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ON DP-COLORING OF GRAPHS AND MULTIGRAPHS

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Abstract: While solving a question on the list coloring of planar graphs, Dvořák and Postle introduced the new notion of DP-coloring (they called it *correspondence coloring*). A *DP-coloring* of a graph G reduces the problem of finding a coloring of G from a given list L to the problem of finding a "large" independent set in the auxiliary graph $H(G, L)$ with vertex set $\{(v, c) : v \in V(G) \text{ and } c \in L(v)\}$. It is similar to the old reduction by Plesnevič and Vizing of the k-coloring problem to the problem of finding an independent set of size $|V(G)|$ in the Cartesian product $G\square K_k$, but DP-coloring seems more promising and useful than the Plesnevič–Vizing reduction. Some properties of the DP-chromatic number $\chi_{DP}(G)$ resemble the properties of the list chromatic number $\chi_\ell(G)$ but some differ quite a lot. It is always the case that $\chi_{DP}(G) \geq \chi_\ell(G)$. The goal of this note is to introduce DP-colorings for multigraphs and to prove for them an analog of the result of Borodin and Erdős–Rubin–Taylor characterizing the multigraphs that do not admit DP-colorings from some DP-degree-lists. This characterization yields an analog of Gallai's Theorem on the minimum number of edges in n-vertex graphs critical with respect to DP-coloring.

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1. Introduction

All graphs in this note are assumed simple; i.e., they cannot have parallel edges or loops; multigraphs may have multiple edges but not loops. The complete n-vertex graph is denoted by K_n, and the n-vertex cycle is denoted by C_n. If G is a (multi)graph and v, $u \in V(G)$, then $E_G(v, u)$ denotes the set of all edges in G connecting v and u, $e_G(v, u) := |E_G(v, u)|$, and $\deg_G(v) := \sum_{u \in V(G)\setminus\{v\}} e_G(v, u)$. For $A \subseteq V(G)$, $G[A]$ denotes the sub(multi)graph of G induced by A and for disjoint $A, B \subseteq V(G)$, $G[A, B]$ denotes the maximal bipartite sub(multi)graph of G with parts A and B. If $G_1, \dots, G_k$ are (multi)graphs, then $G_1 + \dots + G_k$ denotes the (multi)graph with vertex set $V(G_1) \cup \dots \cup V(G_k)$ and edge set $E(G_1) \cup \dots \cup E(G_k)$. The independence number of G is denoted by $\alpha(G)$. Given $k \in \mathbb{Z}_{>0}$, let $[k]$ denote the set $\{1, \dots, k\}$.

Recall that a (proper) *k-coloring* of G is a mapping $f : V(G) \to [k]$ such that $f(v) \neq f(u)$ whenever $vu \in E(G)$. The smallest k such that G has a k-coloring is called the *chromatic number of* G and is denoted by $\chi(G)$. Plesnevič and Vizing [1] proved that G has k-coloring if and only if the Cartesian product $G\square K_k$ includes an independent set of size $|V(G)|$, i.e., $\alpha(G\square K_k) = |V(G)|$.

In order to tackle some graph coloring problems, Vizing [2] and independently Erdős, Rubin, and Taylor [3] introduced a more general notion of *list coloring.* A *list* L for a graph G is a map $L : V(G) \to \mathrm{Pow}(\mathbb{Z}_{>0})$ that assigns to each vertex $v \in V(G)$ a set $L(v) \subseteq \mathbb{Z}_{>0}$. An *$L$-coloring* of G is a mapping $f : V(G) \to \mathbb{Z}_{>0}$ such that $f(v) \in L(v)$ for each $v \in V(G)$ and $f(v) \neq f(u)$ whenever $vu \in E(G)$. The *list chromatic number*, $\chi_\ell(G)$, is the minimum k such that G has an L-coloring for each L satisfying $|L(v)| = k$ for every $v \in V(G)$.

Since G is k-colorable if and only if G is L-colorable with the list $L : v \mapsto [k]$, we have $\chi_\ell(G) \geq \chi(G)$ for every G; however, the difference $\chi_\ell(G) - \chi(G)$ can be arbitrarily large. Moreover, graphs with

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chromatic number 2 may have arbitrarily high list chromatic number. While 2-colorable graphs may have arbitrarily high minimum degree, Alon [4] showed that $\chi_\ell(G) \geq (1/2 - o(1)) \log_2 \delta$ for each graph G with minimum degree δ. On the other hand, some well-known upper bounds for $\chi(G)$ in terms of vertex degrees hold for $\chi_\ell(G)$ as well. For example, Brooks' Theorem and the degeneracy upper bound hold for $\chi_\ell(G)$. Furthermore, Borodin [5, 6] and independently Erdős, Rubin, and Taylor [3] generalized Brooks' Theorem to degree lists. Recall that a list L for G is a *degree list* if $|L(v)| = \deg_G(v)$ for every $v \in V(G)$.

