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COUNTABLY CATEGORICAL WEAKLY O-MINIMAL STRUCTURES OF FINITE CONVEXITY RANK

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Abstract: We completely describe countably categorical weakly o-minimal theories of finite convexity rank.

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1. Preliminaries

Denote a countable first-order language by L. We consider L-structures and assume throughout that L contains the binary relation symbol $<$ that is interpreted as a linear order on these structures. This article addresses the concept of *weak o-minimality* which was firstly perused in [1]. A subset A of a linearly ordered structure M is called *convex* whenever $a < c < b$ for all $a, b \in A$ and $c \in M$ implies that $c \in A$. Refer as a *weakly o-minimal structure* to a linearly ordered structure $M = \langle M, =, <, \dots \rangle$ such that every definable (with parameters) subset of M is a union of finitely many convex subsets of M.

Given two subsets A and B of a linearly ordered structure M, the record $A < B$ means that $a < b$ for all $a \in A$ and $b \in B$. The record $A < b$ means that $A < \{b\}$. Denote by A^+ and A^- the sets of elements b satisfying $A < b$ and $b < A$ respectively.

Definition 1.1 [2]. Consider a weakly o-minimal theory T and a sufficiently saturated model M of T. Take an M-definable formula $\phi(x)$ with one free variable.

The *convexity rank* $RC(\phi(x))$ of $\phi(x)$ is defined as follows:

(1) $RC(\phi(x)) \geq 1$ whenever $\phi(M)$ is infinite;

(2) $RC(\phi(x)) \geq \alpha + 1$ whenever there exist a parametrically definable equivalence $E(x, y)$ and infinitely many elements b_i, for $i \in \omega$, such that

- $M \models \neg E(b_i, b_j)$ for all $i, j \in \omega$ with $i \neq j$;
- $RC(E(x, b_i)) \geq \alpha$ and $E(M, b_i)$ is a convex subset of $\phi(M)$ for each $i \in \omega$;

(3) $RC(\phi(x)) \geq \delta$ whenever $RC(\phi(x)) \geq \alpha$ for all $\alpha < \delta$, where δ is a limit ordinal.

If $RC(\phi(x)) = \alpha$ for some α then say that $RC(\phi(x))$ is *defined*. If $RC(\phi(x)) \geq \alpha$ for all α, then put $RC(\phi(x)) = \infty$.

Definition 1.2 [3]. Given a weakly o-minimal structure M, take $A, B \subseteq M$ such that M is $|A|^+$-saturated and take nonalgebraic types $p, q \in S_1(A)$. Say that p is not *weakly orthogonal* to q and write $p \not\perp^w q$ whenever there exist an A-definable formula $H(x, y)$, $\alpha \in p(M)$, and $\beta_1, \beta_2 \in q(M)$ such that $\beta_1 \in H(M, \alpha)$ and $\beta_2 \notin H(M, \alpha)$.

Lemma 1.3 [3, Corollary 34(iii)]. *The relation* $\not\perp^w$ *is an equivalence on* $S_1(A)$.

Recall some concepts from [1]. Denote by $\pi : M^{n+1} \to M^n$ the projection forgetting the last coordinate. Take an $\varnothing$-definable set $Y \subset M^{n+1}$ and put $Z := \pi(Y)$. Given $\bar{a} \in Z$, put $Y_{\bar{a}} := \{y : (\bar{a}, y) \in Y\}$. Suppose that $Y_{\bar{a}}$ is bounded above but has no supremum in M for each $\bar{a} \in Z$. Denote by $\sim$ the $\varnothing$-definable equivalence on M^n that is defined as

$$\bar{a} \sim \bar{b} \text{ for all } \bar{a}, \bar{b} \in M^n \setminus Z, \quad \bar{a} \sim \bar{b} \Leftrightarrow \sup Y_{\bar{a}} = \sup Y_{\bar{b}}, \text{ if } \bar{a}, \bar{b} \in Z.$$

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Put $\overline{Z} := Z/\sim$. Given a tuple $\bar{a} \in Z$, denote by $[\bar{a}]$ the $\sim$-coset of $\bar{a}$. There exists a natural $\varnothing$-definable linear order on $M \cup \overline{Z}$ defined as follows. Given $\bar{a} \in Z$ and $c \in M$, we have $[\bar{a}] < c$ if and only if $w < c$ for all $w \in Y_{\bar{a}}$. If $\bar{a} \not\sim \bar{b}$, then there exists $x \in M$ with $[\bar{a}] < x < [\bar{b}]$ or $[\bar{b}] < x < [\bar{a}]$; therefore, $<$ induces a linear order on $M \cup \overline{Z}$. Call $\overline{Z}$ a *sort*, in this case an *$\varnothing$-definable sort*, in $\overline{M}$, which is the Dedekind completion of M, and assume that $\overline{Z}$ is embedded naturally into $\overline{M}$. Similarly we can obtain a sort in $\overline{M}$ by considering infima instead of suprema.

Definition 1.4 [1]. Given a linearly ordered structure M, take an infinite $D \subseteq M$ and an arbitrary $K \subseteq \overline{M}$. Say that a function $f : D \to K$ is *locally increasing* (*locally decreasing or locally constant*) on D provided that for every $x \in D$ there exists an infinite interval $J \subseteq D$ containing x such that f is strictly increasing (strictly decreasing or constant) on J.

Say also that a function f is *locally monotone* on $D \subseteq M$ whenever f is either locally increasing or locally decreasing on D.

Proposition 1.5 [4]. *Given a weakly o-minimal structure M, take $A \subseteq M$ and a nonalgebraic type $p \in S_1(A)$. Then every function into an A-definable sort whose domain includes $p(M)$ is locally monotone or locally constant on $p(M)$.*

Given an A-definable function f on $D \subseteq M$ and an A-definable equivalence E on D, say that f is *strictly increasing* (or *decreasing*) *on D/E* whenever for all $a, b \in D$ with $a < b \wedge \neg E(a, b)$ we have $f(a) < f(b)$ (or $f(a) > f(b)$).

