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MULTIDIMENSIONAL EXACT SOLUTIONS TO THE REACTION-DIFFUSION SYSTEM WITH POWER-LAW NONLINEAR TERMS

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Abstract: We study a nonlinear reaction-diffusion system that is modeled by a system of parabolic equations with power-law nonlinear terms. The proposed construction of exact solutions enables us to split the process of finding the components depending on time and the spatial coordinates. We construct multiparametric families of exact solutions in elementary functions. The cases are elaborated of blow-up solutions as well as exact solutions time-periodic but spatially anisotropic.

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1. Introduction

Consider the system of N quasilinear parabolic equations

$$\frac{\partial u_k}{\partial t} = \nabla \cdot \left(u_k^{\lambda_k} \nabla u_k\right) + u_k^{1-\lambda_k} \sum_{j \neq k} \alpha_{kj}(t)\, u_j^{\lambda_j}, \tag{1}$$

where $u_k \overset{\triangle}{=} u_k(\mathbf{x}, t)$ are the sought functions, $\mathbf{x} \in \mathbb{R}^n$ is the vector of independent variables, $n \in \mathbb{N}$ with $n \geq 2$, $t \in [0, +\infty)$ is time, $k = 1, 2, \dots, \mathrm{N}$ with $\mathrm{N} \in \mathbb{N}$, ∇ is the gradient operator, $\alpha_{kj}(t)$ are known functions, and the real parameters $\lambda_k \neq 0$ represent nonlinearity of the medium.

The equations of this type describe nonstationary diffusion kinetic processes in distributed multicomponent systems, i.e., the evolution of a large class of nonlinear systems with diffusion and interactions among components [1–5]. Mathematical models, resting on the equations of the form (1), find applications in biology, environmental science, and economics. System (1) is particularly widely used in chemical kinetics to describe the processes of reaction-diffusion type [1, 6–8]. Thus, we regard the required functions $u_k(\mathbf{x}, t)$ as the concentrations of interacting components of some mixture of substances, while the known functions $\alpha_{kj}(t)$ characterize the rates of occurring reactions. We cover the situation when $\alpha_{kj}(t)$ vanish identically for some k and j. The case of constant coefficients $\alpha_{kj}(t)$ deserves particular attention.

In order to construct exact solutions to (1), we use the ansatz

$$u_k(\mathbf{x}, t) = \psi_k(t)[W(\mathbf{x}) + \varphi_k(t)]^{1/\lambda_k}. \tag{2}$$

Here

$$W(\mathbf{x}) = \frac{1}{2}(A\mathbf{x}, \mathbf{x}) + (\mathbf{B}, \mathbf{x}) + C, \tag{3}$$

where the nonzero symmetric numerical matrix A of size n, the constant vector $\mathbf{B} \in \mathbb{R}^n$ and the constant $C \in \mathbb{R}$ are to be determined.

Ansatzes of type (2) and (3) have been used successfully in the series of articles [9–15] to construct exact solutions to nonlinear heat equations and other parabolic equations. Somewhat more general ansatzes were applied in [16–18]. However, this approach has not been applied previously to constructing exact solutions to systems of the form (1).

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The main goal of this article consists in constructing exact solutions of the form (2), (3) for system (1), namely, in finding a description, as complete as possible, of the coefficients of a quadratic polynomial in the spatial variables as well as the functions $\psi_k(t)$ and $\varphi_k(t)$ of time and the methods for deriving them from the exponents and coefficients of (1). We obtain parametric families of exact solutions and select among them: both spatially isotropic and anisotropic; blowing up in one or both directions; time-periodic; oscillating with increasing amplitude and frequency; expressed in terms of elementary functions.

2. Reduction to a System of ODEs

Inserting (2) into (1) and rearranging, we arrive at

$$\psi_k'[W(\mathbf{x})+\varphi_k]+\frac{1}{\lambda_k}\,\psi_k\varphi_k'=\frac{1}{\lambda_k}\,\psi_k^{1+\lambda_k}[W(\mathbf{x})+\varphi_k]\Delta W(\mathbf{x})$$
$$+\frac{1}{\lambda_k^2}\,\psi_k^{1+\lambda_k}|\nabla W(\mathbf{x})|^2+\psi_k^{1-\lambda_k}\sum_{j\neq k}\alpha_{kj}(t)\,\psi_j^{\lambda_j}[W(\mathbf{x})+\varphi_j].$$

Here $\psi_k=\psi_k(t)$, $\varphi_k=\varphi_k(t)$, $\psi_k'=\frac{d\psi_k}{dt}$, and $\varphi_k'=\frac{d\varphi_k}{dt}$ for $k=1,2,\dots,\mathrm{N}$. By straightforward calculations we find from (3) that

$$|\nabla W(\mathbf{x})|^2=(A^2\mathbf{x},\mathbf{x})+2(A\mathbf{B},\mathbf{x})+|\mathbf{B}|^2,\quad \Delta W(\mathbf{x})=\operatorname{tr}A,$$

the trace of the matrix A. With these relations and (3), the last N equalities become

$$\left(\psi_k'-\frac{\operatorname{tr}A}{\lambda_k}\,\psi_k^{1+\lambda_k}-\psi_k^{1-\lambda_k}\sum_{j\neq k}\alpha_{kj}(t)\,\psi_j^{\lambda_j}\right)\left(\frac{1}{2}(A\mathbf{x},\mathbf{x})+(\mathbf{B},\mathbf{x})+C\right)+\psi_k'\varphi_k+\frac{1}{\lambda_k}\psi_k\varphi_k'$$
$$=\frac{\operatorname{tr}A}{\lambda_k}\,\psi_k^{1+\lambda_k}\varphi_k+\frac{1}{\lambda_k^2}\psi_k^{1+\lambda_k}((A^2\mathbf{x},\mathbf{x})+2(A\mathbf{B},\mathbf{x})+|\mathbf{B}|^2)+\psi_k^{1-\lambda_k}\sum_{j\neq k}\alpha_{kj}(t)\,\psi_j^{\lambda_j}\varphi_j. \tag{4}$$

It is straightforward to verify that if the symmetric matrix A, the vector $\mathbf{B}$, and the constant C satisfy the system of algebraic equations

$$A=2\sigma A^2, \tag{5}$$

$$\mathbf{B}=2\sigma A\mathbf{B}, \tag{6}$$

$$C=\sigma|\mathbf{B}|^2, \tag{7}$$

where $\sigma\neq 0$ is the separation constant, then (4) reduces to the system of ordinary differential equations

$$\psi_k'=\left(\frac{\operatorname{tr}A}{\lambda_k}+\frac{1}{\sigma\lambda_k^2}\right)\psi_k^{1+\lambda_k}+\psi_k^{1-\lambda_k}\sum_{j\neq k}\alpha_{kj}(t)\,\psi_j^{\lambda_j}, \tag{8}$$

$$\varphi_k'=\left(\operatorname{tr}A\,\psi_k^{\lambda_k}-\lambda_k\frac{\psi_k'}{\psi_k}\right)\varphi_k+\lambda_k\psi_k^{-\lambda_k}\sum_{j\neq k}\alpha_{kj}(t)\,\psi_j^{\lambda_j}\varphi_j. \tag{9}$$

These arguments justify following statement.