Theorem 1 [3, 5, 6] (a simple proof in [7]). *Suppose that G is a connected graph. Then G is not L-colorable for some degree list L if and only if each block of G is either a complete graph or an odd cycle.*

This result yields an extension of Gallai's bound [8] on the minimum number of edges in n-vertex k-critical graphs (i.e., graphs G with $\chi(G) = k$ such that after deletion of any edge or vertex the chromatic number decreases) to n-vertex list-k-critical graphs (i.e., graphs G with $\chi_\ell(G) = k$ such that after deletion of any edge or vertex the list chromatic number decreases).

List coloring proved useful in establishing a number of results for ordinary graph coloring; however, generally it is often much harder to prove upper bounds on the list chromatic number than on the chromatic number. In order to prove such an upper bound for a class of planar graphs, Dvořák and Postle [9] introduced and heavily used a new generalization of list coloring; they called it *correspondence coloring*, and we will call it *DP-coloring*, for short.

First, we show how to reduce to DP-coloring the problem of L-coloring of a graph G. Given a list L for G, the vertex set of the auxiliary graph $H = H(G, L)$ is $\{(v, c) : v \in V(G) \text{ and } c \in L(v)\}$, and two distinct vertices (v, c) and (v', c') are adjacent in H if and only if either $c = c'$ and $vv' \in E(G)$, or $v = v'$. Note that the independence number of H is at most $|V(G)|$, since $V(H)$ is covered by $|V(G)|$ cliques. If H has an independent set I with $|I| = |V(G)|$, then, for each $v \in V(G)$, there is a unique $c \in L(v)$ such that $(v, c) \in I$. Moreover, the same color c is not chosen for every two adjacent vertices. In other words, the map $f : V(G) \to \mathbb{Z}_{>0}$ defined by $(v, f(v)) \in I$ is an L-coloring of G. On the other hand, if G has an L-coloring f, then the set $\{(v, f(v)) : v \in V(G)\}$ is an independent set of size $|V(G)|$ in H.

By construction, for every distinct v, $v' \in V(G)$, the set of edges of H connecting $\{(v, c) : c \in L(v)\}$ and $\{(v', c') : c' \in L(v')\}$ is empty if $vv' \notin E(G)$ and forms a matching (possibly empty) if $vv' \in E(G)$. Based on these properties of $H(G, L)$, Dvořák and Postle [9] introduced the DP-coloring. The phrasing below is slightly different, but the essence and the spirit are theirs.

Definition 2. Let G be a graph. A *cover* of G is a pair (L, H), where L is an assignment of pairwise disjoint sets to the vertices of G and H is a graph with vertex set $\bigcup_{v \in V(G)} L(v)$ satisfying the following conditions:

1. $H[L(v)]$ is a complete graph for each $v \in V(G)$.
2. For each $uv \in E(G)$ the edges between $L(u)$ and $L(v)$ form a matching (possibly empty).
3. For every two distinct $u, v \in V(G)$ with $uv \notin E(G)$ no edges of H connect $L(u)$ and $L(v)$.

Definition 3. Suppose that G is a graph and (L, H) is a cover of G. An *(L, H)-coloring* of G is an independent set $I \subseteq V(H)$ of size $|V(G)|$. In this context, we refer to the vertices of H as the *colors*. G is said to be *(L, H)-colorable* if G admits an (L, H)-coloring.

Note that if (L, H) is a cover of G and I is an (L, H)-coloring, then $|I \cap L(v)| = 1$ for all $v \in V(G)$. Fig. 1 shows an example of two distinct covers of $G \cong C_4$.

Definition 4. Let G be a graph and let $f : V(G) \to \mathbb{Z}_{\geq 0}$ be an assignment of nonnegative integers to the vertices of G. Say that G is *DP-f-colorable* if G is (L, H)-colorable whenever (L, H) is a cover of G and $|L(v)| \geq f(v)$ for all $v \in V(G)$. If G is DP-$\deg_G$-colorable, then G is said to be *DP-degree-colorable*.

Definition 5. The *DP-chromatic number*, $\chi_{DP}(G)$, is the minimum k such that G is (L, H)-colorable for each choice of (L, H) with $|L(v)| \geq k$ for all $v \in V(G)$.

