D'.$$

Integrating the inequality over D', we obtain

$$(K_{\varphi,\beta})^\gamma \le C_1 \int\limits_{D\setminus Z} \left(\frac{(M(|D\varphi|))^{1/\alpha}}{|J(x,\varphi)|^{1/\beta}}\right)^{\gamma} dx = C_1 \int\limits_{D\setminus Z} \left(\frac{M(|D\varphi|)}{|J(x,\varphi)|}\right)^{\gamma/\alpha} |J(x,\varphi)|\,dx$$
$$= C_1 \int\limits_{D'} \left(\frac{M(|D\varphi|)(\varphi^{-1}(y))}{|J(\varphi^{-1}(y),\varphi)|}\right)^{\gamma/\alpha} dy \le C \int\limits_{D'} \Phi'(y)\,dy \le C\Phi(D') \le C\|\varphi^*\|^\gamma.$$

Sufficiency: Show that $\|\varphi^* f \mid L^1_M(D)\| \le K\|f \mid L^1_{M_1}(D')\|$ for all $f \in L^1_{M_1}(D') \cap \mathrm{Lip}(D')$. Using the analog of Hölder's inequality (3), we have

$$\|\varphi^* f \mid L^1_M(D)\| \le \left\| |Df||D\varphi| \frac{M_1^{-1}(|J(x,\varphi)|)}{M_1^{-1}(|J(x,\varphi)|)} \mid L_M(D) \right\|$$
$$\le 2 \left\| \frac{|D\varphi|}{M_1^{-1}(|J(x,\varphi)|)} \mid L_{M_2}(D) \right\| \||Df|M_1^{-1}(|J(x,\varphi)|) \mid L_{M_1}(D)\|.$$

Consider the second factor:

$$\big\||Df|M_1^{-1}(|J(x,\varphi)|) \mid L_{M_1}(D)\big\|$$
$$\le C_1\left(\int_{D\setminus Z} M_1(|Df|(\varphi(x))M_1^{-1}(|J(x,\varphi)|))\,dx\right)^{1/\beta}$$
$$\le C_2\left(\int_{D'} M_1(|Df|(\varphi(x)))\,dy\right)^{1/\beta} \le C\||Df| \mid L_{M_1}(D')\|.$$

Inserting the so-obtained inequality in the initial one, we deduce the desired result. Moreover, $K = C\big\|\frac{|D\varphi|}{M_1^{-1}(|J(x,\varphi)|)} \mid L_{M_2}\big\|$. If M and M_1 coincide, then $K = C\big\|\frac{|D\varphi|}{M_1^{-1}(|J(x,\varphi)|)} \mid L_\infty\big\|$. $\square$

In [37], it is demonstrated how to extend the operator $\varphi^* : L^1_{M_1}(D') \cap \mathrm{Lip}(D') \to L^1_M(D)$ of Theorem 3 to an operator $\varphi^* : L^1_{M_1}(D') \to L^1_M(D)$ under some additional conditions on the N-functions. This method is also applicable to the operator φ^* of Theorem 9 since substantial in it is the Δ_2-condition on M and M_1 rather than the stronger Δ'-condition. This yields

Proposition. *Suppose that $\varphi : D \to D'$ induces the bounded composition operator $\varphi^* : L^1_{M_1}(D') \cap \mathrm{Lip}(D') \to L^1_M(D)$. Then the extension of φ^* by continuity coincides with the composition operator $\varphi^* : L^1_{M_1}(D') \to L^1_M(D)$ provided that the N-function $M_1(u)$ either grows slower than u^n or satisfies the condition*

$$\int_1^\infty \frac{M^{-1}(t)}{t^{1+1/n}}\, dt < \infty.$$

Remark. In [42], some study was carried out of composition operators in Sobolev–Orlicz spaces but the operators $\varphi^* : W^1_M(D') \to W^1_M(D)$ in the normed spaces $W^1_M(D)$ (and not in the seminormed spaces L^1_M) were considered. Moreover, more rigid constraints were imposed on the N-functions defining these functions: the N-functions $M : \mathbb{R}^+ \to \mathbb{R}^+$ in [42] satisfy the condition

