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GROUPS WITH THE QUASICYCLIC CENTRALIZER OF A FINITE INVOLUTION

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Abstract: We prove that a group with the not maximal quasicyclic centralizer of a finite involution is locally finite.

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As was proved by Burnside, all elements of odd order constitute a normal subgroup in a finite group with cyclic Sylow 2-subgroup.

For infinite periodic groups, this is false even in the case when the Sylow 2-subgroup has order 2 and is the center of the group [1]. As was recently discovered [2, 3], such key results of the theory of finite groups connected with involutions as Glauberman's Z^*-Theorem and the Baer–Suzuki Theorem (in its even version) are false either. On the other hand, some interesting results of the theory of finite groups are carried over without any loss to the class of groups with finite involution [4–7]. Groups with finite involutions and periodic groups with elementary abelian, primary abelian, and abelian centralizers of involutions were studied in [4, 5, 7] excluding groups of 2-rank 1. In 2002, Mazurov posed the following question in *The Kourovka Notebook* [8]:

15.54. *Suppose that G is a periodic group containing an involution i such that the centralizer $C_G(i)$ is a locally cyclic 2-group. Does the set of all elements of odd order in G that are inverted by i form a subgroup?*

In the present article, we give the affirmative answer to this question for the groups in which the centralizer of an involution i is not a maximal subgroup.

Recall that an involution i in G is called *finite* if the order of the commutator $[i, g] = ig^{-1}ig$ is finite for every $g \in G$. It is clear that each involution of a periodic group is finite.

Theorem. *Suppose that a group G containing a finite involution and the centralizer of some involution i is a locally cyclic 2-group not maximal in G. Then the set of all elements of odd order in G inverted by i constitutes a subgroup. In particular, G is locally finite, and either $G = C_G(i)$ or G is a Frobenius group with abelian kernel $[i, G]$ and noninvariant factor $C_G(i)$.*

Since every locally cyclic 2-group is either finite or countable, the theorem implies the solution of Mazurov's Question 15.54 for uncountable groups.

Corollary. *If an uncountable group G containing a finite involution and the centralizer of some involution i is a locally cyclic 2-group then i inverts each element of odd order in G and G is a locally finite Frobenius group with abelian kernel $[i, G]$.*

1. Proof of the Theorem

Lemma 1. *Suppose that a group G contains a finite involution and the centralizer of some involution i is a locally cyclic 2-group. Then the subgroup $C_G(i)$ is strongly isolated, all involutions in G are conjugate, and the order of the product of every two involutions in G is finite and odd.*

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Proof. Since $C_G(u) \leq C_G(i)$ for every nonunit element $u \in C_G(i)$, the subgroup $C_G(i)$ is strongly isolated by definition.

Let k be an arbitrary finite involution in G different from i. The definition of finite involution implies that the subgroup $D = \langle i, k \rangle = \langle k, k^i \rangle \cdot \langle i \rangle$ is a finite dihedral group. By hypotheses, $C_G(i)$ contains no elementary abelian subgroups of order 4, and so the subgroup D does not contain such subgroups either. This means that $|ik|$ is odd and the involutions i and k are conjugate in D. In particular, i is finite in G. Let k be an arbitrary involution in G different from i. Then $D = \langle i, k \rangle = \langle i, i^k \rangle \cdot \langle k \rangle$ is a finite dihedral group, as above, the order of the element ik is odd, and the involutions i and k are conjugate in D. This proves the lemma.

A proper subgroup B in a group G is called *strongly embedded* if B contains an involution and the subgroup $B \cap B^g$ does not contain involutions for any $g \in G \setminus B$.

Lemma 2. *Let i be a finite involution of a group G and let $C_G(i)$ be a locally finite cyclic 2-group. Then every proper subgroup B in G containing $C_G(i)$ is strongly embedded in G.*

Proof. The lemma is obvious for $B = C_G(i)$. Suppose that $B \neq C_G(i)$, $g \in G \setminus B$, and $k = j^g \in B \cap B^g$ for some involution $j \in B$. By Lemma 1, $k, j \in i^B = j^B$ and $k^t = j$ for some $t \in B$. But then $gt \in C_G(j) < B$, which contradicts the choice of g. Hence, B is strongly embedded in G. The lemma is proved.