Dvořák and Postle observed that $\chi_{DP}(G) \leq k + 1$ for every k-degenerate graph G and that Brooks' Theorem almost holds for DP-colorings, with the exception that $\chi_{DP}(C_n) = 3$ for every cycle C_n and

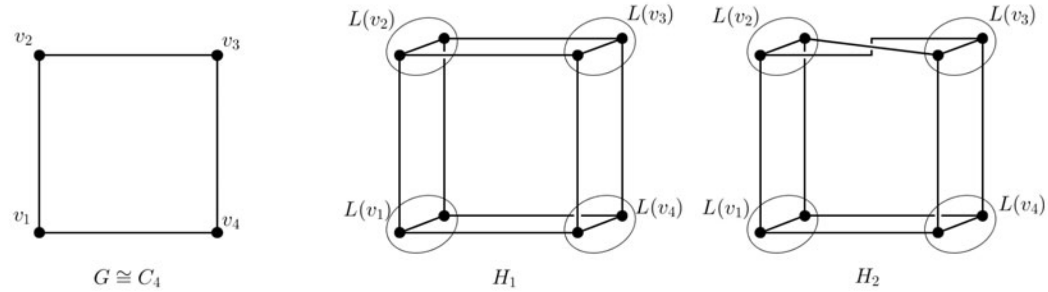

Fig. 1. The graph C_4 and two covers of C_4 such that C_4 is (L, H_1)-colorable but not (L, H_2)-colorable

not only for odd n, as for list coloring. The fact that $\chi_{DP}(C_4) = 3$ marks an important difference between DP-coloring and list coloring, since it implies that the orientation theorems of Alon–Tarsi [10] and the Bondy–Boppana–Siegel Lemma (see [10]) on list coloring do not extend to DP-coloring. Dvořák and Postle also point out that the proof of Thomassen's Theorem on list-5-colorability of planar graphs extends to DP-coloring. The first author of this note showed [11] that the lower bound on the DP-chromatic number of a graph G with minimum degree δ is much stronger than Alon's bound [4] for list coloring; namely, $\chi_{DP}(G) \geq \Omega(\delta/\log \delta)$. On the other hand, he proved an analog of Johansson's upper bound [12] on the list chromatic number of triangle-free graphs with given maximum degree.

The goal of this article is to naturally extend the notion of DP-coloring to multigraphs and to derive some simple properties of DP-colorings of multigraphs. The main result is an analog of Theorem 1: a characterization of connected multigraphs that are not DP-degree-colorable. This result also yields a lower bound on the number of edges in n-vertex DP-critical graphs (we define such graphs in the next section).

The structure of the article is as follows: In the next section we define DP-coloring of multigraphs and related notions, discuss some examples, and state our main result, Theorem 9. In Section 3 we prove the main result. In Section 4 we briefly discuss DP-critical (multi)graphs and show a bound on the number of edges in them which is implied by the main result. For completeness, in the Appendix we present a DP-version of Gallai's proof [8] of his lemma on the number of edges in so-called Gallai trees (the original paper [8] is in German).

2. Definitions and the Main Result

To define DP-coloring for multigraphs, we only need to change Definition 2 as below and replace the word *graph* with the word *multigraph* in Definitions 3–5. The new version of Definition 2 is

Definition 6. Let G be a multigraph. A *cover* of G is a pair (L, H), where L is an assignment of pairwise disjoint sets to the vertices of G and H is a graph with vertex set $\bigcup_{v \in V(G)} L(v)$ satisfying the following conditions:

1. $H[L(v)]$ is a complete graph for each $v \in V(G)$.
2. The set of edges between $L(u)$ and $L(v)$ is the union of $e_G(u, v)$ (possibly empty) matchings for every two distinct $u, v \in V(G)$.

Given a positive integer k and a multigraph G, let G^k denote the multigraph that is obtained from G by replacing each edge in G with a set of k parallel edges. In particular, $G^1 = G$ for every G. The next two lemmas demonstrate the two classes of multigraphs that are not DP-degree-colorable; the first of them exhibits multigraphs whose DP-chromatic number exceeds the number of vertices. In particular, for each $k \geq 2$, the 2-vertex multigraph K_2^k has DP-chromatic number $k + 1$.

Lemma 7. *K_n^k is not DP-degree-colorable.*

Proof. Let $G := K_n^k$. For each $v \in V(G)$, let $L(v) := \{(v, i, j) : i \in [n-1],\ j \in [k]\}$, and

$$(v_1, i_1, j_1)(v_2, i_2, j_2) \in E(H) \iff v_1 = v_2 \ \vee\ i_1 = i_2.$$

Then (L, H) is a cover of G and $|L(v)| = k(n-1) = \deg_G(v)$ for all $v \in V(G)$. We claim that G is not (L, H)-colorable. Indeed, if $I \subseteq V(H)$ is such that $|I \cap L(v)| = 1$ for all $v \in V$, then for some distinct (v_1, i_1, j_1), $(v_2, i_2, j_2) \in I$, we have $i_1 = i_2$. Thus, I is not an independent set. □