Theorem 1. *Nonlinear reaction-diffusion system* (1) *admits exact solutions* (2)*, where $W(\mathbf{x})$ can be an arbitrary polynomial of the form* (3) *with coefficients satisfying* (5)–(7)*, while $\psi_k(t)$ and $\varphi_k(t)$ are solutions to* (8) *and* (9).

REMARK 1. This theorem enables us to separate the construction of components of the solution depending on the spatial variables and time. This splitting substantially simplifies the problem, replacing the original nonlinear system of parabolic PDEs with systems of algebraic and ordinary differential equations.

REMARK 2. Conditions of type (5)–(7) on the coefficients of the quadratic function (3) of spatial variables were obtained in [12] for one parabolic equation allowing the coefficients to depend on time, and the conditions appeared in a more general form as ODE systems. However, the separation constant in [12] was fixed and determined by a single exponent in the original equation. In Theorem 1, rather than one parabolic equation, we consider a system of equations of this type with nonlinear coupling in the form of power-law terms and use constant coefficients of the function (3), but the separation constant is now regarded as a free parameter.

The algebro-differential system (5)–(9) consists of the block (5)–(7) of algebraic equations and the block of $2N$ ordinary differential equations (8), (9). We study both blocks separately.

First of all, consider the solvability of (5). The latter always admits the trivial solution $A = 0$. Henceforth we consider only the nontrivial solutions of (5). It is easy to verify that $A = \frac{1}{2\sigma} P$ is a solution to (5), where P is an arbitrary idempotent matrix, i.e., $P^2 = P$. It is known [19] that each idempotent matrix P can be expressed as $P = M E_m M^{-1}$ with a nondegenerate matrix M of size n and the diagonal matrix E_m with $m \in \{1, 2, \dots, n\}$ ones and $n - m$ zeros on the diagonal in arbitrary order; E_m is also idempotent. Since only symmetric matrices A are relevant, we have to take symmetric idempotent matrices P, and so $P = S^T E_m S$, where S is an arbitrary orthogonal matrix. Thus, the matrix

$$A = \frac{1}{2\sigma} S^T E_m S \tag{10}$$

is a solution to (5). Moreover,

$$\operatorname{tr} A = \frac{m}{2\sigma}, \quad m \le n, \ n \in \mathbb{N}, \ n \ge 2. \tag{11}$$

Vector equation (6) amounts to a system of n homogeneous linear algebraic equations on the components $b_1, \dots, b_n$ of the required vector $\mathbf{B}$. For every fixed matrix (10) with $\operatorname{rank} A = m < n$ the homogeneous linear system always has a nontrivial solution; furthermore, we can choose the components $b_1, \dots, b_n$ of $\mathbf{B}$ arbitrarily in a linear m-dimensional subspace. In the case that $\operatorname{rank} A = m \equiv n$; i.e., for $E_m \equiv E$, the homogeneous linear system of equations has as a solution for every vector $\mathbf{B} \in \mathbb{R}^n$. Once we have successively found solutions to the matrix and vector equations, the constant C is uniquely determined from (7).

EXAMPLE 1. Take $n = 3$ and suppose that the square 3×3 matrices E_m are of rank 3, 2, and 1 respectively. We can express every orthogonal 3×3 matrix S as

$$S = \begin{bmatrix} \cos(s_1)\cos(s_3) - \sin(s_1)\sin(s_2)\sin(s_3) & -\sin(s_1)\cos(s_2) & a_{13} \\ \cos(s_1)\sin(s_2)\sin(s_3) + \sin(s_1)\cos(s_3) & \cos(s_1)\cos(s_2) & a_{23} \\ \cos(s_2)\sin(s_3) & -\sin(s_2) & \cos(s_2)\cos(s_3) \end{bmatrix}.$$

For brevity, here we use the notation

$$a_{13} = -\cos(s_1)\sin(s_3) - \sin(s_1)\sin(s_2)\cos(s_3),$$

$$a_{23} = \cos(s_1)\sin(s_2)\cos(s_3) - \sin(s_1)\sin(s_3).$$

To construct particular examples, specify the numerical values of the parameters s_1, s_2, and s_3. For instance, take $s_1 = -\frac{\pi}{4}$, $s_2 = \sin^{-1}\left(\frac{1}{3}\right)$, and $s_3 = \frac{3\pi}{4}$; then the orthogonal matrix S becomes

$$S = \begin{bmatrix} -1/3 & 2/3 & -2/3 \\ 2/3 & 2/3 & 1/3 \\ 2/3 & -1/3 & -2/3 \end{bmatrix}.$$

The matrix E_m of rank 3 coincides with the identity matrix, and in this case we obtain the quadratic form $\frac{1}{2}(A\mathbf{x}, \mathbf{x}) = \frac{1}{4\sigma}[x^2 + y^2 + z^2]$. For the matrix A of this quadratic form every vector $\mathbf{B} = (b_1, b_2, b_3)$ is the solution to (6). In this case (3) becomes

$$W_0(x, y, z) = \frac{1}{4\sigma}[x^2 + y^2 + z^2] + b_1 x + b_2 y + b_3 z + \sigma\left(b_1^2 + b_2^2 + b_3^2\right), \tag{12}$$

and the resulting solutions are radially symmetric in the spatial variables x, y, and z.

Take the rank 2 matrices:

$$E_{1m} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad E_{2m} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_{3m} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

For these matrices we find from (10) that

$$A_1 = \frac{1}{18\sigma} \begin{bmatrix} 5 & 2 & -4 \\ 2 & 8 & 2 \\ -4 & 2 & 5 \end{bmatrix}, \quad A_2 = \frac{1}{18\sigma} \begin{bmatrix} 5 & -4 & 2 \\ -4 & 5 & 2 \\ 2 & 2 & 8 \end{bmatrix}, \quad A_3 = \frac{1}{18\sigma} \begin{bmatrix} 8 & 2 & 2 \\ 2 & 5 & -4 \\ 2 & -4 & 5 \end{bmatrix}.$$

From (6) we obtain the corresponding vectors $\mathbf{B}_1$, $\mathbf{B}_2$, and $\mathbf{B}_3$:

$$\mathbf{B}_1 = (k_1, 2k_1 + 2k_3, k_3), \quad \mathbf{B}_2 = (k_1, k_2, 2k_1 + 2k_2), \quad \mathbf{B}_3 = (2k_2 + 2k_3, k_2, k_3),$$

where k_1, k_2, and k_3 are arbitrary constants. With these vectors, the constants C_1, C_2, and C_3 are defined as

$$C_1 = \sigma\big(5k_1^2 + 8k_1k_3 + 5k_3^2\big), \quad C_2 = \sigma\big(5k_1^2 + 8k_1k_2 + 5k_2^2\big), \quad C_3 = \sigma\big(5k_2^2 + 8k_2k_3 + 5k_3^2\big).$$

Finally,

$$\begin{aligned} W_1(x, y, z) &= \frac{1}{180\sigma}[(5x + 2y - 4z)^2 + 9(2y + z)^2] \\ &+k_1x + (2k_1 + 2k_3)y + k_3z + \sigma\big(5k_1^2 + 8k_1k_3 + 5k_3^2\big), \end{aligned} \tag{13}$$

$$\begin{aligned} W_2(x, y, z) &= \frac{1}{180\sigma}[(5x - 4y + 2z)^2 + 9(y + 2z)^2] \\ &+k_1x + k_2y + (2k_1 + 2k_2)z + \sigma\big(5k_1^2 + 8k_1k_2 + 5k_2^2\big), \end{aligned} \tag{14}$$

$$\begin{aligned} W_3(x, y, z) &= \frac{1}{72\sigma}[(4x + y + z)^2 + 9(y - z)^2] \\ &+(2k_2 + 2k_3)x + k_2y + k_3z + \sigma\big(5k_2^2 + 8k_2k_3 + 5k_3^2\big). \end{aligned} \tag{15}$$

Thus, in this case the solution is anisotropic in x, y, and z.