Definition 1.6 [5, 6]. Given a weakly o-minimal structure M, take $B, D \subseteq M$, a B-definable sort $A \subseteq \overline{M}$, and a B-definable function $f : D \to A$ locally increasing (decreasing) on D. Say that *f has depth n* on D whenever there are equivalences $E_1(x, y), \dots, E_n(x, y)$ partitioning D into infinitely many infinite convex cosets such that for all $2 \leq i \leq n$ each coset of E_i is subdivided into infinitely many infinite convex subcosets of E_{i-1} and the following hold:

• f is strictly increasing (decreasing) on each coset of E_1;

• f is locally decreasing (increasing) on D/E_k for every odd $k \leq n$ or, which is the same, f is strictly decreasing (increasing) on each $E_{k+1}(a, M)/E_k$ for all $a \in D$;

• f is locally increasing (decreasing) on D/E_k for every even $k \leq n$;

• f is strictly monotone on D/E_n.

In this case call f a *locally increasing* (*decreasing*) *function of depth n*.

Obviously, each strictly increasing (decreasing) function is a locally increasing (decreasing) function of depth 0.

Theorem 1.7 [6]. *If T is a weakly o-minimal theory then every function into a definable sort has finite depth.*

Extend Definition 1.6 in a natural way by introducing the concept of *locally constant function of depth n* when in this definition f is a constant function on each coset of E_1. Observe that in this case f can be locally increasing or locally decreasing on D/E_1. In the three examples below f is locally constant.

Example 1.8 [1]. Take a linearly ordered structure $M = \langle M, <, P_1^1, P_2^1, f^1 \rangle$ that is the disjoint union of interpretations of unary predicates P_1 and P_2, and in addition $P_1(M) < P_2(M)$. Identify the interpretation of P_2 with $\mathbb{Q}$, the rationals, with the standard order and that of P_1 with $\mathbb{Q} \times \mathbb{Q}$ ordered lexicographically. Interpret f as the partial unary function with $\mathrm{Dom}(f) = P_1(M)$ and $\mathrm{Range}(f) = P_2(M)$ defined by the rule $f((n, m)) = n$ for all $(n, m) \in \mathbb{Q} \times \mathbb{Q}$.

We can show that M is a countably categorical weakly o-minimal structure. Put $p := \{P_1(x)\}$ and $q := \{P_2(x)\}$. It is obvious that $p, q \in S_1(\varnothing)$. Given $a \in p(M)$, there exists a unique $b \in q(M)$ with $f(a) = b$, i.e. $b \in \mathrm{dcl}(\{a\})$.

Consider the formula $E(x, y) := P_1(x) \wedge P_1(y) \wedge \exists z\, [P_2(z) \wedge f(x) = z \wedge f(y) = z]$. It is an $\varnothing$-definable equivalence partitioning $p(M)$ into infinitely many infinite convex cosets.

We assert that f is locally constant of depth 1 on $P_1(M)$; i.e., f is constant on each coset of E and strictly increasing on $P_1(M)/E$.

Example 1.9. Consider a linearly ordered structure $M = \langle M, <, P_1^1, P_2^1, E_1^p, E_2^p, E_1^q, f^1 \rangle$ which is the disjoint union of interpretations of some unary predicates P_1 and P_2, and in addition $P_1(M) < P_2(M)$. Identify the interpretation of P_1 with $\mathbb{Q} \times \mathbb{Q} \times \mathbb{Q}$ ordered lexicographically, and that of P_2 with $\mathbb{Q} \times \mathbb{Q}$ also ordered lexicographically. The interpretations of the binary predicates $E_1^p(x, y)$ and $E_2^p(x, y)$ are the equivalences on $P_1(M)$ such that $E_1^p(x, y) \Leftrightarrow n_1 = n_2 \wedge m_1 = m_2$ and $E_2^p(x, y) \Leftrightarrow n_1 = n_2$ for all $x = (n_1, m_1, l_1)$ and $y = (n_2, m_2, l_2) \in \mathbb{Q} \times \mathbb{Q} \times \mathbb{Q}$. Similarly we define an interpretation of the binary predicate $E_1^q(x, y)$: it is an equivalence on $P_2(M)$ such that $E_1^q(x, y) \Leftrightarrow n_1 = n_2$ for all $x = (n_1, m_1)$ and $y = (n_2, m_2) \in \mathbb{Q} \times \mathbb{Q}$. Interpret f as a partial unary function with $\mathrm{Dom}(f) = P_1(M)$ and $\mathrm{Range}(f) = P_2(M)$ defined by the rule $f((n, m, l)) = (-n, m)$ for all $(n, m, l) \in \mathbb{Q} \times \mathbb{Q} \times \mathbb{Q}$.

We can show that M is a countably categorical weakly o-minimal structure. Put $p := \{P_1(x)\}$ and $q := \{P_2(x)\}$. It is obvious that $p, q \in S_1(\varnothing)$.

We assert that f is a locally constant function of depth 2 on $P_1(M)$, i.e., f is constant on each coset of E_1^p, strictly increasing on $E_2(a, M)/E_1$ for all $a \in P_1(M)$, and strictly decreasing on $P_1(M)/E_2$.

In Example 1.9, defining f as $f((n, m, l)) = (n, -m)$ for all $(n, m, l) \in \mathbb{Q} \times \mathbb{Q} \times \mathbb{Q}$, we obtain a locally constant function f of depth 2 on $P_1(M)$; furthermore, f is constant on each coset of E_1^p, strictly decreasing on $E_2(a, M)/E_1$ for all $a \in P_1(M)$, and strictly increasing on $P_1(M)/E_2$.