Lemma 8. *C_n^k is not DP-degree-colorable.*

PROOF. Let $G := C_n^k$. Without loss of generality, assume that $V(G) = [n]$ and $e_G(u, v) = k$ if and only if $|u - v| = 1$ or $\{u, v\} = \{1, n\}$. For each $v \in [n]$, let

$$L(v) := \{(v, i, j) \,:\, i \in [2], j \in [k]\},$$

and let

$$(v_1, i_1, j_1)(v_2, i_2, j_2) \in E(H) :\Longleftrightarrow v_1 = v_2 \vee (|v_1 - v_2| = 1 \wedge i_1 = i_2) \vee (\{v_1, v_2\} = \{1, n\} \wedge i_1 = i_2 + 1 + n \pmod 2).$$

Then (L, H) is a cover of G and $|L(v)| = 2k = \deg_G(v)$ for all $v \in [n]$. We claim that G is not (L, H)-colorable. Indeed, suppose that $I \subset V(H)$ is an (L, H)-coloring of G. Let $I = \{(v, i_v, j_v)\}_{v=1}^n$. Without loss of generality, assume that $i_1 = 1$. Then for each $v \in [n]$, $i_v = v \pmod 2$. Thus, $i_1 = i_n + 1 + n \pmod 2$, so $(1, i_1, j_1)(n, i_n, j_n) \in E(H)$; therefore, I is not independent. □

Our main result shows that the above lemmas describe all 2-connected multigraphs that are not DP-degree-colorable.

Theorem 9. *Suppose that G is a connected multigraph. Then G is not DP-degree-colorable if and only if each block of G is one of the graphs K_n^k and C_n^k for some n and k.*

The result has an implication for the number of edges in DP-k-*critical* graphs and multigraphs, i.e., (multi)graphs G with $\chi_{DP}(G) = k$ such that every proper sub(multi)graph of G has a smaller DP-chromatic number. It is easy to show (and follows from the above lemmas and Lemma 12 in the next section) that K_n^k is DP-$(k(n-1)+1)$-critical and C_n^k is DP-$(2k+1)$-critical. It is also easy to show (and follows from Theorem 9) that

$$2|E(G)| \geq (k-1)n \text{ for every } n\text{-vertex DP-}k\text{-critical multigraph } G. \tag{1}$$

The examples of C_n^k show that, for each odd $k \geq 3$, there are infinitely many 2-connected DP-k-critical multigraphs G with equality in (1). However, if we consider only simple graphs, then Theorem 9 implies a stronger bound than (1), which is an analog of Gallai's bound [8] for ordinary coloring (see [7] for list coloring):

Corollary 10. *Let $k \geq 4$ and let G be a DP-k-critical graph distinct from K_k. Then*

$$2|E(G)| \geq \left(k - 1 + \frac{k-3}{k^2-3}\right)n. \tag{2}$$

We will prove Theorem 9 in the next section and derive Corollary 10 in Section 4.

3. Proof of Theorem 9

We proceed by a series of lemmas.

Lemma 11. *Suppose that G is a regular n-vertex multigraph whose underlying simple graph is a cycle. Then G is not DP-degree-colorable if and only if $G \cong C_n^k$ for some k.*

PROOF. Without loss of generality, assume that $V(G) = [n]$ and $e_G(u, v) > 0$ if and only if $|u-v| = 1$ or $\{u, v\} = \{1, n\}$. Suppose that $G \ncong C_n^k$. Since G is regular, this implies that n is even and for some distinct positive r, s, $e_G(v, v+1) = r$ for all odd $v \in [n]$ and $e_G(1, n) = e_G(v, v+1) = s$ for all even $v \in [n-1]$. Without loss of generality, assume that $s > r$.

Let (L, H) be a cover of G such that $|L(v)| = \deg_G(v) = r + s$ for all $v \in [n]$. We will show that G is (L, H)-colorable. For $x \in L(1)$, say that a color $y \in L(v)$ is *x-admissible* if there exists