Consider the rank 1 matrices:

$$E_{1m} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad E_{2m} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad E_{3m} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

For these matrices we find from (10) that

$$A_1 = \frac{1}{18\sigma} \begin{bmatrix} 1 & -2 & -2 \\ -2 & 4 & 4 \\ -2 & 4 & 4 \end{bmatrix}, \quad A_2 = \frac{1}{18\sigma} \begin{bmatrix} 4 & 4 & -2 \\ 4 & 4 & -2 \\ -2 & -2 & 1 \end{bmatrix}, \quad A_3 = \frac{1}{18\sigma} \begin{bmatrix} 4 & -2 & 4 \\ -2 & 1 & -2 \\ 4 & -2 & 4 \end{bmatrix}.$$

From (6) we obtain the corresponding vectors $\mathbf{B}_1$, $\mathbf{B}_2$, and $\mathbf{B}_3$:

$$\mathbf{B}_1 = (k, -2k, -2k), \quad \mathbf{B}_2 = \left(k, k, -\frac{1}{2}\,k\right), \quad \mathbf{B}_3 = \left(k, -\frac{1}{2}\,k, k\right),$$

where k is an arbitrary constant. With these vectors, the constants C_1, C_2, and C_3 are defined as $C_1 = 9\sigma k^2$, $C_2 = \frac{9}{4}\,\sigma k^2$, and $C_3 = \frac{9}{4}\,\sigma k^2$. Finally,

$$\begin{aligned} W_1(x, y, z) &= \frac{1}{36\sigma}\,\omega_1^2 + k\omega_1 + 9\sigma k^2, \quad \text{where } \omega_1 = x - 2y - 2z, \\ W_2(x, y, z) &= \frac{1}{36\sigma}\,\omega_2^2 + \frac{k}{2}\,\omega_2 + \frac{9}{4}\,\sigma k^2, \quad \text{where } \omega_2 = 2x + 2y - z, \\ W_3(x, y, z) &= \frac{1}{36\sigma}\,\omega_3^2 + \frac{k}{2}\,\omega_3 + \frac{9}{4}\,\sigma k^2, \quad \text{where } \omega_3 = 2x - y + 2z. \end{aligned} \tag{16}$$

In this case we actually obtain one-dimensional solutions in the spatial variables ω_1, ω_2, and ω_3.

For $n = 2$ the square 2×2-matrices E_m of rank 1 also yield one-dimensional solutions. The identity matrix E_m of rank 2 for every orthogonal 2×2-matrix S leads to

$$W(x, y) = \frac{1}{4\sigma}(x^2 + y^2) + b_1 x + b_2 y + \sigma\big(b_1^2 + b_2\big),$$

where b_1 and b_2 are arbitrary constants. In this case we have a radially symmetric solution in the spatial variables x and y.

3. Solvability of (8), (9)

Let us begin studying system (8), (9) consisting of N nonlinear equations on the functions $\psi_k(t)$ and N linear equations on the functions $\varphi_k(t)$.

Introduce $Z_k(t) = \psi_k^{\lambda_k}(t)$ and $S_k = \big(\frac{\operatorname{tr} A}{\lambda_k} + \frac{1}{\sigma\lambda_k^2}\big)$ for $k = \overline{1,\mathrm{N}}$ and use them to rearrange (8), (9) as

$$Z_k' = \lambda_k S_k Z_k^2 + \lambda_k \sum_{j \neq k} \alpha_{kj}(t)\, Z_j, \tag{17}$$

$$\varphi_k' = \left(\operatorname{tr} A\, Z_k - \frac{Z_k'}{Z_k}\right)\varphi_k + \frac{\lambda_k}{Z_k} \sum_{j \neq k} \alpha_{kj}(t)\, Z_j \varphi_j. \tag{18}$$

Note that (17) is a vector Bernoulli equation for the new required functions $Z_k(t)$. Recall that we are interested in explicit analytical solutions expressible as compositions of finite collections of elementary and special functions. Since nontrivial stationary solutions to (17) are also of interest for our study, firstly we concentrate on constructing them.

3.1. Constant reaction rates and stationary solutions to (17). Assume that the coefficients α_{kj} characterizing reaction rates are constant for all k and j, where $j, k = \overline{1,\mathrm{N}}$ with $k \neq j$. Then we seek nontrivial stationary solutions to (17) from the system of N nonlinear algebraic equations

$$S_k Z_k^2 + \sum_{j \neq k} \alpha_{kj}\, Z_j = 0. \tag{19}$$

Here Z_k are the required constants. Using (11), express S_k as

$$S_k = \frac{1}{\sigma\lambda_k}\left(\frac{m}{2} + \frac{1}{\lambda_k}\right). \tag{20}$$

System (19) is linear-quadratic, and in general for arbitrary coefficients it is difficult to obtain conditions for the solvability of (19). Thus, we consider a series of interesting particular cases.

1. Suppose that for some positive integer m with $1 \leq m \leq n$ and all $k = \overline{1,\mathrm{N}}$ the equalities $\lambda_k = -\frac{2}{m}$ hold. Then $S_k = 0$, system (19) becomes linear, and we can express it as

$$Q\mathbf{Z} = 0, \tag{21}$$

where $\mathbf{Z} = \operatorname{col}(Z_1, Z_2, \ldots, Z_\mathrm{N})$ is the required vector, while the matrix $Q = [q_{kj}]_{k,j=\overline{1,\mathrm{N}}}$ is defined as $q_{kj} = \alpha_{kj}$ for $k \neq j$ and $q_{kj} = 0$ for $k = j$. For (21) to admit nontrivial solutions, it is necessary and sufficient that $\det Q = 0$. This condition can be tested directly from (1). If it holds then, using some solution to (21), we obtain a system of linear ODEs with constant coefficients for the functions $\varphi_k(t)$.

2. If $S_k = -\sum_{j \neq k} \alpha_{kj}$ for all $k = 1, 2, \ldots, \mathrm{N}$ then $\mathbf{Z} = \operatorname{col}(Z_1, Z_2, \ldots, Z_\mathrm{N}) = \operatorname{col}(1, 1, \ldots, 1)$ is a solution to (19). If $S_k = \sum_{j \neq k} \alpha_{kj}$ for all $k = \overline{1,\mathrm{N}}$ then $\mathbf{Z} = \operatorname{col}(Z_1, Z_2, \ldots, Z_\mathrm{N}) = \operatorname{col}(-1, -1, \ldots, -1)$ is a solution to (19).

3. Suppose that for some positive integer m with $1 \le m \le n$ and some $\sigma \ne 0$ the inequalities $S_k \ne 0$ hold for all $k = \overline{1,\mathrm{N}}$. If the matrix Q has a real eigenvalue $\mu(Q) \ne 0$ with associated eigenvector

$$\mathbf{Z} = \mathrm{col}(Z_1, Z_2, \dots, Z_{\mathrm{N}}) = \mathrm{col}\left(-\frac{\mu(Q)}{S_1}, -\frac{\mu(Q)}{S_2}, \dots, -\frac{\mu(Q)}{S_{\mathrm{N}}} \right)$$

then this vector is a stationary solution to (19).

