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THEOREMS OF COMPARISON AND STABILITY WITH PROBABILITY 1 FOR ONE-DIMENSIONAL STOCHASTIC DIFFERENTIAL EQUATIONS

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Abstract: We prove the comparison theorems for scalar stochastic differential equations in the case of different diffusion coefficients. Conditions are given of stability with probability 1 with respect to the trivial solution to stochastic differential equations with random coefficients. The results remain valid for deterministic analogs of stochastic differential equations with symmetric integrals.

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Introduction

Given are the two scalar equations with symmetric integrals

$$d\xi_t^{(1)} = \sigma^{(1)}\big(t, \xi_t^{(1)}\big) * dX(t) + b^{(1)}\big(t, \xi_t^{(1)}\big)\,dt, \quad \xi_t^{(1)}|_{t=t_0} = \xi_0^{(1)}, \tag{1}$$

$$d\xi_t^{(2)} = \sigma^{(2)}\big(t, \xi_t^{(2)}\big) * dX(t) + b^{(2)}\big(t, \xi_t^{(2)}\big)\,dt, \quad \xi_t^{(2)}|_{t=t_0} = \xi_0^{(2)}, \tag{2}$$

where $X(t)$, $t \in R^+$, is an arbitrary continuous function not assumed to be differentiable. These equations are deterministic analogs of stochastic differential equations with Stratonovich integral and so the theorems proven for these equations are valid with probability 1 for stochastic differential equations (in what follows, SDEs) as well.

One of the aims of this article is the proof of comparison theorems for (1) and (2). For the first time, a comparison theorem was proven by Skorokhod in [1] for the equations of the form $d\xi(t) = b(t, \xi(t))\,dt + \sigma(t, \xi(t))\,dW(t)$, where $W(t)$ is a standard Wiener process, and the most general form of this theorem is presented in [2]. The result of [1] was developed in [3, 4]. The comparison theorems for SDEs with a multidimensional Wiener process and for stochastic partial differential equations (SPDEs) are given in [5]. Earlier results on SPDEs can be found in [6]. However, all these results are proven in the case of $\sigma^{(1)}(t,u) = \sigma^{(2)}(t,u)$ for all (t,u). The case that this condition fails is examined for a special Ito equation in [7]. The essential particularities of the comparison theorems of this article is the fact that they are proven for more general equations than those of [7] and the coefficients of the equations can be random. Some preliminary results, being the base of this article, were announced on the International Youth Scientific Forum "Lomonosov-2015" [8]. Our approach relies on the fact that the structure is known of a solution to the deterministic analog of an SDE.

Application of the comparison theorems to SDEs of the form

$$d\xi_t = \sigma(t, \xi_t) * dW(t) + b(t, \xi_t)\,dt, \quad \xi_t|_{t=t_0} = \xi_0, \tag{3}$$
$$\sigma(t,0) = 0, \quad b(t,0) = 0,$$

allows us to establish stability conditions for SDEs with probability 1. As a rule, for SDEs with nonrandom $\sigma(t,u)$ and $b(t,u)$, stability is understood in the weaker sense: in probability and p-stability. A perturbed solution ξ_t to (3) with the initial condition $\xi_0 = x_0$ is *stable in probability* whenever, for every $\varepsilon > 0$,

$$\lim_{\xi_0 \to 0} P\{\sup_{t>0} |\xi_t| > \varepsilon\} = 0.$$

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A perturbed solution ξ_t to (3) with the initial condition $\xi_0 = x_0$ is *p-stable* if $\sup_{|x|\le\delta, t\ge 0} E|\xi_t|^p \to 0$ as $\delta \to 0$. Given (3), introduce the Lyapunov operator

$$L = \frac{\partial}{\partial t} + \left(b^{(1)}(t,u) + \frac{\partial \sigma^{(1)}(t,u)}{\partial u} \sigma^{(1)}(t,u) \right) \frac{\partial}{\partial u} + \frac{1}{2} (\sigma^{(1)})^2(t,u) \frac{\partial^2}{\partial u^2}.$$

The conditions of stability in probability and p-stability are given in [9, 10] and they rely on the construction of the Lyapunov function $V(t,x)$ whose Lyapunov operator satisfies some inequalities. A perturbed solution ξ_t to (3) with the initial condition $\xi_0 = x_0$ is *stable with probability* 1 if, for a.a. ω and every $\varepsilon > 0$, there exists $\delta(\varepsilon, \omega) > 0$ such that the inequality $|x_0| < \delta$ yields $|\xi_t| < \varepsilon$ for all $t > 0$. The authors are unaware of the corresponding theorems for stability with probability 1 for SDEs.