In Example 1.9, putting $f((n, m, l)) = (-n, -m)$ for all $(n, m, l) \in \mathbb{Q} \times \mathbb{Q} \times \mathbb{Q}$, we obtain a locally constant function f of depth 1 on $P_1(M)$; furthermore, f is constant on each coset of E_1^p and strictly decreasing on $P_1(M)/E_1$.

Example 1.10. Consider the linearly ordered structure

$$M = \langle M, <, P_1^1, P_2^1, E_1^p, E_2^p, \dots, E_{n-1}^p, E_1^q, E_2^q, \dots, E_{k-1}^q, f^1 \rangle,$$

where $k < n$, which is the disjoint union of interpretations of unary predicates P_1 and P_2, and in addition $P_1(M) < P_2(M)$. Identify the interpretation of P_1 with $\mathbb{Q}^n$ ordered lexicographically, and that of P_2 with $\mathbb{Q}^k$ also ordered lexicographically. The interpretations of the binary predicates $E_1^p(x, y)$, $\dots$, $E_{n-1}^p(x, y)$ are the equivalences on $P_1(M)$ such that $E_i^p(x, y) \Leftrightarrow x_1 = y_1 \wedge \dots \wedge x_{n-i} = y_{n-i}$ for all $x = (x_1, x_2, \dots, x_n)$, $y = (y_1, y_2, \dots, y_n) \in \mathbb{Q}^n$ and all $1 \le i \le n-1$. Similarly we define the interpretations of the binary predicates $E_1^q(x, y), \dots, E_{k-1}^q(x, y)$: they are the equivalences on $P_2(M)$ such that $E_i^q(x, y) \Leftrightarrow x_1 = y_1 \wedge \dots \wedge x_{k-i} = y_{k-i}$ for all $x = (x_1, x_2, \dots, x_k)$, $y = (y_1, y_2, \dots, y_k) \in \mathbb{Q}^k$ and all $1 \le i \le k-1$. Interpret f as a partial unary function with $\mathrm{Dom}(f) = P_1(M)$ and $\mathrm{Range}(f) = P_2(M)$ defined as

$$f((x_1, x_2, \dots, x_n)) = ((-1)^{k-1} x_1, (-1)^{k-2} x_2, \dots, (-1)^2 x_{k-2}, (-1)^1 x_{k-1}, x_k)$$

for all $(x_1, x_2, \dots, x_n) \in \mathbb{Q}^n$.

It is obvious that $E_1^p(a, M) \subset E_2^p(a, M) \subset \dots \subset E_{n-1}^p(a, M)$ for all $a \in P_1(M)$ and $E_1^q(b, M) \subset E_2^q(b, M) \subset \dots \subset E_{k-1}^q(b, M)$ for all $b \in P_2(M)$.

We assert that f is a locally constant function of depth k on $P_1(M)$; i.e., f is constant on each coset of E_{n-k}^p, strictly increasing on each $E_{n-k+1}^p(a, M)/E_{n-k}^p$, and strictly decreasing on each $E_{n-k+2}^p(a, M)/E_{n-k+1}^p, \dots$. Finally, if $n-k$ is odd then f is strictly decreasing on $P_1(M)/E_{n-1}^p$, and if $n-k$ is even then f is strictly increasing on $P_1(M)/E_{n-1}^p$.

Definition 1.11 [7]. Given a weakly o-minimal structure M, take $A \subseteq M$ and a nonalgebraic type $p \in S_1(A)$.

(1) An A-definable formula $F(x, y)$ is called *p-stable* whenever there exist $\alpha, \gamma_1, \gamma_2 \in p(M)$ with $F(M, \alpha) \setminus \{\alpha\} \ne \varnothing$ and $\gamma_1 < F(M, \alpha) < \gamma_2$.

(2) A p-stable formula $F(x, y)$ is called *right-convex* (*left-convex*) whenever there exists $\alpha \in p(M)$ such that $F(M, \alpha)$ is convex, α is the left (right) endpoint of $F(M, \alpha)$, and $\alpha \in F(M, \alpha)$.

Given two p-stable right-convex (left-convex) formulas $F_1(x, y)$ and $F_2(x, y)$, say that $F_2(x, y)$ is *greater than* $F_1(x, y)$ whenever there exists $\alpha \in p(M)$ such that $F_1(M, \alpha) \subset F_2(M, \alpha)$.

Definition 1.12 [8]. Call a p-stable right-convex (left-convex) formula $F(x,y)$ *equivalence-generating* whenever for all $\alpha,\beta \in p(M)$ with $M \models F(\beta,\alpha)$ we have

$$M \models \forall x\,[x \geq \beta \to [F(x,\alpha) \leftrightarrow F(x,\beta)]] \quad (M \models \forall x\,[x \leq \beta \to [F(x,\alpha) \leftrightarrow F(x,\beta)]]).$$

Example 1.13. Consider the linearly ordered structure $M = \langle \mathbb{Q}, =, <, R^2 \rangle$, where $\mathbb{Q}$ is as before, $M \models R(b,a) \Leftrightarrow a \leq b < a + \sqrt{2}$ for all $a, b \in M$, and so $R(M,a) = \{b \in M \mid a \leq b < a + \sqrt{2}\}$ and $R(a,M) = \{b \in M \mid a - \sqrt{2} < b \leq a\}$.

We can show that M is a weakly o-minimal structure. We can also see that $R(x,y)$ is p-stable and right-convex, but not equivalence-generating.