4. Suppose that $S_k \ne 0$ for all $k = \overline{1,\mathrm{N}}$. If the matrix of the linear part of (19) is of the form

$$\Lambda = \begin{bmatrix} 0 & \alpha_{12} & 0 & 0 & \dots & 0 \\ 0 & 0 & \alpha_{23} & 0 & \dots & 0 \\ 0 & 0 & 0 & \alpha_{34} & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & \alpha_{\mathrm{N}-1,\mathrm{N}} \\ \alpha_{\mathrm{N}1} & 0 & 0 & 0 & \dots & 0 \end{bmatrix}, \tag{22}$$

where $\alpha_{12} \ne 0$, $\alpha_{23} \ne 0, \dots, \alpha_{\mathrm{N}-1,\mathrm{N}} \ne 0$, and $\alpha_{\mathrm{N}1} \ne 0$, then it reduces to the single equation $\Omega Z_1^{2^{\mathrm{N}}} + Z_1 = 0$, where

$$\Omega = \frac{S_{\mathrm{N}}}{\alpha_{\mathrm{N}1}} \left(\frac{S_{\mathrm{N}-1}}{\alpha_{\mathrm{N}-1,\mathrm{N}}} \right)^2 \left(\frac{S_{\mathrm{N}-2}}{\alpha_{\mathrm{N}-2,\mathrm{N}-1}} \right)^4 \left(\frac{S_{\mathrm{N}-3}}{\alpha_{\mathrm{N}-3,\mathrm{N}-2}} \right)^8 \dots \left(\frac{S_2}{\alpha_{23}} \right)^{2^{\mathrm{N}-2}} \left(\frac{S_1}{\alpha_{12}} \right)^{2^{\mathrm{N}-1}},$$

which has nontrivial real solution $Z_1 = \left(-\frac{1}{\Omega} \right)^{1/(2^{\mathrm{N}}-1)}$. Furthermore, $Z_2, \dots, Z_{\mathrm{N}}$ are defined as

$$Z_2 = -\frac{S_1}{\alpha_{12}} Z_1^2, \quad Z_3 = -\frac{S_2}{\alpha_{23}} \left(\frac{S_1}{\alpha_{12}} \right)^2 Z_1^4, \quad Z_4 = -\frac{S_3}{\alpha_{34}} \left(\frac{S_2}{\alpha_{23}} \right)^2 \left(\frac{S_1}{\alpha_{12}} \right)^4 Z_1^8,$$

and so forth.

5. Take $\mathrm{N} = 2$. In this case (19) becomes

$$S_1 Z_1^2 + \alpha_{12} Z_2 = 0, \quad S_2 Z_2^2 + \alpha_{21} Z_1 = 0. \tag{23}$$

Solving this system, we obtain

$$Z_1 = -\left(\frac{\alpha_{21}}{S_2} \right)^{1/3} \left(\frac{\alpha_{12}}{S_1} \right)^{2/3}, \quad Z_2 = -\left(\frac{\alpha_{12}}{S_1} \right)^{1/3} \left(\frac{\alpha_{21}}{S_2} \right)^{2/3}.$$

6. Take $\mathrm{N} = 3$. Then (19) becomes

$$\begin{gathered} S_1 Z_1^2 + \alpha_{12} Z_2 + \alpha_{13} Z_3 = 0, \quad S_2 Z_2^2 + \alpha_{21} Z_1 + \alpha_{23} Z_3 = 0, \\ S_3 Z_3^2 + \alpha_{31} Z_1 + \alpha_{32} Z_2 = 0. \end{gathered} \tag{24}$$

Consider the two cases: In the first case one of the coefficients α_{kj} vanishes; for definiteness, assume that $\alpha_{13} = 0$. In the second case one of the coefficients of the quadratic terms vanishes; for definiteness, assume that $S_3 = 0$. To start with, consider the first case. If together with $\alpha_{13} = 0$ we also have $\alpha_{12} = 0$ then the first equation in (24) yields $Z_1 = 0$, while Z_2 and Z_3 satisfy a system of two equations similar to the one studied in Subsection 5. Therefore, assume henceforth that α_{12} is nonzero. Then

$$Z_2 = -\frac{S_1 Z_1^2}{\alpha_{12}} \tag{25}$$

from the first equation in (24). Inserting Z_2 into the second equation yields

$$\frac{S_2 S_1^2}{\alpha_{12}^2} Z_1^4 + \alpha_{21} Z_1 + \alpha_{23} Z_3 = 0. \tag{26}$$

If $\alpha_{23} = 0$ then (26) leads to a cubic equation for the nonzero solution Z_1. Inserting this solution into (25), we find Z_2, and then the third equation yields the corresponding value of Z_3. Thus, assume henceforth that α_{23} is nonzero. Solve (26) for Z_3 and insert the result into the third equation of (24), which leads to a degree 7 equation for Z_1. It has at least one real root, which is not zero provided that neither is α_{31}.

Thus, for $\alpha_{13} = 0$ and nonzero α_{12}, α_{23}, and α_{31} system (24) has a nontrivial real solution.

Consider the second case: $S_3 = 0$ in (24). If in addition one of the coefficients α_{31} and α_{32} vanishes then the system decouples; hence, assume that both coefficients are nonzero. Then the third equation of (24) implies that

$$Z_2 = -\frac{\alpha_{31} Z_1}{\alpha_{32}} \tag{27}$$

if one of the coefficients α_{13} and α_{23} vanishes then, using the corresponding equation of (24) together with (27), we find the solutions Z_1 and Z_2, and then we must only find Z_3 from a linear equation, or make a verification in the case that Z_3 does not appear in the system. Thus, assume that the coefficients α_{13} and α_{23} are nonzero. Then solve the first equation of for Z_3:

$$Z_3 = \frac{\alpha_{12}\alpha_{31}}{\alpha_{13}\alpha_{32}} Z_1 - \frac{S_1}{\alpha_{13}} Z_1^2. \tag{28}$$

Inserting (28) and (27) into (24), we find its solution

$$Z_1 = \frac{\alpha_{32}(\alpha_{12}\alpha_{23}\alpha_{31} + \alpha_{13}\alpha_{21}\alpha_{32})}{S_2\alpha_{13}\alpha_{31}^2 - S_1\alpha_{23}\alpha_{32}^2}, \quad Z_2 = \frac{\alpha_{31}(\alpha_{12}\alpha_{23}\alpha_{31} + \alpha_{13}\alpha_{21}\alpha_{32})}{S_2\alpha_{13}\alpha_{31}^2 - S_1\alpha_{23}\alpha_{32}^2},$$

$$Z_3 = -\frac{(\alpha_{12}\alpha_{23}\alpha_{31} + \alpha_{13}\alpha_{21}\alpha_{32})\bigl(S_2\alpha_{21}\alpha_{32}^3 + S_1\alpha_{12}\alpha_{31}^3\bigr)}{\bigl(S_2\alpha_{13}\alpha_{31}^2 - S_1\alpha_{23}\alpha_{32}^2\bigr)^2}.$$

Thus, for $N = 3$ nontrivial solutions to (24) exist and can be written out explicitly in a large class of cases determined by weak conditions. One of these conditions consists in the vanishing of one of the coefficients α_{kj} of the linear terms and corresponds to the absence of interaction between the pair of components of the mixture. The second consists in the vanishing of one of the coefficients S_k of the quadratic terms and sometimes we can fulfill it by choosing the number m, which is a parameter in the solution being constructed.

