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THE RIEMANN–ROCH THEOREM FOR THE DYNNIKOV–NOVIKOV DISCRETE COMPLEX ANALYSIS

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Abstract: We prove an analog of the Riemann–Roch Theorem for the Dynnikov–Novikov discrete complex analysis.

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We prove a discrete analog of the Riemann–Roch Theorem for the discretization of complex analysis which was proposed by Dynnikov and Novikov in [1]. Earlier discrete analogs of the Riemann–Roch Theorem were considered for another discretization of complex analysis in the article [2] by Bobenko and Skopenkov and for graphs in the article [3] by Baker and Norine.

We will consider triangulations of compact Riemann surfaces, where the triangles are colored in black and white, and to each vertex of a simplicial complex there is attached an even number of faces which is at least four. Such finite simplicial complexes will be called *discrete Riemann surfaces*.

Let us insignificantly change the definition of discrete Dolbeault operator of [1].

Refer to real functions on the vertices and faces of a Riemann surface C as *discrete functions* and 1-*forms on* C respectively.

The vector spaces of functions and 1-forms on C will be denoted by $\Omega^0(C)$ and $\Omega^1(C)$ respectively. The space of 1-forms equal to zero on all black or white simplices will be denoted by Ω^1_W or Ω^1_B respectively: $\Omega^1 = \Omega^1_W \oplus \Omega^1_B$.

The operator $Q : \Omega^0(C) \to \Omega^1_W(C)$ assigns the sum of the numbers at the vertices to each white triangle and zero to each black triangle. The operator Q^+ acts similarly with the interchange of the colors.

Consider the dual complex C^* (not simplicial in general) to whose vertices and faces there correspond the faces and vertices of C and whose vertices are joined by an edge whenever the corresponding faces of C have a common edge. A coloring of the faces of C induces a coloring of the vertices of C^*.

Real functions on the faces of C^* will be called 2-*forms on* C.

The discrete Dolbeault operators Q and Q^+ act from $\Omega^1(C)$ into $\Omega^2(C)$ by the following rule: Q assigns to each face in C^* the sum of the vertices at its black vertices, while Q^+ assigns to each face the sum at the white vertices.

It is not hard to notice that $Q^2 = (Q^+)^2 = 0$ and $QQ^+ = Q^+Q$, and so $(Q + Q^+)^2 \neq 0$.

By analogy with the continuous case, we say that a discrete function is *holomorphic on* C if $Q^+f = 0$ on all black triangles in C. If $f \neq 0$ on some black triangle Q^+ then we assume that the black triangle is a *pole* of f and the function itself is *meromorphic* (the idea of the definition is taken from [4]).

Obviously, every discrete function is meromorphic.

By analogy with functions, define holomorphic and meromorphic 1-forms under the additional condition that they belong to $\Omega^1_W(C)$. The zeros and poles of 1-forms lie on white vertices and faces of C^* respectively.

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Since the zeros and poles belong to different spaces and the higher orders are not defined, change the classical definition of divisor.

Refer as a *divisor* (of functions) D on a surface C to the pair $D = (V(D), F(D))$, where $V(D)$ and $F(D)$ are the sets of pairwise distinct vertices and simplices in C respectively.

Throughout the sequel, we consider only "holomorphic" divisors D such that $F(D)$ contains only black triangles.

Denote by $h^0(C, D)$ the dimension of the vector space of meromorphic functions $f \in \Omega^0(C)$ such that (a) the set of the zeros of f includes $V(D)$; (b) the set of the poles of f is a subset in $F(D)$.

Let $\overline{C}$ stand for the discrete Riemann surface obtained by inverting the colors of faces. Designate as $\overline{D}$ the image of a divisor D under the mapping onto $\overline{C}$.

Denote by $h^1(C, D)$ the dimension of the vector space of meromorphic 1-forms $\mu \in \Omega^1_W(\overline{C})$ such that (a) the set of the zeros of μ contains $F(\overline{D})$; (b) the set of the poles of μ is a subset in $V(\overline{D})$. Here we have implicitly used the duality between C and C^*.

Discrete Riemann–Roch Theorem. *Let C be a discrete Riemann surface and let D be a divisor on C. Then*

$$h^0(C, D) - h^1(C, D) = \deg D + \chi(C), \qquad (\star)$$

where $\deg D = |F(D)| - |V(D)|$ *and $\chi(C)$ is the Euler characteristic of C.*

Proof. The definition of $h^0(C, D)$ implies that it is the dimension of the solution space to the system of linear equations $Q^+ f = 0$, in which there are as many variables as C contains vertices minus those in $V(D)$, and there are as many equations as there are black triangles minus those in $F(D)$. The rank-nullity theorem of linear algebra implies that

$$h^0(C, D) = \#\,\{\text{vertices}\} - |V(D)| - \operatorname{rk} Q^+.$$

Show that $h^1(C, D)$ is the dimension of the solution space for the system with the transposed matrix. To this end, observe that

$$(\alpha^0, (Q^+)^t \beta^1) = (Q^+ \alpha^0, \beta^1) = (\alpha^0, *Q^+ * \beta^1), \quad \alpha^0 \in \Omega^0(C),\ \beta^1 \in \Omega^1(C),$$

where $(\cdot,\cdot)$ is the standard Euclidean inner product; $* : \Omega^i(C) \to \Omega^{2-i}(\overline{C})$ is the composition of the duality mapping and the inversion of colors (the "Hodge star"). Consequently,

$$h^1(C, D) = \#\,\{\text{black triangles}\} - |F(D)| - \operatorname{rk} Q^+.$$

Thus,

$$h^0 - h^1 = \#\{\text{vertices}\} - \#\{\text{triangles}\}/2 + \deg D = \chi(C) + \deg D,$$

where we have used the equality $2\#\{\text{edges}\} = 3\#\{\text{triangles}\}$ valid for the simplicial complex C. □

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