Lemma 1.14 [8]. *Given a weakly o-minimal structure M take $A \subseteq M$ such that the structure M is $|A|^+$-saturated and a nonalgebraic type $p \in S_1(A)$. Suppose that $F(x,y)$ is a p-stable right-convex formula, so that $F(x,y)$ is equivalence-generating. Then*
(1) *$G(x,y) := F(y,x)$ is a p-stable left-convex formula which is also equivalence-generating.*
(2) *$E(x,y) := F(x,y) \vee F(y,x)$ is an equivalence partitioning $p(M)$ into convex cosets.*

Theorem 1.15 [8]. *Given a countably categorical weakly o-minimal theory T and $M \models T$, take $A \subseteq M$ and a nonalgebraic type $p \in S_1(A)$. Then every p-stable right-convex (left-convex) formula is equivalence-generating.*

Corollary 1.16 [9, 8]. *Given a countably categorical weakly o-minimal theory T and $M \models T$, take a nonalgebraic type $p \in S_1(\varnothing)$. Consider the complete list $\{F_1(x,y), \dots, F_m(x,y)\}$ of all p-stable right-convex formulas such that $F_1(M,\alpha) \subset \dots \subset F_m(M,\alpha)$ for every $\alpha \in p(M)$. Then all $\varnothing$-definable equivalences with infinite convex cosets on $p(M)$ are precisely E_i for $1 \leq i \leq m$ defined as $E_i(x,y) := F_i(x,y) \vee F_i(y,x)$, and so the following hold:*
• *E_m partitions $p(M)$ into infinitely many cosets of E_m, each of which is convex and open, so that the induced order on the cosets is a dense order without endpoints;*
• *for each $i \in \{1, \dots, m-1\}$, E_i partitions every coset of E_{i+1} into infinitely many cosets of E_i, each of which is convex and open, so that the subcosets of E_i in all cosets of E_{i+1} are densely ordered without endpoints.*

Recall that a complete theory T is called *binary* whenever every formula is equivalent to a Boolean combination of formulas with at most two free variables.

Theorem 1.17 [10]. *If T is a countably categorical weakly o-minimal theory then T is binary if and only if T is of finite convexity rank.*

2. The Main Theorem

Definition 2.1 [10]. Refer as the *convexity rank of* 1-*type* p to $RC(p) := \inf\{RC(\phi(x)) \mid \phi(x) \in p\}$.

In Example 1.8 we have $p \not\perp^w q$, $\operatorname{dcl}(\{a\}) \cap q(M) \neq \varnothing$, and $\operatorname{dcl}(\{b\}) \cap p(M) = \varnothing$ for some $a \in p(M)$ and $b \in q(M)$ with $RC(p) = 2$ and $RC(q) = 1$.

Throughout this section we consider only countably categorical weakly o-minimal theories of finite convexity rank. Denote by n_p the convexity rank of type p, i.e., $RC(p)$, since Theorem 1.17 yields $RC(p) < \omega$ for every nonalgebraic type $p \in S_1(\varnothing)$.

Proposition 2.2. *Given a countably categorical weakly o-minimal theory T of finite convexity rank and $M \models T$, take nonalgebraic types $p, q \in S_1(\varnothing)$ and suppose that $\operatorname{dcl}(\{a\}) \cap q(M) \neq \varnothing$ for some $a \in p(M)$. Then the following are equivalent:*
(1) *$RC(p) > RC(q)$;*
(2) *there is no $\varnothing$-definable bijection $f : p(M) \to q(M)$;*
(3) *$\operatorname{dcl}(\{b\}) \cap p(M) = \varnothing$ for every $b \in q(M)$;*
(4) *there is an $\varnothing$-definable function $f : p(M) \to q(M)$ locally constant on $p(M)$.*

Proof of Proposition 2.2. (1) ⇒ (2) Assume on the contrary that there is an ∅-definable function $f : p(M) \to q(M)$ presenting a bijection between $p(M)$ and $q(M)$. Since $RC(p) = n_p$, there exist ∅-definable equivalences $E_1(x, y), \dots, E_{n_p-1}(x, y)$ partitioning $p(M)$ into infinitely many infinite convex cosets such that $E_1(a, M) \subset \dots \subset E_{n_p-1}(a, M)$ for some (arbitrary) $a \in p(M)$. For each $1 \le i \le n_p - 1$ consider the formula

$$E'_i(x, y) := U_q(x) \wedge U_q(y) \wedge \exists t_1 \exists t_2 [U_p(t_1) \wedge U_p(t_2) \wedge E_i(t_1, t_2) \wedge f(t_1) = x \wedge f(t_2) = y].$$

It is obvious that $E'_1(x, y), \dots, E'_{n_p-1}(x, y)$ are equivalences partitioning $q(M)$ into infinitely many infinite convex cosets and $E'_1(b, M) \subset \dots \subset E'_{n_p-1}(b, M)$, whence $RC(q) \ge n_p$; which contradicts the assumption.

(2) ⇒ (3) Since $\mathrm{dcl}(\{a\}) \cap q(M) \neq \varnothing$, there are $b \in q(M)$ and an ∅-definable formula $\phi(x, y)$ such that $M \models \exists! y \phi(a, y) \wedge \phi(a, b)$. Assume on the contrary that $\mathrm{dcl}(\{b\}) \cap p(M) \neq \varnothing$. Verify that $a \in \mathrm{dcl}(\{b\})$. Otherwise there is $a_1 \in p(M)$ such that $a_1 \neq a$ and $a_1 \in \mathrm{dcl}(\{b\})$. However, then $b \in \mathrm{dcl}(\{a\})$ yields $a_1 \in \mathrm{dcl}(\{a\})$. We can show that $\mathrm{dcl}(\{a\})$ is infinite; this contradicts the countable categoricity. Therefore, $a \in \mathrm{dcl}(\{b\})$. Thus, there exists an ∅-definable formula $\phi'(x, y)$ such that $M \models \exists! y \phi'(a, y) \wedge \exists! x \phi'(x, b) \wedge \phi'(a, b)$.

Define the function f as $f(a) = b \Leftrightarrow \phi'(a, b)$. It is not difficult to understand that f maps $p(M)$ bijectively onto $q(M)$; this contradicts the assumption.

(3) ⇒ (4) Assume on the contrary that $f : p(M) \to q(M)$ is an ∅-definable function which is not locally constant on $p(M)$. Then f must be locally monotone on $p(M)$, i.e. either locally increasing or locally decreasing. But in this event f maps $p(M)$ bijectively onto $q(M)$; this contradicts the above.