$I \subseteq V(H)$ independent in $H - E_H(L(1), L(n))$ such that $|I \cap L(u)| = 1$ for all $u \in [v]$ and $\{x, y\} \subseteq I$. Let $A_x(v) \subseteq L(v)$ denote the set of all x-admissible colors in $L(v)$. Clearly, $|A_x(2)| \geq s$ and $|A_x(3)| \geq r$ for each $x \in L(1)$. Suppose that $|A_x(3)| > r$ for some $x \in L(1)$. Since each color in $L(4)$ has at most r neighbors in $L(3)$, $A_x(4) = L(4)$. Similarly, $A_x(v) = L(v)$ for all $v \geq 4$. In particular, $A_x(n) = L(n)$. Take any $y \in L(n) \setminus N_H(x)$. Since $y \in A_x(n)$, there is a set $I \subseteq V(H)$ independent in $H - E_H(L(1), L(n))$ such that $|I \cap L(u)| = 1$ for all $u \in [n]$ and $\{x, y\} \subseteq I$. But then I is independent in H, and so I is an (L, H)-coloring of G. Thus, we may assume that $|A_x(3)| = r$ for all $x \in L(v)$. Note that

$$L(3) \setminus A_x(3) = L(3) \cap \bigcap_{y \in A_x(2)} N_H(y).$$

Therefore, $L(3) \cap N_H(y)$ is the same set of size s for all $y \in A_x(2)$. Since each vertex in $L(3)$ has at most s neighbors in $L(2)$, the graph $H[A_x(2) \cup (L(3) \setminus A_x(3))]$ is a complete $2s$-vertex graph. Since every vertex in $L(2)$ is x-admissible for some $x \in L(1)$, $H[L(2) \cup L(3)]$ includes a disjoint union of at least two complete $2s$-vertex graphs. Hence, $|L(2) \cup L(3)| \geq 4s$. But $|L(2)| = |L(3)| = r + s < 2s$; a contradiction. □

Lemma 12. *Let G be a connected multigraph and suppose that (L, H) is a cover of G such that $|L(v)| \geq \deg_G(v)$ for all $v \in V(G)$, and $|L(v_0)| > \deg_G(v_0)$ for some $v_0 \in V(G)$. Then G is (L, H)-colorable.*

Proof. If $|V(G)| = 1$ then the claim is obvious. Suppose now that G is a counterexample with the fewest vertices. Consider the multigraph $G' := G - v_0$. For each $v \in V(G')$, let $L'(v) := L(v)$, and let $H' := H - L(v_0)$. By construction, (L', H') is a cover of G' such that for all $v \in V(G')$, $|L(v)| \geq \deg_{G'}(v)$. Moreover, since G is connected, each connected component of G' contains a vertex u adjacent in G to v_0 and thus satisfying $\deg_{G'}(u) < \deg_G(u)$. Hence, by the minimality assumption, G' is (L', H')-colorable. Let $I' \subseteq V(H')$ be an (L', H')-coloring of G'. Then $|N_G(I') \cap L(v_0)| \leq \deg_G(v_0)$, so $L(v_0) \setminus N_G(I') \neq \varnothing$. Thus, I' can be extended to an (L, H)-coloring I of G; a contradiction. □

Lemma 13. *Let G be a connected multigraph and let (L, H) be a cover of G. Suppose that there are $v_1 \in V(G)$ and $x_1 \in L(v_1)$ such that $G - v_1$ is connected and for some $v_2 \in V(G) \setminus \{v_1\}$, x_1 has fewer than $e_G(v_1, v_2)$ neighbors in $L(v_2)$. Then G is (L, H)-colorable.*

Proof. Put $G' := G - v_1$. For each $v \in V(G')$, let $L'(v) := L(v) \setminus N_H(x_1)$, and $H' := H - L(v_1) - N_H(x_1)$. Then (L', H') is a cover of G'. Moreover, for each $v \in V(G')$,

$$|L'(v)| = |L(v)| - |L(v) \cap N_H(x_1)| \geq \deg_G(v) - e_G(v, v_1) = \deg_{G'}(v),$$

and

$$|L'(v_2)| = |L(v_2)| - |L(v_2) \cap N_H(x_1)| > \deg_G(v_2) - e_G(v_2, v_1) = \deg_{G'}(v_2).$$

Since G' is connected, Lemma 12 implies that G' is (L', H')-colorable. But if $I' \subseteq V(H')$ is an (L', H')-coloring of G', then $I' \cup \{x_1\}$ is an (L, H)-coloring of G, as desired. □

Lemma 14. *Suppose that G is a 2-connected multigraph and (L, H) is a cover of G with $|L(v)| \geq \deg_G(v)$ for each $v \in V(G)$. If G is not (L, H)-colorable, then G is regular and for each pair of adjacent vertices $v_1, v_2 \in V(G)$, the bipartite graph $H[L(v_1), L(v_2)]$ is $e_G(v_1, v_2)$-regular.*

Proof. Consider any two adjacent $v_1, v_2 \in V(G)$. By Lemma 13, $H[L(v_1), L(v_2)]$ is an $e_G(v_1, v_2)$-regular bipartite graph with parts $L(v_1)$ and $L(v_2)$. Therefore, $|L(v_1)| = |L(v_2)|$, and so $\deg_G(v_1) = \deg_G(v_2)$, as desired. Since G is connected while v_1 and v_2 are arbitrary adjacent vertices in G, this implies that G is regular. □