To close this subsection, we make two observations. Firstly, our list of the cases when (19) admits nontrivial solutions is incomplete and can be extended. Secondly, the nontrivial solvability of (19) for Z_k does not mean necessarily that it is solvable for ψ_k. If $Z_k > 0$ for all $k = \overline{1,\mathrm{N}}$ then for arbitrary nonzero real λ_k we find $\psi_k = Z_k^{1/\lambda_k}$. If among Z_j with $j = \overline{1,\mathrm{N}}$ there is a negative number $Z_\nu < 0$ then the real solution $\psi_\nu = Z_\nu^{1/\lambda_\nu}$ exists only in the case that $\lambda_\nu = \frac{p_\nu}{q_\nu}$ is a rational number with odd p_ν.

Example 2. Consider a system of type (1) with three equations in the case of three spatial coordinates

$$\frac{\partial u_1}{\partial t} = \nabla \cdot \bigl(u_1^3 \nabla u_1\bigr) + u_1^{-2}\bigl(u_2^3 + u_3^3\bigr), \quad \frac{\partial u_2}{\partial t} = \nabla \cdot \bigl(u_2^3 \nabla u_2\bigr) + u_2^{-2}\bigl(u_1^3 + u_3^3\bigr),$$

$$\frac{\partial u_3}{\partial t} = \nabla \cdot \bigl(u_3^3 \nabla u_1\bigr) + u_3^{-2}\bigl(u_1^3 + u_2^3\bigr). \tag{29}$$

For this system the matrix $Q = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ has $\mu = 2$ as an eigenvalue with associated eigenvector $q = \begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix}$. To construct spatially isotropic solutions, put $m = 3$ and take $\sigma = \frac{11}{36}$. Then we obtain $S_1 = S_2 = S_3 = 2$ and the vector q is a solution to the corresponding linear-quadratic system of type (19). Integrating the corresponding linear system of ODEs of type (9) with constant coefficients yields the following parametric family of spatially isotropic exact solutions to (29):

$$u_i(x,y,z,t) = -[W_0(x,y,z) + \varphi_i(t)]^{1/3}, \quad i = 1,2,3,$$

where $W_0(x,y,z)$ is defined in (12), while the functions $\varphi_i(t)$, as

$$\varphi_1(t) = C_2 \exp\left(-\frac{87}{11}\,t\right) + C_3 \exp\left(\frac{12}{11}\,t\right),$$

$$\varphi_2(t) = (C_1 + C_2) \exp\left(-\frac{87}{11}\,t\right) + C_3 \exp\left(\frac{12}{11}\,t\right),$$

$$\varphi_3(t) = -(C_1 + 2C_2) \exp\left(-\frac{87}{11}\,t\right) + C_3 \exp\left(\frac{12}{11}\,t\right).$$

Here C_1, C_2, and C_3 are arbitrary real parameters.

To construct spatially anisotropic solutions, take $m = 2$ and $\sigma = \frac{2}{9}$. Then we again obtain $S_1 = S_2 = S_3 = 2$ and the vector q is a solution to the corresponding linear-quadratic system of type (19). Integrating the corresponding linear system of ODEs of type (9) with constant coefficients yields the following parametric family of spatially anisotropic exact solutions to (29):

$$u_1(x,y,z,t) = -\left[W(x,y,z) + C_2 \exp\left(\frac{3}{2}\,t\right) + C_3 \exp\left(-\frac{15}{2}\,t\right)\right]^{1/3},$$

$$u_2(x,y,z,t) = -\left[W(x,y,z) + (C_1 + C_3) \exp\left(-\frac{15}{2}\,t\right) + C_2 \exp\left(\frac{3}{2}\,t\right)\right]^{1/3},$$

$$u_3(x,y,z,t) = -\left[W(x,y,z) - (C_1 + 2C_3) \exp\left(-\frac{15}{2}\,t\right) + C_2 \exp\left(\frac{3}{2}\,t\right)\right]^{1/3},$$

where $W(x,y,z)$ is any of the functions defined in (13)–(15), while C_1, C_2, and C_3 are arbitrary real parameters.

Example 3. Consider the following system of type (1) consisting of three equations in the case of three spatial coordinates:

$$\frac{\partial u_1}{\partial t} = \nabla \cdot \left(u_1^{\lambda_1} \nabla u_1\right) + \alpha_{12} u_1^{1-\lambda_1} u_2^{\lambda_2}, \quad \frac{\partial u_2}{\partial t} = \nabla \cdot \left(u_2^{\lambda_2} \nabla u_2\right) + \alpha_{23} u_2^{1-\lambda_2} u_3^{\lambda_3},$$
$$\frac{\partial u_3}{\partial t} = \nabla \cdot \left(u_3^{\lambda_3} \nabla u_3\right) + \alpha_{31} u_3^{1-\lambda_3} u_1^{\lambda_1}. \tag{30}$$

It admits the multiparametric exact solutions

$$u_1(x,y,x,t) = [W(x,y,z) + \varphi_1(t)]^{\frac{1}{\lambda_1}}, \quad u_2(x,y,x,t) = [W(x,y,z) + \varphi_2(t)]^{\frac{1}{\lambda_2}},$$

$$u_3(x,y,x,t) = [W(x,y,z) + \varphi_3(t)]^{\frac{1}{\lambda_2}},$$

where $W(x,y,z)$ is one of the functions defined in (12)–(16), while $\varphi_i(t)$ for $i = 1,2,3$ satisfy the linear system of ODEs

$$\begin{aligned}
\varphi_1'(t) &= \frac{m}{2\sigma}\, Z_1 \varphi_1(t) + \lambda_1 \alpha_{12} \frac{Z_2}{Z_1}\, \varphi_2(t), \\
\varphi_2'(t) &= \frac{m}{2\sigma}\, Z_2 \varphi_2(t) + \lambda_2 \alpha_{23} \frac{Z_3}{Z_2}\, \varphi_3(t), \\
\varphi_3'(t) &= \frac{m}{2\sigma}\, Z_3 \varphi_3(t) + \lambda_3 \alpha_{31} \frac{Z_1}{Z_3}\, \varphi_1(t).
\end{aligned} \tag{31}$$

Here the constants Z_i are defined as

$$Z_1 = -\left[\frac{S_3}{\alpha_{31}}\left(\frac{S_2}{\alpha_{23}}\right)^2\left(\frac{S_1}{\alpha_{12}}\right)^4\right]^{-1/7}, \quad Z_2 = -\frac{S_1}{\alpha_{12}}\, Z_1^2, \quad Z_3 = -\frac{S_2}{\alpha_{23}}\left(\frac{S_1}{\alpha_{12}}\right)^2 Z_1^4,$$

while S_i for $i = 1, 2, 3$, as (20). Take $m = 2$ and $\sigma = \frac{\sqrt{3}}{2}$ and suppose that the parameters α_{12}, α_{23}, α_{31}, λ_1, λ_2, λ_3 satisfy

$$\alpha_{12} = -\frac{2\sqrt{3}}{3}\frac{(1+\lambda_1)}{\lambda_1^2}, \quad \alpha_{23} = -\frac{2\sqrt{3}}{3}\frac{(1+\lambda_2)}{\lambda_2^2}, \quad \alpha_{31} = \frac{2\sqrt{3}}{3}\frac{(1+\lambda_1+\lambda_2)\lambda_1\lambda_2}{(1+\lambda_1)^2(1+\lambda_2)^2},$$

$$\lambda_1 \neq -1, \quad \lambda_2 \neq -1, \quad \lambda_1 \neq -1-\lambda_2, \quad \lambda_3 = -\frac{(1+\lambda_1)(1+\lambda_2)}{1+\lambda_1+\lambda_2}.$$