The Main Part

1. This article is based on the ideas of [11]. Since the method we use is essentially pathwise and so remains valid for pathwise analogs of SDEs, recall the necessary notions (for details see [11]).

Let $\mathbb{R} = (-\infty, +\infty)$ and $\mathbb{R}^+ = [0, +\infty)$. Consider partitions T_n, $n \in \mathbb{N}$, of the segment $[0,t]$: $T_n = \{t_k^{(n)}\}$, $0 = t_0^{(n)} \le t_1^{(n)} \le \dots \le t_k^{(n)} \le \dots \le t_{m_n}^{(n)} = t$, $n \in \mathbb{N}$, such that $T_n \subset T_{n+1}$, $n \in \mathbb{N}$, and $\lambda_n = \max_k \big| t_k^{(n)} - t_{k-1}^{(n)} \big| \to 0$ as $n \to \infty$. Denote by $X^{(n)}(s)$, $s \in [0,t]$, the broken line constructed from a function $X(s)$ and corresponding to T_n. Put

$$\Delta t_k^{(n)} = t_k^{(n)} - t_{k-1}^{(n)}, \quad \big[\Delta t_k^{(n)}\big] = \big[t_{k-1}^{(n)}, t_k^{(n)}\big], \quad \Delta X_k^{(n)} = X\big(t_k^{(n)}\big) - X\big(t_{k-1}^{(n)}\big).$$

By a *symmetric integral* we mean the expression

$$\int_0^t f(s, X(s)) * dX(s) = \lim_{n\to\infty} \sum_k \frac{1}{\Delta t_k^{(n)}} \int_{[\Delta t_k^{(n)}]} f(s, X^{(n)}(s))\, ds \Delta X_k^{(n)}$$

if the limit on the right-hand side exists and is independent of the choice of the sequence of T_n, $n \in \mathbb{N}$.

The following condition (S) is sufficient for a symmetric integral to exist.

We say that a pair of functions $X(s)$, $s \in \mathbb{R}^+$, and $f(s,u)$, $s \in \mathbb{R}^+$, $u \in \mathbb{R}$, *satisfies condition* (S) *on* $[0,t]$ if

(a) $X(s)$, $s \in [0,t]$, is continuous;

(b) $f(s,u)$, $s \in [0,t]$, has bounded variation and is right-continuous for $s \in [0,t]$ for a.a. u;

(c) for a.a. u we have

$$\int_0^t \mathbf{1}(X(s) = u) |f|(ds, u) = 0,$$

where the function $|f|(s,u)$ is the total variation of $f(\tau, u)$ with respect to τ on $[0,s]$;

(d) the total variation $|f|(t,u)$ of $f(s,u)$ with respect to s on $[0,s]$ is locally summable with respect to u.

If $F(s,u)$ has the continuous partial derivatives $F'_s(s,u)$ and $F'_u(s,u)$, while $(X(s), F'_u(s,u))$ satisfy condition (S) on $[0,t]$; then the following equality [11, Chapter 2, Section 7, Theorem 7.2] is valid:

$$F(t, X(t)) - F(0, X(0)) = \int_0^t F'_u(s, X(s)) * dX(s) + \int_0^t F'_s(s, X(s))\, ds.$$

In particular, the last formula implies that a symmetric integral in the case of a typical trajectory of a Wiener process, i.e., in case $X(t) = W(t)$, within the framework of the Ito formula with probability 1 coincides with the stochastic Stratonovich integral.

Let the equation

$$\xi(t) - \xi(0) = \int\limits_0^t \sigma(s, \xi(s)) * dX(s) + \int\limits_0^t b(s, \xi(s))\, ds \tag{4}$$

be given. By a *solution* to (4) we mean a function of the form $\xi(s) = \phi(s, X(s))$, $s \in [0, T]$, satisfying the conditions:

(a) $(X(s), \sigma(s, \phi(s, u)))$ satisfy condition (S) on $[0, T]$;

(b) $b(s, \phi(s, X(s)))$ is summable on $[0, T]$;

(c) $\xi(s)$ turns (4) into equality.