Lemma 15. *Let G be a 2-connected multigraph. Suppose that $u_1, u_2, w \in V(G)$ are distinct vertices such that $G - u_1 - u_2$ is connected, $e_G(u_1, u_2) < e_G(u_1, w)$, and $e_G(u_2, w) \geq 1$. Then G is DP-degree-colorable.*

Proof. Suppose that G is not (L, H)-colorable for some cover (L, H) with $|L(v)| = \deg_G(v)$ for all $v \in V(G)$. We show first that

$$\text{there are nonadjacent } x_1 \in L(u_1) \text{ and } x_2 \in L(u_2) \text{ with } N_H(x_1) \cap N_H(x_2) \cap L(w) \neq \varnothing. \qquad (3)$$

Indeed, consider any $x_2 \in L(u_2)$. By Lemma 14, $|L(w) \cap N_H(x_2)| = e_G(u_2, w) \geq 1$. Similarly, for each $y \in L(w) \cap N_H(x_2)$, $|L(u_1) \cap N_H(y)| = e_G(u_1, w) > e_G(u_1, u_2) = |L(u_1) \cap N_H(x_2)|$. Thus, there exists $x_1 \in (L(u_1) \cap N_H(y)) \setminus (L(u_1) \cap N_H(x_2))$. By the choice, x_1 and x_2 are nonadjacent and $y \in N_H(x_1) \cap N_H(x_2) \cap L(w)$. This proves (3).

Let x_1 and x_2 satisfy (3). Let $G' := G - u_1 - u_2$. Given $v \in V(G')$, put $L'(v) := L(v) \setminus (N_H(x_1) \cup N_H(x_2))$, and $H' := H - L(u_1) - L(u_2) - N_H(x_1) - N_H(x_2)$. Then G' is connected and (L', H') is a cover of G' satisfying the conditions of Lemma 12 with w in the role of v_0. Thus G' is (L', H')-colorable, and hence G is (L, H)-colorable; a contradiction. □

Lemma 16. *Suppose that G is an n-vertex 2-connected multigraph that contains a vertex adjacent to all other vertices. Then either $G \cong K_n^k$ for some k or G is DP-degree-colorable.*

Proof. Suppose that G is an n-vertex multigraph that is not DP-degree-colorable and assume that $w \in V(G)$ is adjacent to all other vertices. If some distinct $u_1, u_2 \in V(G) \setminus \{w\}$ are nonadjacent, then the triple u_1, u_2, w satisfies the conditions of Lemma 15, and so G is DP-degree-colorable. Hence every two vertices in G are adjacent; in other words, the underlying simple graph of G is K_n. It remains to show that every two vertices in G are connected by the same number of edges. Indeed, if u_1, u_2, $u_3 \in V(G)$ are such that $e_G(u_1, u_2) < e_G(u_1, u_3)$, then, by Lemma 15 again, G is DP-degree-colorable. □

Lemma 17. *Suppose that G is a 2-connected n-vertex multigraph in which each vertex has at most 2 neighbors. Then either $G \cong C_n^k$ for some k, or G is DP-degree-colorable.*

Proof. Suppose that G is a 2-connected n-vertex multigraph in which each vertex has at most 2 neighbors and that is not DP-degree-colorable. Then the underlying simple graph of G is a cycle and Lemma 14 implies that G is regular. Hence, $G \cong C_n^k$ by Lemma 11. □

Lemma 18. *Suppose that G is a 2-connected n-vertex multigraph that is not DP-degree-colorable. Then $G \cong K_n^k$ or C_n^k for some k.*

Proof. By Lemmas 16 and 17, we may assume that G contains a vertex u such that $3 \leq |N_G(u)| \leq n - 2$. Since G is 2-connected, $G - u$ is connected. However, $G - u$ is not 2-connected. Indeed, let u_1 be any vertex in $V(G) \setminus (\{u\} \cup N_G(u))$ that shares a neighbor w with u. By Lemma 15 with u in place of u_2, $G - u_1 - u$ is disconnected, so u_1 is a cut vertex in $G - u$.