Then we can express the general solution to (31) as

$$\varphi_1(t) = C_1 + e^{\sqrt{3}t}(C_2\sin(t) + C_3\cos(t)),$$

$$\varphi_2(t) = \frac{\lambda_1(2C_1 + e^{\sqrt{3}t}(\sqrt{3}C_3 - C_2)\sin(t) - e^{\sqrt{3}t}(\sqrt{3}C_2 + C_3)\cos(t))}{2(1+\lambda_1)},$$

$$\varphi_3(t) = \frac{\lambda_1\lambda_2(2C_1 - e^{\sqrt{3}t}(\sqrt{3}C_3 + C_2)\sin(t) + e^{\sqrt{3}t}(\sqrt{3}C_2 - C_3)\cos(t))}{2(1+\lambda_1)(1+\lambda_2)}.$$

Here C_1, C_2, and C_3 are arbitrary constants. The value $m = 2$ corresponds to the functions $W(x, y, z)$ defined in (13)–(15), which leads to spatially anisotropic exact solutions.

3.2. The existence of periodic solutions to (9). In this subsection we consider the important question of existence of periodic solutions to the system of ODEs (9) and, accordingly, periodic solutions to the original nonlinear reaction-diffusion system (1) on assuming that the functions $\psi_k(t) = \psi_{k0} \equiv \text{const}$ are constant.

Assume that $\text{N} = 2$. Then the characteristic equation for (9) becomes

$$l^2 - \frac{m}{2\sigma}(Z_1 + Z_2)l + \frac{m^2}{4\sigma^2}\, Z_1Z_2 - \alpha_{12}\alpha_{21}\lambda_1\lambda_2 = 0.$$

Recall that here $Z_i = \psi_{i0}^{\lambda_i}$ for $i = 1, 2$. For the roots of this equation to be pure imaginary, we must impose the condition $Z_1 + Z_2 = 0$. In this case (23) decouples into the equations $S_1Z_1^2 - \alpha_{12}Z_1 = 0$ and $S_2Z_2^2 - \alpha_{21}Z_2 = 0$. Hence, for $S_1 \neq 0$ and $S_2 \neq 0$ we find that

$$Z_1 = \psi_{10}^{\lambda_1} = \frac{\alpha_{12}}{S_1}, \quad Z_2 = \psi_{20}^{\lambda_2} = \frac{\alpha_{21}}{S_2}, \quad \frac{\alpha_{12}}{S_1} = -\frac{\alpha_{21}}{S_2}.$$

With these formulas, the characteristic equation becomes $l^2 + K = 0$, where

$$K = \alpha_{12}\alpha_{21}\lambda_1\lambda_2\left[\frac{m^2}{\left(m + \frac{2}{\lambda_1}\right)\left(m + \frac{2}{\lambda_2}\right)} - 1\right].$$

For positive λ_1 and λ_2 the sign of K is opposite to that of the product $\alpha_{12}\alpha_{21}$. Hence, for $\alpha_{12}\alpha_{21} < 0$ we have $K > 0$ and the characteristic equation has pure imaginary roots $l = \pm i\sqrt{K}$. Summarizing the argument, we obtain the following

Proposition 1. *If* $\lambda_1 > 0$, $\lambda_2 > 0$, *and* $\alpha_{12}\alpha_{21} < 0$, *while for some* $m \in \{1, 2, \dots, n\}$ *we have*

$$\left(m + \frac{2}{\lambda_2}\right)\frac{\alpha_{12}}{\alpha_{21}} + \left(m + \frac{2}{\lambda_1}\right)\frac{\lambda_2}{\lambda_1} = 0, \tag{32}$$

then system (1) *has a family of time-periodic solutions* (2) *with constant* $\psi_1(t) \equiv \psi_{10}$ *and* $\psi_2(t) \equiv \psi_{20}$.

Consider the existence of periodic solutions in the case $\mathrm{N} = 3$ on assuming that the solutions $\psi_k(t) = \psi_{k0}$ are constant and the system is of the form corresponding to the one indicated in Subsection 4. Then we can express the characteristic equation for (9) as

$$l^3 + a_1 l^2 + a_2 l + a_3 = 0, \tag{33}$$

where

$$a_1 = -\frac{m}{2\sigma}(Z_1 + Z_2 + Z_3), \quad a_2 = \left(\frac{m}{2\sigma}\right)^2 (Z_1Z_2 + Z_1Z_3 + Z_2Z_3),$$

$$a_3 = -\left(\frac{m}{2\sigma}\right)^3 Z_1Z_2Z_3 - \lambda_1\lambda_2\lambda_3\alpha_{12}\alpha_{23}\alpha_{31}.$$

Here the solution $Z_k = \psi_k^{\lambda_k}$ to the system of type (19) is calculated as

$$Z_1 = -|\Omega|^{-1/7}, \quad Z_2 = -\frac{S_1}{\alpha_{12}} Z_1^2, \quad Z_3 = -\left(\frac{S_2}{a_{23}}\right)\left(\frac{S_1}{\alpha_{12}}\right)^2 Z_1^4, \tag{34}$$

$$\Omega = \frac{(\lambda_3 m + 2)(\lambda_2 m + 2)^2(\lambda_1 m + 2)^4}{128\,\sigma^7\lambda_1^8\lambda_2^4\lambda_3^2\alpha_{12}^4\alpha_{23}^2\alpha_{31}}.$$

In order for (33) to have a pair of pure imaginary roots, it is necessary and sufficient that $a_2 > 0$ and $a_1a_2 - a_3 = 0$. Thus, the following statement holds.

Proposition 2. *Assume that* $\mathrm{N} = 3$, *the matrix of the linear part of* (19) *is of the form* (22), *the numbers* λ_1, λ_2, *and* λ_3 *are rational with odd numerators, and the coefficients of the system and the solution* Z_k *to the system of type* (19), *calculated using* (34), *satisfy the conditions*

$$Z_1Z_2 + Z_2Z_3 + Z_3Z_1 > 0, \quad m^3(Z_1 + Z_2)(Z_2 + Z_3)(Z_3 + Z_1) - 8\sigma^3\lambda_1\lambda_2\lambda_3\alpha_{12}\alpha_{23}\alpha_{31} = 0.$$

Then system (1) *has time-periodic solutions.*

Example 4. In the 3-dimensional coordinate space consider the system of three equations

$$\frac{\partial u_1}{\partial t} = \nabla \cdot \left(u_1^3 \nabla u_1\right) - u_1^{-2}u_2^3, \quad \frac{\partial u_2}{\partial t} = \nabla \cdot \left(u_2^3 \nabla u_2\right) - u_2^{-2}u_3^3,$$

$$\frac{\partial u_3}{\partial t} = \nabla \cdot \left(u_3^3 \nabla u_3\right) + p^7 u_3^{-2}u_1^3, \tag{35}$$

where p is an arbitrary real parameter. Verify that there exists a value $p = p_*$ of the parameter such that (35) admits time-periodic spatially anisotropic solutions.