(4) ⇒ (1) Take an ∅-definable function $f : p(M) \to q(M)$ locally constant on $p(M)$. Consider the formula

$$E(x, y) := U_q(x) \wedge U_q(y) \wedge [x < y \to \forall t(x < t < y \to f(x) = f(t) = f(y))]$$
$$\wedge [x > y \to \forall t(x > t > y \to f(x) = f(t) = f(y))].$$

It is not difficult to understand that $E(x, y)$ is an equivalence partitioning $p(M)$ into infinitely many infinite convex cosets.

Since $RC(p) = n_p$, there exist ∅-definable equivalences $E_1(x, y), \dots, E_{n_p-1}(x, y)$ partitioning $p(M)$ into infinitely many infinite convex cosets such that $E_1(a, M) \subset \dots \subset E_{n_p-1}(a, M)$ for some (arbitrary) $a \in p(M)$.

It is obvious that $E(x, y) \equiv E_i(x, y)$ for some $1 \le i \le n_p - 1$. Then we assert that $RC(q) = n_p - i$. Indeed, f is constant on each coset of E_i. Consider the behavior of f on each $E_{i+1}(a, M)/E_i$, where $a \in p(M)$. It must be strictly monotone on each $E_{i+1}(a, M)/E_i$, otherwise there appears an ∅-definable equivalence $\overline{E}(x, y)$ with $E_i(a, M) \subset \overline{E}(a, M) \subset E_{i+1}(a, M)$; this contradicts the property that E_{i+1} directly succeeds the relation $E_i(x, y)$ in the set of all ∅-definable relations on $p(M)$. Similarly we show that f is strictly monotone on each $E_{k+1}(a, M)/E_k$, where $i \le k \le n_p - 2$, and that f is strictly monotone on $p(M)/E_{n_p-1}$. Given each $i + 1 \le j \le n_p - 1$, consider the formula

$$E'_j(x, y) := U_q(x) \wedge U_q(y) \wedge \exists t_1 \exists t_2 [U_p(t_1) \wedge U_p(t_2) \wedge E_j(t_1, t_2) \wedge f(t_1) = x \wedge f(t_2) = y].$$

It is obvious that $E'_{i+1}(x, y), \dots, E'_{n_p-1}(x, y)$ are equivalences partitioning $q(M)$ into infinitely many infinite convex cosets and that $E'_{i+1}(b, M) \subset \dots \subset E'_{n_p-1}(b, M)$. Hence, $RC(q) \ge n_p - i$. Moreover, if there exists an ∅-definable equivalence $E^q(x, y)$ partitioning $q(M)$ into infinitely many infinite convex cosets and, furthermore, $E^q(b, M) \subset E'_{i+1}(b, M)$, then consider the formula

$$\widehat{E}(x, y) := U_p(x) \wedge U_p(y) \wedge \exists t_1 \exists t_2 [E^q(t_1, t_2) \wedge f(x) = t_1 \wedge f(y) = t_2].$$

It is obvious that $E_i(a, M) \subset \widehat{E}(a, M) \subset E_{i+1}(a, M)$; once again, this contradicts the property that E_{i+1} directly succeeds $E_i(x, y)$ in the set of all ∅-definable relations on $p(M)$. Similarly we show that there exists no ∅-definable equivalence $E^q(x, y)$ partitioning $q(M)$ into infinitely many infinite convex cosets and, furthermore, $E'_k(b, M) \subset E^q(b, M) \subset E'_{k+1}(b, M)$ for all $i + 1 \le k \le n_p - 2$ or $E'_{n_p-1}(b, M) \subset E^q(b, M)$. Thus, $RC(q) = n_p - i$. □

Corollary 2.3. *Given a countably categorical weakly o-minimal theory T of finite convexity rank and $M \models T$, take nonalgebraic types $p, q \in S_1(\varnothing)$ and suppose that $\mathrm{dcl}(\{a\}) \cap q(M) \neq \varnothing$ for some $a \in p(M)$. The following hold:*

(1) If $RC(p) = RC(q)$ then there exists a unique $\varnothing$-definable locally monotone bijection $f : p(M) \to q(M)$ of depth k for some $0 \leq k \leq n_p - 1$.

(2) If $RC(p) > RC(q)$ then there is a unique $\varnothing$-definable function $f : p(M) \to q(M)$ locally constant and having depth k for some $1 \leq k \leq n_q$.

Proof of Corollary 2.3. (1) By Proposition 2.2, there is an $\varnothing$-definable bijection $f : p(M) \to q(M)$, which must be locally monotone on $p(M)$ by Proposition 1.5. Since $RC(p) = n_p$, it follows that f is of depth k for some $0 \leq k \leq n_p - 1$. Verify that f is unique. Assume on the contrary that there exists an $\varnothing$-definable function g with $g(a) \neq f(a)$ for some $a \in p(M)$. Then $f(a) = b$ and $g(a) = b_1$ for some $b, b_1 \in q(M)$. Consider the formula $\phi(b, y) := \exists x[f(x) = b \wedge g(x) = y]$. Obviously, $M \models \exists! \phi(b, y) \wedge \phi(b, b_1)$, i.e., $b_1 \in \mathrm{dcl}(\{b\})$. This implies that $\mathrm{dcl}(\{b\})$ is infinite; a contradiction with the countable categoricity of T.