Therefore, $G - u$ has at least two leaf blocks, say B_1 and B_2. For $i \in [2]$, let x_i be the cut vertex of $G - u$ contained in B_i. Since G itself is 2-connected, u has a neighbor $u_i \in B_i - x_i$ for each $i \in [2]$. Then u_1 and u_2 are nonadjacent and $G - u - u_1 - u_2$ is connected. Since u has at least 3 neighbors, $G - u_1 - u_2$ is also connected. Hence, we are done by Lemma 15 with u in the role of w. □

Lemma 19. *Suppose that $w \in V(G)$, $G = G_1 + G_2$, and $V(G_1) \cap V(G_2) = \{w\}$. If G_1 and G_2 are not DP-degree-colorable, then G is not DP-degree-colorable.*

Proof. Suppose that G_1 is not (L_1, H_1)-colorable and G_2 is not (L_2, G_2)-colorable, where for each $i \in [2]$, (L_i, H_i) is a cover of G_i such that $|L(v)| = \deg_{G_i}(v)$ for all $v \in V(G_i)$. Without loss of generality, assume that $L_1(v_1) \cap L_2(v_2) = \varnothing$ for all $v_1 \in V(G_1)$ and $v_2 \in V(G_2)$. For each $v \in V(G)$, let

$$L(v) := \begin{cases} L_1(v), & \text{if } v \in V(G_1) \setminus \{w\}, \\ L_2(v), & \text{if } v \in V(G_2) \setminus \{w\}, \\ L_1(w) \cup L_2(w), & \text{if } v = w, \end{cases}$$

and let $H := H_1 + H_2 + K(L(w))$, where $K(L(w))$ denotes the complete graph with vertex set $L(w)$. Then (L, H) is a cover of G and for each $v \in V(G)$, $|L(v)| = \deg_G(v)$. Suppose that G is (L, H)-colorable and let I be an (L, H)-coloring of G. Without loss of generality, assume that $I \cap L(w) \subseteq L_1(w)$. Then $I \cap V(H_1)$ is an (L_1, H_1)-coloring of G_1; a contradiction. □

Proof of Theorem 9. Lemmas 7, 8, and 19 show that if each block of G is isomorphic to one of the multigraphs K_n^k and C_n^k for some n and k, then G is not DP-degree-colorable.

Now assume that G is a connected multigraph that is not DP-degree-colorable. If G is 2-connected, then we are done by Lemma 18. Therefore, we may assume that G has a cut vertex $w \in V(G)$. Let G_1 and G_2 be nontrivial connected subgraphs of G such that $G = G_1 + G_2$ and $V(G_1) \cap V(G_2) = \{w\}$. It remains to show that neither G_1 nor G_2 is DP-degree-colorable, since then we will be done by induction. Suppose towards a contradiction that G_1 is DP-degree-colorable. Let (L, H) be a cover of G such that $|L(v)| = \deg_G(v)$ for all $v \in V(G)$. By Lemma 12 applied to the connected components of $G_2 - w$, there exists an independent set $I_2 \subseteq \bigcup_{v \in V(G_2)\setminus\{w\}} L(v)$ such that $|L(v) \cap I_2| = 1$ for all $v \in V(G_2) \setminus \{w\}$. Given $v \in V(G_1)$, put $L_1(v) := L(v) \setminus N_H(I_2)$. (Note that $L_1(v) = L(v)$ for all $v \in V(G_1) \setminus \{w\}$.) Also, let

$$H_1 := H\Big[\bigcup_{v \in V(G_1)} L_1(v)\Big].$$

Then (L_1, H_1) is a cover of G_1. Note that $|L_1(v)| = |L(v)| = \deg_G(v) = \deg_{G_1}(v)$ for each $v \in V(G_1)\setminus\{w\}$; and we have $|L_1(w)| = |L(v)| - |N_H(I_2) \cap L(w)| \geq \deg_G(w) - \deg_{G_2}(w) = \deg_{G_1}(w)$ for w. Since G_1 is DP-degree-colorable, it is (L_1, H_1)-colorable. But if I_1 is an (L_1, H_1)-coloring of G_1, then $I_1 \cup I_2$ is an (L, H)-coloring of G. $\square$

4. On DP-Critical Graphs

Gallai in [8] proved (2) for ordinary k-critical n-vertex graphs using an upper bound on the number of edges in *Gallai trees*—the connected graphs in which every block is a complete graph or an odd cycle. We will need the same statement for *GDP-trees*—the graphs in which each block is a complete graph or a cycle (not necessarily odd).

Lemma 20. *Let $k \geq 4$ and let T be an n-vertex GDP-tree with maximum degree $\Delta(T) \leq k-1$ not containing K_k. Then*

$$2|E(T)| \leq \left(k - 2 + \frac{2}{k-1}\right)n. \tag{4}$$

The proof is the same as Gallai's. We present the proof in the Appendix, since Gallai's paper is in German. Below is the rest of the proof of Corollary 10. It is based on Gallai's ideas but is shorter.