Take $m = 2$ and choose the corresponding A, $\mathbf{B}$, and C for the solution in the form (3), obtaining $W(x, y, z)$. In particular, we can take any of the functions (13)–(15). Choose $\sigma = \frac{4}{9}$. Then $S_k = 1$ for $k = 1, 2, 3$. Calculating the stationary solutions to the system of type (8), we find that

$$\Omega = p^{-7}, \quad Z_1 = -p, \quad Z_2 = p^2, \quad Z_3 = -p^4.$$

The characteristic equation of the corresponding system of type (9) is

$$l^3 + \frac{9}{4}(p^4 - p^2 + p)l^2 - \frac{81}{16}(p^6 - p^5 + p^3)l + \frac{999}{64}\,p^7 = 0.$$

Applying Proposition 2, we arrive at

$$27p^6 - 27p^5 - 27p^4 + 118p^3 - 27p^2 - 27p + 27 = 0.$$

The greatest negative root $p_* = -0.6574451\dots$ of this equation satisfies all hypotheses of Proposition 2; therefore, the corresponding linear system of type (9) has periodic solutions. This establishes the existence of time-periodic spatially anisotropic solutions to (35) with $p = p_*$.

EXAMPLE 5. Consider the system of type (1) consisting of four equations in the case of four spatial coordinates:

$$\begin{gathered} \frac{\partial u_1}{\partial t} = \nabla \cdot \big(u_1^3 \nabla u_1\big) + u_1^{-2} u_3^3, \\ \frac{\partial u_2}{\partial t} = \nabla \cdot \big(u_2^3 \nabla u_2\big) + u_2^{-2}\bigg(-\frac{485}{886}\,u_1^3 + \frac{1371}{886}\,u_3^3\bigg), \\ \frac{\partial u_3}{\partial t} = \nabla \cdot \big(u_3^3 \nabla u_3\big) + u_3^{-2} u_4^3, \\ \frac{\partial u_4}{\partial t} = \nabla \cdot \big(u_4^3 \nabla u_4\big) + u_4^{-2}\bigg(-\frac{443}{144}\,u_2^3 + \frac{587}{144}\,u_3^3\bigg). \end{gathered} \tag{36}$$

To construct spatially anisotropic solutions, take $m = 2$ and $\sigma = -\frac{4}{9}$. This yields $S_1 = S_2 = S_3 = S_4 = -1$, and the vector $Z = \mathrm{col}(1,1,1,1)$ is a solution to the corresponding linear-quadratic system of type (19). Integrating the corresponding linear system of ODEs of type (9) with constant coefficients, we obtain the following parametric family of exact solutions to (36) which are time-periodic and spatially anisotropic:

$$u_k(x_1, x_2, x_3, x_4, t) = [W(x_1, x_2, x_3, x_4) + \varphi_k(t)]^{1/3}, \quad k = \overline{1,4}, \tag{37}$$

where

$$\begin{aligned} W(x_1, x_2, x_3, x_4) = &-\frac{9}{32}\bigg(\sum_{j=1}^{4} x_j^2 + \sqrt{2}\,x_1x_2 + \sqrt{2}\,x_1x_3 - \sqrt{2}\,x_2x_4 + \sqrt{2}\,x_3x_4\bigg) \\ &+(\sqrt{2}\,k_1 + k_2)x_1 + k_1x_2 + (\sqrt{2}\,k_2 + k_1)x_3 + k_2x_4 \\ &-\frac{4}{9}\big((\sqrt{2}\,k_1 + k_2)^2 + k_1^2 + (\sqrt{2}\,k_2 + k_1)^2 + k_2^2\big), \end{aligned} \tag{38}$$

$$\begin{gathered} \varphi_1(t) = \bigg(\frac{108}{97}\,C_2 - \frac{48}{97}\,C_1\bigg)\cos t + \bigg(\frac{48}{97}\,C_2 + \frac{108}{97}\,C_1\bigg)\sin t, \\ \varphi_2(t) = \bigg(\frac{522}{443}\,C_2 - \frac{72}{443}\,C_1\bigg)\cos t + \bigg(\frac{72}{443}\,C_2 + \frac{522}{443}\,C_1\bigg)\sin t, \\ \varphi_3(t) = C_2\cos t + C_1\sin t, \\ \varphi_4(t) = \bigg(\frac{3}{4}\,C_2 + \frac{1}{3}\,C_1\bigg)\cos t + \bigg(\frac{3}{4}\,C_2 - \frac{1}{3}\,C_1\bigg)\sin t. \end{gathered}$$

Here k_1, k_2, C_1, and C_2 are arbitrary real parameters.

If we consider system (36) in three-dimensional coordinate space then the parametric family of exact time-periodic spatially anisotropic solutions is defined in (37), where instead of (38) we have to use the expression

$$\begin{aligned} W(x_1, x_2, x_3) = &-\frac{9}{32}\bigg(\frac{3}{2}\,x_1^2 + x_2^2 + \frac{3}{2}\,x_3^2 + \sqrt{2}\,x_1x_2 + x_1x_3 - \sqrt{2}\,x_2x_3\bigg) \\ &+(\sqrt{2}\,k_1 + k_2)x_1 + k_1x_2 + k_2x_3 - \frac{4}{9}\big((\sqrt{2}\,k_1 + k_2)^2 + k_1^2 + k_2^2\big). \end{aligned} \tag{39}$$

Observe also that, by choosing other matrices of quadratic forms in (38) and (39), we can extend the indicated parametric families of exact solutions.

We should note that, following the discovery of Belousov's periodic chemical reaction [20], the existence and construction of periodic solutions to the corresponding mathematical models has come under scrutiny [21]. As the survey [1] indicates, the existence and construction of periodic solutions for models of reaction-diffusion systems with distributed parameters are pertinent and interesting both for applications in chemical technology and from the theoretical points of view for the qualitative theory of differential equations. Studies of these questions basing on various methods and approaches are successful and continuing presently; for instance, see [22] and the references therein. Thus, the periodic solutions to the reaction-diffusion system (1) constructed here may turn out useful in applications to modeling chemical reactions.

3.3. Nonstationary solutions to (17). The system is nonlinear, and in general for arbitrary coefficients $\alpha_{kj}(t)$ it is difficult to obtain conditions for its solvability; thus, we consider a series of interesting particular cases. To simplify exposition, assume henceforth that all λ_j for $j = \overline{1,\mathrm{N}}$ are rational numbers with odd numerators.