(2) By Proposition 2.2, there is an $\varnothing$-definable function $f : p(M) \to q(M)$ locally constant on $p(M)$. Then f is constant on each coset of $E^p_{n_p - n_q}$, locally monotone on $p(M)/E^p_{n_p - n_q}$, and of depth k for some $1 \leq k \leq n_q$. Assume on the contrary that there exists an $\varnothing$-definable function $g : p(M) \to q(M)$ distinct from f. Then there is $a \in p(M)$ with $f(a) \neq g(a)$. The function g could be bijective, which would force $RC(p) = RC(q)$. Thus, it must be locally constant on $p(M)$ and, furthermore, constant on each coset of $E^p_{n_p - n_q}$. Considering the formula $\phi(b, y)$ from the proof of claim (1), we see that $\mathrm{dcl}(\{b\})$ is infinite; this contradicts the countable categoricity of T. $\square$

Below we need the concept of (p_1, p_2)-splitting formula introduced in [10]. Take $A \subseteq M$ and nonalgebraic types $p_1, p_2 \in S_1(A)$ with $p_1 \not\perp^w p_2$. Refer to an A-definable formula $\phi(x, y)$ as a (p_1, p_2)-*splitting formula* if there exists $a \in p_1(M)$ such that $\phi(a, M)$ is convex, $\phi(a, M) \subset p_2(M)$, and $\phi(a, M)^- = p_2(M)^-$. Given two (p_1, p_2)-splitting formulas $\phi_1(x, y)$ and $\phi_2(x, y)$, say that $\phi_1(x, y)$ is *smaller* than $\phi_2(x, y)$ if there exists $a \in p_1(M)$ with $\phi_1(a, M) \subset \phi_2(a, M)$. Note that if $p_1, p_2 \in S_1(A)$ are nonalgebraic types with $p_1 \not\perp^w p_2$ then a (p_1, p_2)-splitting formula exists and the set of all (p_1, p_2)-splitting formulas is linearly ordered. It is also obvious that for every (p_1, p_2)-splitting formula $\phi(x, y)$ the function $f(x) := \sup \phi(x, M)$ is not constant on $p_1(M)$.

Proposition 2.4. *For a countably categorical weakly o-minimal theory T of finite convexity rank take nonalgebraic types $p, q \in S_1(\varnothing)$ with $p \not\perp^w q$. Then $RC(p) > RC(q)$ if and only if for every (p, q)-splitting formula $R(x, y)$ there is an $\varnothing$-definable equivalence $E(x, y)$ partitioning $p(M)$ into infinitely many infinite convex cosets such that $f(x) := \sup R(x, M)$ is constant on each coset of E.*

Proof of Proposition 2.4. Since $RC(p) = n_p$, there are $\varnothing$-definable equivalences $E_1(x, y), \ldots, E_{n_p - 1}(x, y)$ partitioning $p(M)$ into infinitely many infinite convex cosets such that $E_1(a, M) \subset \cdots \subset E_{n_p - 1}(a, M)$ for whatever $a \in p(M)$.

($\Rightarrow$) Suppose that $RC(p) > RC(q)$. Assume on the contrary that there is a (p, q)-splitting formula $R(x, y)$ such that for every $\varnothing$-definable equivalence $E(x, y)$ partitioning $p(M)$ into infinitely many infinite convex cosets the function $f(x) := \sup R(x, M)$ is not constant on each coset of E. Then f is not constant on each coset of E_1. Therefore, it must be strictly monotone (strictly increasing or strictly decreasing) on each coset of E_1. Indeed, f cannot be locally monotone (not strictly monotone) on each coset of E_1, as otherwise there would appear an $\varnothing$-definable equivalence $E_0(x, y)$ partitioning $p(M)$ into infinitely many infinite convex cosets such that $E_0(a, M) \subset E_1(a, M)$ for some (arbitrary) $a \in p(M)$; this contradicts the property that $E_1(x, y)$ is minimal among the $\varnothing$-definable nontrivial equivalences on $p(M)$.

Consider the behavior of f on each $E_2(a, M)/E_1$ for $a \in p(M)$. It must be strictly monotone on each $E_2(a, M)/E_1$, otherwise there would appear an $\varnothing$-definable equivalence $\overline{E}(x, y)$ such that $E_1(a, M) \subset \overline{E}(a, M) \subset E_2(a, M)$ although E_2 directly succeeds the relation $E_1(x, y)$ in the set of all $\varnothing$-definable relations on $p(M)$. Similarly we show that f is strictly monotone on each $E_{k+1}(a, M)/E_k$, where $1 \leq$

$k \leq n_p - 2$, and strictly monotone on $p(M)/E_{n_p-1}$. For each $1 \leq i \leq n_p - 1$ consider the formula

$$E'_i(x,y) := [x \leq y \to \exists t_1 \exists t_2 (E_i(t_1,t_2) \wedge f(t_1) < x \leq y < f(t_2))]$$
$$\wedge [x > y \to \exists t_1 \exists t_2 (E_1(t_1,t_2) \wedge f(t_1) < y < x < f(t_2))].$$

It can be seen that $E'_1(x,y), \dots, E'_{n_p-1}(x,y)$ are equivalences partitioning $q(M)$ into infinitely many infinite convex cosets and, furthermore, $E'_1(b,M) \subset \dots \subset E'_{n_p-1}(b,M)$. Hence, $RC(q) \geq n_p$, which contradicts the assumption.