Recall that $[i, G]$ is by definition the group generated by all commutators $[i, g] = i^{-1}g^{-1}ig$, where $g \in G$. It is well known that the subgroup $[i, G]$ is normal in G (see, for example, [9, Proposition 2.16]).

Belyaev's Theorem in [10] (see also [11; 9, Corollary 2.30]) implies

Lemma 3. *If the subgroup $C_G(i)$ is finite then either $C_G(i) = G$ or $G = [i, G] \leftthreetimes C_G(i)$ is a locally finite Frobenius group with kernel $[i, G]$ and complement $C_G(i)$.*

In view of Lemma 3, we will assume that the subgroup $C_G(i)$ is infinite and so it is a quasicyclic group C_{2^∞} [12]. Denote by J the set of involutions of G.

Lemma 4. *Suppose that G is a group with the quasicyclic centralizer of a finite involution i and $C_G(i) < B < G$ ($C_G(i) \neq B \neq G$). Then $B = [i, B] \leftthreetimes C_G(i)$ is a locally finite Frobenius group with kernel $[i, B]$ and complement $C_G(i)$; moreover, $[i, B] = B \cap B^j$ for every involution $j \in G \setminus B$.*

Proof. By Lemma 2, B is strongly embedded in G and, by the definition of strongly embedded subgroup, $J \setminus B$ is not empty. Let j be an arbitrary involution in $G \setminus B$. By Lemma 2.1 of [5], the subgroup B includes a set M_j of strictly real elements with respect to j of the same cardinality as B (under the assumptions made, $|J \cap B| = |B|$). In particular, the subgroup $H = B \cap B^j$ is nonidentity; moreover, $H^j = H$. Using the conditions $C_H(j) = 1$ and Busarkin's Lemma [9, Lemma 2.20], we see that H is a periodic abelian group without involutions. By Lemma 2.2 of [5], $B = H \cdot C_G(i)$; i.e., B is factorized by two abelian subgroups H and $C_G(i)$. By the well-known Ito's Theorem in [13], the commutant T of B is abelian. The relation $i \in T$ would imply that $T \leq C_G(i)$ and $B \leq C_G(i)$ contrary to $B \neq C_G(i)$. Hence, $i \notin T$ and, by hypotheses, $T \cap C_G(i) = 1$. By Busarkin's Lemma, T is a periodic abelian group without involutions whose every element is inverted by i. Since the quotient group B/T is abelian, $[i, B] = T$ and $B = [i, B] \leftthreetimes C_G(i)$ is a locally finite Frobenius group with kernel $[i, B]$ and complement $C_G(i)$. In particular, $H \leq [i, B]$ and $H = B \cap B^j = [i, B]$ since $B = H \cdot C_G(i)$. The lemma is proved.

Let us now finish the proof of the theorem. The case of $G = C_G(i)$ is trivial; when $C_G(i)$ is finite, the theorem follows from Lemma 3. Suppose that $C_G(i)$ is infinite, B is the subgroup of Lemma 4, and j is an arbitrary involution in $G \setminus B$. By Lemma 4, $B \cap B^j = [i, B] \neq 1$ and j inverts each element b in $[i, B]$. Suppose that $b \neq 1$ and $A = C_G(b)$. Then $i, j \in N_G(A)$, $A \cap C_G(i) = 1$. By Busarkin's Lemma, A is a periodic abelian group without involutions and the involutions i and j invert each element in A. Since the involution $j \in G \setminus B$ is arbitrary, we conclude that $J = Ai$, $J = i^A$, and $A = [i, G]$. By Frattini's argument, $G = [i, G] \leftthreetimes C_G(i)$. By Schmidt's Theorem in [12], G is locally finite, and it is obvious that it is a Frobenius group with abelian kernel $[i, G]$ and complement $C_G(i)$. The theorem is proved.

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A. I. Sozutov
Siberian State Aerospace University, Krasnoyarsk, Russia
E-mail address: sozutov_ai@mail.ru