$$\lim_{u\to\infty} \frac{M(u)}{u^q \log^\alpha u} = 1$$

for $q \geq n$ and $\alpha \geq 0$ or $q \leq n$ and $\alpha \leq 0$. It is proved in [42] that if a homeomorphism φ induces the bounded composition operator of the Sobolev–Orlicz spaces W^1_M then φ is a q-quasiconformal mapping ($\varphi \in W^{1,1}_{\mathrm{loc}}(D)$, $|D\varphi(x)|^q \leq K|J(x,\varphi)|$ for almost all $x \in D$). In the case when the functions defining the Sobolev–Orlicz spaces are equal and have the same form as in [42], the necessary condition of this article coincides with the necessary condition of [42].

Let us state one more result about the properties of the composition operator induced by the inverse to some homeomorphism of a Sobolev–Orlicz space. The proof of this result is based on Theorems 9 and 1.

Theorem 10. *Suppose that $Q(u) = Cu^q(\log u)^{a_1}(\log\log u)^{a_2}...(\log ... \log u)^{a_n}$, $q > 1$, $a_i \in \mathbb{R}$, is the principal part of an N-function M, and M_1 satisfies the Δ_2-condition. Assume that the N-function M satisfies the conditions of Theorem 5. If a homeomorphism $\varphi : D \to D'$ induces the bounded composition operator $\varphi^* : L^1_{M_1}(D') \to L^1_M(D)$ and has finite codistortion then the inverse $\varphi^{-1} : D' \to D$ induces the bounded composition operator ${\varphi^{-1}}^* : L^1_{\alpha'}(D) \to L^1_{\beta'}(D')$ and has bounded distortion.*

Proof. Observe first of all that, by Theorem 9, $\varphi : D \to D'$ possesses the properties:

(1) $\varphi \in \mathrm{ACL}(D)$;

(2) φ has finite distortion;

(3) $K_{\varphi,\beta} = \left\| \frac{(M(|D\varphi|))^{1/\alpha}}{(|J(x,\varphi)|)^{1/\beta}} \mid L_\gamma \right\|$ ($K_{\varphi,\beta} = \left\| \frac{M(|D\varphi|)}{|J(x,\varphi)|} \mid L_\infty \right\|$ for $M = M_1$) is finite.

For the function $M(u)$ with the above-given principal part $Q(u)$, the exponent α equals q. Consequently, $(M(u))^{1/\alpha} \geq u^{1/\rho}$, where $\rho \geq 1$ depends on the exact form of $M(u)$. Then the following holds for φ:

(4) $K_{\varphi,\beta\rho} = \left\| \frac{|D\varphi|}{(|J(x,\varphi)|)^{\rho/\beta}} \mid L_{\gamma\rho} \right\|$ is finite.

This fact and Theorem 7 allow us to apply Theorem 1 to $\varphi : D \to D'$. Thus, the inverse $\varphi^{-1} : D' \to D$ possesses the properties:

(5) $\varphi^{-1} \in W^1_{\beta',\mathrm{loc}}(D')$, where $\beta' = \beta/(\beta - \rho(n-1))$;

(6) φ^{-1} has finite distortion;

(7) $K_{\varphi^{-1},\alpha'} = \left\| \frac{|D\varphi^{-1}(y)|}{|J(y,\varphi^{-1})|^{1/\alpha'}} \mid L_{\gamma\rho}(D') \right\|$, where $\alpha' = \alpha/(\alpha - \rho(n-1))$, $q' = \infty$ for $q = n-1$.

In [1], the question is also studied of determining the conditions under which a homeomorphism of Euclidean domains induces the bounded composition operator. The above-obtained conditions on the mapping $\varphi^{-1} : D' \to D$ make it possible to establish this fact. We finally conclude that, in our notations, the mapping $\varphi^{-1} : D' \to D$ induces the bounded composition operator ${\varphi^{-1}}^* : L^1_{\alpha'}(D) \to L^1_{\beta'}(D')$ and has finite distortion. $\square$

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