($\Leftarrow$) Suppose that for every (p,q)-splitting formula $R(x,y)$ there is an $\varnothing$-definable equivalence $E(x,y)$ partitioning $p(M)$ into infinitely many infinite convex cosets such that $f(x) := \sup R(x,M)$ is constant on each coset of E. Verify that $RC(p) > RC(q)$. Take an arbitrary (p,q)-splitting formula $R(x,y)$. By assumption, there is an $\varnothing$-definable equivalence $E(x,y)$ partitioning $p(M)$ into infinitely many infinite convex cosets such that $f(x) := \sup R(x,M)$ is constant on each coset of E. Suppose that $E(x,y)$ is maximal with this property. It is obvious that $E(x,y) \equiv E_i(x,y)$ for some $1 \leq i \leq n_p - 1$. Consider now the behavior of f on each $E_{i+1}(a,M)/E_i$ for $a \in p(M)$. The function f could be constant on each $E_{i+1}(a,M)/E_i$, as otherwise it would be constant on each coset of E_{i+1}, while $E_i(x,y)$ is maximal with this property. Consequently, f must be strictly monotone on each $E_{i+1}(a,M)/E_i$; otherwise, were it locally monotone (not strictly monotone) on each $E_{i+1}(a,M)/E_i$ there would appear an $\varnothing$-definable equivalence $\overline{E}(x,y)$ such that $E_i(a,M) \subset \overline{E}(a,M) \subset E_{i+1}(a,M)$; this contradicts the property that E_{i+1} directly succeeds $E_i(x,y)$ in the set of all $\varnothing$-definable relations on $p(M)$. Similarly we show that f is strictly monotone on each $E_{k+1}(a,M)/E_k$, where $i \leq k \leq n_p - 2$, and strictly monotone on $p(M)/E_{n_p-1}$. Given $i + 1 \leq j \leq n_p - 1$, consider the formula

$$E'_j(x,y) := U_q(x) \wedge U_q(y) \wedge \exists t_1 \exists t_2 [U_p(t_1) \wedge U_p(t_2) \wedge E_j(t_1,t_2)$$
$$\wedge f(t_1) < x < f(t_2) \wedge f(t_1) < y < f(t_2)].$$

Note that $E'_{i+1}(x,y), \dots, E'_{n_p-1}(x,y)$ is an equivalence partitioning $q(M)$ into infinitely many infinite convex cosets and, furthermore, $E'_{i+1}(b,M) \subset \dots \subset E'_{n_p-1}(b,M)$. Hence, $RC(q) \geq n_p - i$. If there is an $\varnothing$-definable equivalence $E^q(x,y)$ partitioning $q(M)$ into infinitely many infinite convex cosets and, furthermore, $E^q(b,M) \subset E'_{i+1}(b,M)$, then consider the formula

$$\widehat{E}(x,y) := U_p(x) \wedge U_p(y) \wedge \exists t_1 \exists t_2 [E^q(t_1,t_2) \wedge f(x) < t_1 < f(y) \wedge f(x) < t_2 < f(y)].$$

It is obvious that $E_i(a,M) \subset \widehat{E}(a,M) \subset E_{i+1}(a,M)$; once again this contradicts the property that E_{i+1} directly succeeds the relation $E_i(x,y)$ in the set of all $\varnothing$-definable relations on $p(M)$. Similarly we show that there exists no $\varnothing$-definable equivalence $E^q(x,y)$ partitioning $q(M)$ into infinitely many infinite convex cosets and, furthermore, $E'_k(b,M) \subset E^q(b,M) \subset E'_{k+1}(b,M)$ for every $i+1 \leq k \leq n_p - 2$ or $E'_{n_p-1}(b,M) \subset E^q(b,M)$. Thus, $RC(q) = n_p - i$, i.e., $RC(p) > RC(q)$. □

Corollary 2.5. *Given a countably categorical weakly o-minimal theory T of finite convexity rank, take nonalgebraic types $p, q \in S_1(\varnothing)$ with $p \not\perp^w q$. Then $RC(p) = RC(q)$ if and only if there exists a (p,q)-splitting formula $R(x,y)$ such that the function $f(x) := \sup R(x,M)$ is locally monotone (not locally constant) on $p(M)$.*

Lemma 2.6. *Given a countably categorical weakly o-minimal theory T of finite convexity rank and $M \models T$, take nonalgebraic types $p, q \in S_1(\varnothing)$ and suppose that $\mathrm{dcl}(\{a\}) \cap q(M) \neq \varnothing$ for some $a \in p(M)$.*

(1) *If $RC(p) = RC(q)$ then there are precisely $2n_p$ (p,q)-splitting formulas.*

(2) *If $RC(p) > RC(q)$ then there are precisely $2n_q$ (p,q)-splitting formulas.*

Proof of Lemma 2.6. (1) Suppose that $RC(p) = RC(q)$. By Corollary 2.3, there is a unique $\varnothing$-definable locally monotone bijection $f : p(M) \to q(M)$ of depth k for some $0 \le k \le n_p - 1$. Consider the formulas

$$\phi^0_-(x,y) := U_p(x) \wedge U_q(y) \wedge y < f(x), \quad \phi^0_+(x,y) := U_p(x) \wedge U_q(y) \wedge y \le f(x),$$
$$\phi^i_-(x,y) := U_p(x) \wedge U_q(y) \wedge \forall t\big[E^p_i(x,t) \to y < f(t)\big], \quad 1 \le i \le n_p - 1,$$
$$\phi^i_+(x,y) := U_p(x) \wedge U_q(y) \wedge \exists t\big[E^p_i(x,t) \wedge y < f(t)\big], \quad 1 \le i \le n_p - 1.$$

It is obvious that all these formulas are (p,q)-splitting formulas; furthermore,

$$\phi^{n_p-1}_-(a,M) \subset \cdots \subset \phi^1_-(a,M) \subset \phi^0_-(a,M) \subset \phi^0_+(a,M) \subset \phi^1_+(a,M) \subset \cdots \subset \phi^{n_p-1}_+(a,M).$$

We assert that no other (p,q)-splitting formulas exist. Assume on the contrary that there exists a (p,q)-splitting formula $\Phi(x,y)$ distinct from these $2n_p$ (p,q)-splitting formulas. Then the following cases are possible:

$$\phi^{i+1}_-(a,M) \subset \Phi(a,M) \subset \phi^i_-(a,M) \quad \text{for some } 0 \le i \le n_p - 2,$$
$$\phi^i_+(a,M) \subset \Phi(a,M) \subset \phi^{i+1}_+(a,M) \quad \text{for some } 0 \le i \le n_p - 2,$$
$$\Phi(a,M) \subset \phi^{n_p-1}_-(a,M) \quad \text{or} \quad \phi^{n_p-1}_+(a,M) \subset \Phi(a,M).$$