The following theorem clarifies the *structure of a solution* to SDEs and equations with symmetric integrals. This theorem is proven for a finite time interval but it remains valid for unbounded intervals as well and it is given below in modified form.

Theorem A (see [11, Chapter 2, Section 10, Theorem 10.1]). *Assume that*

(a) $X(s)$, $s \in \mathbb{R}^+$, *is a nowhere differentiable continuous function;*

(b) $\sigma(s, \phi)$, $\sigma'_s(s, \phi)$, $\sigma'_\phi(s, \phi)$, *and* $b(s, \phi)$ *are jointly continuous on* $\mathbb{R}^+ \times \mathbb{R}$.

Then

1. *If* $\xi(s) = \phi(s, X(s))$ *is a solution to* (4) *such that* $\phi(s, u)$ *has jointly continuous partial derivatives* $\phi'_s(s, u)$, $\phi'_u(s, u)$, *and* $\phi''_{su}(s, u)$ *then* $\xi(s)$ *satisfies the relations*

$$\phi'_u(s, u) = \sigma(s, \phi(s, u)), \quad \phi'_s(s, X(s)) = b(s, \phi(s, X(s))). \tag{5}$$

2. *Assume that* $\phi(s, u)$ *has jointly continuous partial derivatives* $\phi'_s(s, u)$, $\phi'_u(s, u)$, *and* $\phi''_{su}(s, u)$ *and is a solution to* (5) *with the initial condition* $\xi(0) = \phi(0, X(0))$. *Then the function* $\xi(s) = \phi(s, X(s))$ *is a solution to* (4).

REMARK 1. If we consider the SDE, i.e., $X(s) = W(s)$, having a unique solution then the structure of this solution agrees with the structure of a solution to the deterministic analog of the SDE.

If $\sigma^{-1}(t, u)$ is locally summable then

$$\int\limits_{\xi(0)}^{\xi(t)} \frac{1}{\sigma(t, \psi)}\, d\psi = X(t) + C(t) \quad \forall t > 0, \tag{6}$$

where $C(t)$ is a solution to the ordinary differential equation

$$C'(t) = \frac{b(t, \phi(t, X(t) + C(t))) - \phi'_t(t, v)|_{v = X(t) + C(t)}}{\sigma(t, \phi(t, X(t) + C(t)))}. \tag{7}$$

2. Let us turn back to (1) and (2). In view of (6), a solution to (1) is representable as a function of a solution to (2); namely, we have the following

Theorem 1. *Assume that* $\xi_t^{(1)}$ *and* $\xi_t^{(2)}$ *are solutions to* (1) *and* (2), *respectively, satisfying the conditions of Theorem* A *and* $(\sigma^{(k)}(t, u))^{-1}$, $k = 1, 2$, *are locally summable. Then* $\xi_t^{(1)} = z(t, \xi_t^{(2)})$, *where*

$$z(t, u) = \phi\left(t, \int\limits_{\xi_0^{(2)}}^{u} \frac{1}{\sigma^{(2)}(t, \psi)}\, d\psi - C^{(2)}(t)\right), \tag{8}$$

and, moreover,

$$z'_u(t, u) = \frac{\sigma^{(1)}(t, z(t, u))}{\sigma^{(2)}(t, u)} \tag{9}$$

at all points u *such that* $\sigma^{(2)}(t, u) \neq 0$.

PROOF. Applying (6) to (2) yields

$$X(t) = \int\limits_{\xi_0^{(2)}}^{\xi_t^{(2)}} \frac{1}{\sigma^{(2)}(t,\psi)}\, d\psi - C^{(2)}(t),$$

where $C^{(2)}(t)$ satisfies (7) with $\sigma^{(2)}(t,u)$ and $b^{(2)}(t,u)$.

Hence, Theorem A and the last equality imply that $\xi_t^{(1)} = z\big(t, \xi_t^{(2)}\big)$. Let $\sigma^{(2)}(t,u) \neq 0$ at some point u and so at some neighborhood of u. In view of (8), the derivative $z'_u(t,u)$ is calculated by (9). □

REMARK 2. If we take a standard Wiener process $W(t)$ rather than $X(t)$ then $C^{(k)}(t)$, $k = 1, 2$, are random and $z(t,u)$ is progressively measurable.