Seek a solution to (17) as $z_k(t) = \mu_k f(t)$, where $\mu_k \in \mathbb{R}$, while $f(t)$ is a continuously differentiable function. Inserting $z_k(t) = \mu_k f(t)$ into (17) leads to

$$\mu_k f' = \lambda_k S_k \mu_k^2 f^2 + \lambda_k f \sum_{j\neq k} \alpha_{kj}(t)\mu_j = 0. \tag{40}$$

Suppose that

$$\sum_{j\neq k} \alpha_{kj}(t)\mu_j = \frac{\mu_k}{\lambda_k}\, b(t), \quad k = \overline{1,\mathrm{N}}, \tag{41}$$

for some function $b(t)$. Then (40) becomes

$$f' = \lambda_k S_k \mu_k f^2 + b(t) f, \quad k = \overline{1,\mathrm{N}}. \tag{42}$$

Putting $\mu_k = \frac{1}{\lambda_k S_k}$, we arrive at Bernoulli's equation $f' = f^2 + b(t)f$. Integrating, we find its general solution

$$f(t) = \frac{\exp \int\limits_0^t b(\tau)\, d\tau}{C - \int\limits_0^t \exp \int\limits_0^\xi b(\tau)\, d\tau d\xi} \tag{43}$$

with an arbitrary constant C. This argument justifies

Proposition 3. *If for some $\sigma \neq 0$ the values of S_k for $k = \overline{1,\mathrm{N}}$, calculated using* (20)*, are such that $\mu_k = \frac{1}{\lambda_k S_k}$ satisfy* (41) *then* (1) *has a solution* (2), (3) *with $\psi_k(t) = (\mu_k f(t))^{1/\lambda_k}$, where $f(t)$ is defined by* (43)*, while the functions $\varphi_k(t)$ satisfy the system of ODEs*

$$\varphi_k' = \left(\frac{m\lambda_k}{m\lambda_k + 2} f - \frac{f'}{f}\right)\varphi_k + \lambda_k^2 S_k \sum_{j\neq k} \frac{\alpha_{kj}(t)}{\lambda_j S_j}\, \varphi_j, \quad k = \overline{1,\mathrm{N}}. \tag{44}$$

Suppose that $\alpha_{kj}(t) = \frac{\hat{\alpha}_{kj}}{t}$ for all k, $j = \overline{1,\mathrm{N}}$ with $k \neq j$, where $\hat{\alpha}_{kj} \in \mathbb{R}$. Seek a solution to (17) as $Z_k(t) = g_k t^{-1}$. Then for some constants $g_k \in \mathbb{R}$ the insertion into (17) yields a linear-quadratic system of the form

$$S_k g_k^2 + \frac{1}{\lambda_k}\, g_k + \sum_{j\neq k} \hat{\alpha}_{kj} g_j = 0, \tag{45}$$

where S_k are defined in (20). The system (45) differs from (19) only in one additional linear term, and we can apply to it the approaches of Subsection 3.1 to constructing nontrivial solutions. This justifies

Proposition 4. *If* (45) *admits a nontrivial solution* g_k *for* $k=\overline{1,\mathrm{N}}$, *then* (1) *has a solution* (2), (3) *with* $\psi_k(t)=g_k^{1/\lambda_k}t^{-1/\lambda_k}$, *while the functions* $\varphi_k(t)$ *satisfy the system of ODEs*

$$\varphi_k'=\frac{1}{t}\left(1+\frac{mg_k}{2\sigma}\right)\varphi_k+\frac{1}{t}\frac{\lambda_k}{g_k}\sum_{j\neq k}\hat{\alpha}_{kj}g_j\varphi_j,\quad k=\overline{1,\mathrm{N}}.$$

Example 6. In the n-dimensional coordinate space consider the system of two equations

$$\frac{\partial u_1}{\partial t}=\nabla\cdot\left(u_1^3\nabla u_1\right)+\frac{1}{t}\,u_1^{-2}u_2^3,\qquad \frac{\partial u_2}{\partial t}=\nabla\cdot\left(u_2^3\nabla u_2\right)+\frac{1}{t}\,u_2^{-2}u_1^3.$$

Applying Proposition 4, we obtain a parametric family of exact solutions (2), (3) in which the functions $\psi_k(t)$ and $\varphi_k(t)$ for $k=1,2$ are defined as

$$\psi_k(t)=-\left(\frac{24\sigma}{3m+2}\right)^{1/3}t^{-1/3},$$

$$\varphi_1(t)=C_1t^{a-3}+C_2t^{a+3},\quad \varphi_2(t)=-C_1t^{a-3}+C_2t^{a+3},\quad a=\frac{2-9m}{3m+2}.$$

Using Proposition 3, we obtain a parametric family of exact solutions (2), (3) in which the symmetric matrix A, the vector $\mathbf{B}$, and the constant C satisfy the system of algebraic equations (5)–(7), while the functions $\psi_k(t)$ and $\varphi_k(t)$ for $k=1,2$ are defined as

$$\psi_1(t)=\psi_2(t)=\left(\frac{6\sigma}{3m+2}\right)^{1/3}\frac{t}{\left(C-\frac{1}{4}t^4\right)^{1/3}},$$

$$\varphi_1(t)=(C_1+C_2t^{-6})(4C-t^4)^{\frac{2}{3m+2}},\quad \varphi_2(t)=(C_1-C_2t^{-6})(4C-t^4)^{\frac{2}{3m+2}}.$$

Here $1\le m\le n$ and $\sigma\neq 0$, while C_1 and C_2 are arbitrary real parameters. For $C>0$ this family contains blow-up solutions that are defined only on a finite interval of time [23].

Example 7. In the n-dimensional coordinate space consider the system of three equations

$$\frac{\partial u_1}{\partial t}=\nabla\cdot\left(u_1^3\nabla u_1\right)+t\,u_1^{-2}\left(u_2^3-u_3^3\right),\qquad \frac{\partial u_2}{\partial t}=\nabla\cdot\left(u_2^3\nabla u_2\right)+t\,u_2^{-2}\left(u_3^3-u_1^3\right),$$

$$\frac{\partial u_3}{\partial t}=\nabla\cdot\left(u_3^3\nabla u_3\right)+t\,u_3^{-2}\left(u_1^3-u_2^3\right).$$

Applying Proposition 3, we obtain a parametric family of exact solutions (2), (3) in which the symmetric matrix A, the vector $\mathbf{B}$ and the constant C satisfy the system of algebraic equations (5)–(7), while the functions $\psi_k(t)$ and $\varphi_k(t)$ for $k=1,2,3$ are defined as

$$\psi_1(t)=\psi_2(t)=\psi_3(t)=\left(\frac{6\sigma}{(3m+2)(C-t)}\right)^{1/3},$$

$$\varphi_1(t)=\frac{1}{2}(C-t)^{\frac{2}{3m+2}}\left[(C_1+C_3)\cos\left(\frac{3\sqrt{3}}{2}\,t^2\right)+C_2\sin\left(\frac{3\sqrt{3}}{2}\,t^2\right)+C_1-C_3\right],$$

$$\varphi_2(t)=\frac{1}{4}(C-t)^{\frac{2}{3m+2}}\left[Q_1\cos\left(\frac{3\sqrt{3}}{2}\,t^2\right)+Q_2\sin\left(\frac{3\sqrt{3}}{2}\,t^2\right)+2C_1-2C_3\right],$$

$$\varphi_3(t)=-\frac{1}{4}(C-t)^{\frac{2}{3m+2}}\left[Q_3\cos\left(\frac{3\sqrt{3}}{2}\,t^2\right)+Q_4\sin\left(\frac{3\sqrt{3}}{2}\,t^2\right)-2C_1+2C_3\right],$$

where $Q_1=(\sqrt{3}C_2-C_1-C_3)$, $Q_2=-\sqrt{3}(C_1+C_3)-C_2$, $Q_3=(\sqrt{3}C_2+C_1+C_3)$, and $Q_4=-\sqrt{3}(C_1+C_3)+C_2$. Here $1\le m\le n$ and $\sigma\neq 0$, while C, C_1, C_2, and C_3 are arbitrary real parameters.

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