We use discharging. Let G be an n-vertex DP-k-critical graph distinct from K_k. Note that the minimum degree of G is at least $k-1$. The initial charge of each vertex $v \in V(G)$ is $\mathrm{ch}(v) := \deg_G(v)$. The only discharging rule is this:

(R1) Each vertex $v \in V(G)$ with $\deg_G(v) \geq k$ sends to each neighbor the charge $\frac{k-1}{k^2-3}$.

Denote the new charge of each vertex v by $\mathrm{ch}^*(v)$. We will show that

$$\sum_{v \in V(G)} \mathrm{ch}^*(v) \geq \left(k - 1 + \frac{k-3}{k^2-3}\right)n. \tag{5}$$

Indeed, if $\deg_G(v) \geq k$, then

$$\mathrm{ch}^*(v) \geq \deg_G(v) - \frac{k-1}{k^2-3} \cdot \deg_G(v) \geq k\left(1 - \frac{k-1}{k^2-3}\right) = k - 1 + \frac{k-3}{k^2-3}. \tag{6}$$

Also, if T is a component of the subgraph G' of G induced by the vertices of degree $k-1$, then

$$\sum_{v \in V(T)} \mathrm{ch}^*(v) \geq (k-1)|V(T)| + \frac{k-1}{k^2-3}|E_G(V(T), V(G) \setminus V(T)|.$$

Since T is a GDP-tree and does not contain K_k, by Lemma 20,

$$|E(V(T), V(G) \setminus V(T)| \geq (k-1)|V(T)| - \left(k - 2 + \frac{2}{k-1}\right)|V(T)| = \frac{k-3}{k-1}|V(T)|.$$

Thus for every component T of G' we have

$$\sum_{v \in V(T)} \mathrm{ch}^*(v) \geq (k-1)|V(T)| + \frac{k-1}{k^2-3} \cdot \frac{k-3}{k-1} \cdot |V(T)| = \left(k - 1 + \frac{k-3}{k^2-3}\right)|V(T)|.$$

Together with (6), this implies (5).

5. Appendix

We essentially repeat Gallai's proof of Lemma 20 by induction on the number of blocks. If T is a block, then, since $T \ncong K_k$ and $k \geq 4$, $\Delta(T) \leq k-2$, which is stronger than (4).

Suppose that (4) holds for all GDP-trees with at most s blocks and T is a GDP-tree with $s+1$ blocks. Let B be a leaf block in T and let x be the cut vertex in $V(B)$. Let $D := \Delta(B)$.

Case 1: $D \leq k-3$. Let $T' := T - (V(B) \setminus \{x\})$. Then T' is a GDP-tree with s blocks. So $2|E(T)| = 2|E(T')| + D|V(B)|$ and, by induction,

$$2|E(T')| \leq \left(k-2+\frac{2}{k-1}\right)(n-|V(B)|+1).$$

If $B = K_r$, then $r = D+1 \leq k-2$. So in this case

$$\begin{aligned} &2|E(T)| - \left(k-2+\frac{2}{k-1}\right)n \\ &\leq \left(k-2+\frac{2}{k-1}\right)(n-D) + D(D+1) - \left(k-2+\frac{2}{k-1}\right)n \\ &= D\left(-k+2-\frac{2}{k-1}+D+1\right) \leq -D\frac{2}{k-1} < 0, \end{aligned}$$

as claimed. Similarly, if $B = C_t$, then, by the case, $k \geq 5$ and

$$\begin{aligned} &2|E(T)| - \left(k-2+\frac{2}{k-1}\right)n \\ &\leq \left(k-2+\frac{2}{k-1}\right)(n-t+1) + 2t - n\left(k-2+\frac{2}{k-1}\right) \\ &= (t-1)\left(-k+2-\frac{2}{k-1}+2\right) + 2 < 2\left(-k+4\right) + 2 \leq 0. \end{aligned}$$

Case 2: $D = k-2$. Since $\Delta(T) \leq k-1$, only one block B' apart from B may contain x and this B' must be K_2. Let $T'' = T - V(B)$. Then T'' is a GDP-tree with $s-1$ blocks. So $2|E(T)| = 2|E(T'')| + D|V(B)| + 2$ and, by induction,

$$2|(T'')| \leq \left(k-2+\frac{2}{k-1}\right)(n-|V(B)|).$$

Hence in this case, since $|V(B)| \geq D+1 = k-1$,

$$\begin{aligned} &2|E(T)| - \left(k-2+\frac{2}{k-1}\right)n \\ &\leq \left(k-2+\frac{2}{k-1}\right)(n-|V(B)|) + (k-2)|V(B)| + 2 - \left(k-2+\frac{2}{k-1}\right)n \\ &= |V(B)|\left(-k+2-\frac{2}{k-1}+k-2\right) + 2 \leq -\frac{2}{k-1}|V(B)| + 2 \leq 0, \end{aligned}$$

again. □

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