In what follows we assume that the conditions of Theorem 1 hold and $z(t,u)$ is found as in Theorem 1. Put

$$\Delta(t,u) = z(t,u) - u, \quad M_2(t) = \max_{s\in[0,t]} \xi_s^{(2)}, \quad m_2(t) = \min_{s\in[0,t]} \xi_s^{(2)}.$$

Theorem 2. *Let the conditions of Theorem 1 be fulfilled. If*

$$\inf\{\Delta(t,u), u \in [m_2(t), M_2(t)]\} \geq 0$$

for all $t \geq 0$ then $\xi_t^{(1)} \geq \xi_t^{(2)}$ for all $t \geq 0$.

PROOF. In accord with the conditions of Theorem 2, $\Delta(t,u) \geq 0$ for all $t \geq 0$ and $u \in [m_2(t), M_2(t)]$. By Theorem 1 $\xi_t^{(1)} \geq \xi_t^{(2)}$ for all $t \geq 0$. □

Corollary 1. *Assume that the conditions of Theorem 1 hold and, for all $t \geq 0$, we have*

(a) $\Delta(t, m_2(t)) \geq 0$ *and* $\Delta(t, M_2(t)) \geq 0$;

(b) $\Delta(t,u) \geq 0$ *for all* $u \in \{v : \sigma^{(1)}(t, z(t,v)) = \sigma^{(2)}(t,v)\}$.

Then $\xi_t^{(1)} \geq \xi_t^{(2)}$ for all $t \geq 0$.

PROOF. It suffices to verify that for every $t \geq 0$ the global minimum of $\Delta(t,u)$, $u \in [m_2(t), M_2(t)]$, is nonnegative; in view of condition (a) it suffices to check that the values of $\Delta(t,u)$ are nonnegative at all critical points. Since the necessary condition of an extremum leads to the equality $\sigma^{(1)}(t, z(t,u)) = \sigma^{(2)}(t,u)$, in view of (9), condition (b) yields $\xi_t^{(1)} \geq \xi_t^{(2)}$. □

Examine the case of $\sigma^{(1)}(t,u) = \sigma^{(2)}(t,u)$ for all (t,u).

Theorem 3. *Assume that the conditions of Theorem 1 hold and*

$$\xi_0^{(1)} \geq \xi_0^{(2)}, \quad b^{(1)}(t,u) \geq b^{(2)}(t,u), \quad \sigma^{(1)}(t,u) = \sigma^{(2)}(t,u) \neq 0 \tag{10}$$

for all (t,u). Then $\xi_t^{(1)} \geq \xi_t^{(2)}$ for all $t \geq 0$.

PROOF. Put $\sigma(t,u) = \sigma^{(1)}(t,u) = \sigma^{(2)}(t,u)$. In view of Theorem A, $\xi_t^{(k)} = \phi(t, X(t) + C^{(k)}(t))$ and $C^{(k)}(t)$ are solutions to the ordinary differential equations

$$C^{(k)\prime}(t) = B^{(k)}(t, C^{(k)}(t)),$$

with

$$B^{(k)}(t,y) = \frac{b^{(k)}(t, \phi(t, X(t)+y)) - (\phi)'_t(t,z)|_{z=X(t)+y}}{\sigma(t, \phi(t, X(t)+y))}, \quad k = 1, 2.$$

In this case

$$B^{(1)}(t,y) - B^{(2)}(t,y) = \frac{b^{(1)}(t, \phi(t, X(t)+y)) - b^{(2)}(t, \phi(t, X(t)+y))}{\sigma(t, \phi(t, X(t)+y))}.$$

Conditions (10) yield

$$\phi'_t(t, X(t) + C^{(1)}(t)) \geq \phi'_t(t, X(t) + C^{(2)}(t)). \tag{11}$$

Since $\sigma(t,u) \neq 0$, we have the two cases:

Assume that $\sigma(t,u) > 0$ for all (t,u). In this case

$$\phi'_u(t, X(t) + y) = \sigma(t, \phi(t, X(t) + y)) > 0, \tag{12}$$

and $B^{(1)}(t,y) \geq B^{(2)}(t,y)$. Since $\phi(0, C^{(1)}(0)) \geq \phi(0, C^{(2)}(0))$, we have $C^{(1)}(0) \geq C^{(2)}(0)$ and the comparison theorem for the ordinary differential equations [12, Chapter 1, Section 1.1.2, Theorem 1.1.2] implies that $C^{(1)}(t) \geq C^{(2)}(t)$ for all $t \geq 0$. Hence, (11) and (12) ensure that $\phi(t, X(t) + C^{(1)}(t)) \geq \phi(t, X(t) + C^{(2)}(t))$ which in turn implies that $\xi_t^{(1)} \geq \xi_t^{(2)}$ for $t \geq 0$.

The proof in the case of $\sigma(t,u) < 0$ for all (t,u) is similar. □

REMARK 3. Assume that given are the stochastic versions of (1) and (2) with a Wiener process $X(t) = W(t)$ and coefficients which can be random. Then almost all solutions to these equations satisfy the above statements.

EXAMPLE 1. Consider the two SDEs:

$$d\xi_t^{(1)} = 4\xi_t^{(1)} * dW_t + 3\xi_t^{(1)}\,dt,\ \xi_0^{(1)} = 1, \text{ and } d\xi_t^{(2)} = \frac{1}{1+t} * dW_t - \frac{\xi_t^{(2)}}{1+t}\,dt,\ \xi_0^{(2)} = 0.$$

As is known [13, Chapter 5, Section 5], a solution to the second equation is of the form $\xi_t^{(2)} = W_t/(1+t)$. In this case $z(t,y) = \exp\{4(1+t)y + 3t\}$. Since the conditions of Corollary 1 of Theorem 2 are fulfilled, $\xi_t^{(1)} \geq \xi_t^{(2)}$ for every $t \geq 0$ with probability 1.

3. Demonstrate that the conditions of stability with probability 1 of the SDE can be obtained by comparing the process in question with another already stable process.

EXAMPLE 2. Study stability of a perturbed motion of an SDE of the form

$$d\eta_t = t\eta_t * dW_t - a(t)\eta_t\,dt, \quad t \geq 0, \tag{13}$$

where $a(t)$ is a continuous function. A solution to this equation is representable as

$$\eta_t = \eta_0 \exp\left\{tW_t - \int_0^t \Big(a(s) + W_s\Big)\,ds\right\}$$

and is stable with probability 1 whenever

$$N(\omega) = \sup_{t\geq 0}\left(tW_t - \int_0^t \Big(a(s) + W_s\Big)\,ds\right) < \infty.$$

Let $\frac{1}{s}a(s)$ be integrable on $[0,t]$. Integrating by parts, we find that

$$tW_t - \int_0^t [a(s) + W_s]\,ds = tW_t - \int_0^t \frac{a(s)s}{s}\,ds - \int_0^t W_s\,ds$$

$$= tW_t - t\int_0^t \frac{a(s)}{s}\,ds + \int_0^t\int_0^s \frac{a(\tau)}{\tau}\,d\tau ds - \int_0^t W_s\,ds = -tu(t) + \int_0^t u(s)\,ds,$$

where

$$u(t) = \int_0^t \frac{a(s)}{s}\, ds - W_t.$$

For stability with probability 1 of a perturbed solution to (13), it is necessary to check that

$$u(t) \geq \frac{1}{t} \int_0^t u(s)\, ds - \frac{M_1(\omega)}{t}, \tag{14}$$

where $M_1(\omega)$ is some nonnegative random variable, for t sufficiently large, i.e., for example, for $t \geq t_0(\omega)$. Indeed, put

$$M_2(\omega) = \sup_{t\in[0,t_0(\omega)]} \left(tW_t - \int_0^t (a(s) + W_s)\, ds \right);$$

in result, $N(\omega) \leq M_1(\omega) + M_2(\omega)$.

Let, for instance, $a(t) = t^{\frac{1}{2}+\alpha}$, where $\alpha > 0$. In this case $u(t) = \left(\frac{1}{2} + \alpha\right)^{-1} t^{\frac{1}{2}+\alpha} - W_t$. Inserting $u(t)$ in (14), we see that we need to verify the inequality

$$\frac{2}{3+2\alpha} t^{\frac{1}{2}+\alpha} + \frac{M_1(\omega)}{t} \geq W_t - \frac{1}{t} \int_0^t W_s\, ds. \tag{15}$$

Note that there exists t_1 such that, for $t \geq t_1$, we have

$$2\sqrt{2t \log\log t} + \frac{2}{t} \int_0^t \sqrt{2s \log\log s}\, ds \leq \frac{2t^{\frac{1}{2}+\alpha}}{3+2\alpha}.$$

By the law of the iterated logarithm for a Wiener process, for a.a. ω there exists $t_2(\omega)$ such that $|W_t| \leq 2\sqrt{2t \log\log t}$ for $t > t_2(\omega)$ and so we infer that

$$\left| W_t - \frac{1}{t} \int_{t_2(\omega)}^t W_s\, ds \right| \leq 2\sqrt{2t \log\log t} + \frac{2}{t} \int_{t_2(\omega)}^t \sqrt{2s \log\log s}\, ds + \frac{M_1(\omega)}{t}$$

for $t \geq t_0(\omega) = \max(t_1, t_2(\omega))$, where

$$M_1(\omega) = \int_0^{t_0(\omega)} |W_s|\, ds.$$

Hence, (15) is true. In this case a perturbed solution to (13) for $a(t) = t^{\frac{1}{2}+\alpha}$ is stable with probability 1.

Theorem 4. *Assume that the inequality* $|\xi_t| \leq K \cdot |\eta_t|$ *is valid with probability* 1 *for* $t \geq 0$, *where* ξ_t *is a perturbed solution to* (3) *and* η_t *is a solution stable with probability* 1 *to some* SDE (*for instance, to* (13)), $K = \text{const} > 0$. *Then a perturbed solution to* (3) *is stable with probability* 1.

PROOF. The claim results from the definition of solution stable with probability 1.

Corollary 1. *Assume that* ξ_t *is a perturbed solution to* (3), η_t *is a perturbed nonnegative solution to* (13) *stable with probability* 1, *and both solutions satisfy the conditions of Theorem* 1 *and, hence,* $\xi_t = z(t, \eta_t)$ *and*

$$\sup_{|u|\le\delta, t\ge 0} \left| \frac{1}{tu}\sigma\left(t, \phi\left(t, \frac{1}{t}\left(\log(|u|) - \log(|\eta_0|) + \int_0^t \left(a(s) + W(s)\right) ds\right)\right)\right)\right| \le K$$

for all $t \ge 0$, *where* K *is a positive constant. Then a perturbed solution to* (3) *is stable with probability* 1.

Proof. Theorem 1 and the general form of a solution to (13) imply that

$$\xi_t = z(t, \eta_t) = \phi\left(t, \frac{1}{t}\left(\log|\eta_t| - \log|\eta_0| + \int_0^t \left(a(s) + W(s)\right) ds\right)\right) = \phi\left(t, W(t)\right),$$

where $\phi(t, u)$ is the function used in the construction of a solution to (3). By Theorem 4, for stability with probability 1 of ξ_t it is sufficient that $|\xi_t| \le K\eta_t$ for all $t \ge 0$, where $K = \text{const} > 0$.

From (9) we infer

$$z'_u(t, u) = \frac{1}{tu}\sigma\left(t, \phi\left(t, \frac{1}{t}\left(\log|u| - \log|\eta_0| + \int_0^t \left(a(s) + W_s\right) ds\right)\right)\right).$$

In accord with the assumption of Corollary 1 we have $|z'_u(t, u)| \le K$. The last inequality means that $|\xi_t| \le K\eta_t$ for all $t \ge 0$. □

Corollary 2. *Let* ξ_t *be a perturbed solution to the equation*

$$d\xi_t = t\xi_t * dW(t) + b(t, \xi_t)\, dt,$$

where $b(t, y)$ *is an odd function in* y *for all* $t \ge 0$, η_t *is a perturbed nonnegative solution to* (13) *stable with probability* 1, *and the assumptions of Theorem* 1 *are fulfilled for them. The perturbed solution* ξ_t *is stable with probability* 1 *provided that for all* $t \ge 0$ *we have*

(a) $|\xi_0| \le \eta_0$,

(b) $b(t, y) \le -a(t)y$ *for* $y < 0$.

Proof. By Theorem 4, we need to verify the fulfilment of the condition

$$|\xi_t| \le K \cdot \eta_t. \tag{16}$$

To this end, in accord with Theorem 3 and condition (a), it suffices to check that

$$b(t, y) \le -a(t)y, \quad y > 0, \tag{17}$$

$$b(t, y) \ge a(t)y, \quad y < 0. \tag{18}$$

Inequality (17) is fulfilled because of condition (b) which in view of oddness of $b(t, y)$ validates (18). □

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