Assume without loss of generality that $\phi^{i+1}_-(a,M) \subset \Phi(a,M) \subset \phi^i_-(a,M)$ for some $0 \le i \le n_p - 2$; the remaining cases are analogous. Since f is locally monotone of depth k for some $0 \le k \le n_p-1$, f must be strictly increasing or strictly decreasing on each $E^p_{i+1}(a,M)/E^p_i$ for every $a \in p(M)$. For definiteness, assume the first property. Consider the formula

$$G^\Phi(z,a) := U_p(z) \wedge z \le a \wedge \forall y[U_q(y) \wedge y < f(z) \to \Phi(a,y)].$$

It is not difficult to understand that $G^\Phi(z,x)$ is a p-stable left-convex formula, and, furthermore, $G^\Phi(z,x)$ is smaller than $G_{i+1}(z,x)$ and greater than $G_i(z,x)$, where $G_{i+1}(z,x) := E^p_{i+1}(z,x) \wedge z \le x$ and $G_i(z,x) := E^p_i(z,x) \wedge z \le x$ are also p-stable left-convex formulas. Theorem 1.15 and Lemma 1.14 yield $RC(p) \ge n_p + 1$; this contradicts the assumption. Thus, no other (p,q)-splitting formulas exist.

Suppose that $RC(p) > RC(q)$. By Corollary 2.3, there is a unique $\varnothing$-definable function $f : p(M) \to q(M)$ locally constant of depth k on $p(M)$ for some $1 \le k \le n_q$ and, furthermore, f is constant on each coset of $E^p_{n_p-n_q}$ and locally monotone on $p(M)/E^p_{n_p-n_q}$. Then $\phi^i_-(a,M) = \phi^0_-(a,M)$ and $\phi^0_+(a,M) = \phi^i_+(a,M)$ for each $1 \le i \le n_p - n_q$. □

Lemma 2.7. *Given a countably categorical weakly o-minimal theory T of finite convexity rank and $M \models T$, take nonalgebraic types $p, q \in S_1(\varnothing)$ with $p \not\perp^w q$ and suppose that $\mathrm{dcl}(\{a\}) \cap q(M) = \varnothing$ for some $a \in p(M)$.*

(1) *If $RC(p) = RC(q)$ then there are precisely $2n_p - 1$ (p,q)-splitting formulas.*

(2) *If $RC(p) > RC(q)$ then there are precisely $2n_p - 1$ (p,q)-splitting formulas.*

Proof of Lemma 2.7. (1) Suppose that $RC(p) = RC(q)$. By Corollary 2.5, there exists a (p,q)-splitting formula $\phi(x,y)$ such that the function $f(x) := \sup \phi(x,M)$ is locally monotone on $p(M)$. Since $RC(p) = n_p$, it follows that f is of depth k for some $0 \le k \le n_p - 1$. Consider the formulas

$$\Phi^i_-(x,y) := U_p(x) \wedge U_q(y) \wedge \forall t\big[E^p_i(x,t) \to \phi(t,y)\big], \quad 1 \le i \le n_p - 1,$$

$$\Phi^i_+(x,y) := U_p(x) \wedge U_q(y) \wedge \exists t\big[E^p_i(x,t) \wedge \phi(t,y)\big], \quad 1 \le i \le n_p - 1.$$

It is obvious that these formulas are (p,q)-splitting formulas and, furthermore,

$$\Phi^{n_p-1}_-(a,M) \subset \cdots \subset \Phi^1_-(a,M) \subset \phi(a,M) \subset \Phi^1_+(a,M) \subset \cdots \subset \Phi^{n_p-1}_+(a,M).$$

By analogy with the proof of Lemma 2.6, we can show that no other (p,q)-splitting formulas exist when in $G^{\Phi}(z,x)$ we replace the conjunctive term $y < f(z)$ by $\phi(z,y)$.

(2) Suppose that $RC(p) > RC(q)$. Since $p \not\perp^w q$, by Proposition 2.4 for every (p,q)-splitting formula $R(x,y)$ there is an $\varnothing$-definable equivalence $E(x,y)$ partitioning $p(M)$ into infinitely many infinite convex cosets such that $f(x) := \sup R(x,M)$ is constant on each coset of E. Choose the greatest equivalence $E_i^p(x,y)$ on $p(M)$ such that $f(x) := \sup R(x,M)$ is constant on each coset of E_i^p for every (p,q)-splitting formula $R(x,y)$. Since $E_i^p(x,y)$ is greatest with this property, there is a (p,q)-splitting formula $\phi(x,y)$ such that $f(x) := \sup \phi(x,M)$ is constant on each coset of E_i^p and locally monotone on $p(M)/E_i^p$. It is obvious that $i = n_p - n_q$. Then $\Phi_-^j(a,M) = \phi(a,M) = \Phi_+^j(a,M)$ for each $1 \le j \le n_p - n_q$. Similarly we can show that there are no other (p,q)-splitting formulas. $\square$

Call the (p,q)-splitting formula $\phi(x,y)$ from the proof of Lemma 2.7 the *basis splitting formula* since all remaining (p,q)-splitting formulas are uniquely defined using $\phi(x,y)$.

The following theorem completely describes the countably categorical weakly o-minimal theories of finite convexity rank:

Theorem 2.8. *Given a countably categorical weakly o-minimal theory T of finite convexity rank and $M \models T$ with $|M| = \aleph_0$, the following hold.*

(i) *There is a finite set $C = \{c_0 \dots, c_n\} \subseteq M$, or $M \cup \{-\infty, +\infty\}$ when M lacks the first or the last element, consisting of all $\varnothing$-definable elements of M, with possible exception of $-\infty$ and $+\infty$, such that $M \models c_i < c_j$ for all $i < j \le n$, and for each $j \in \{1,\dots,n\}$ either $M \models \neg(\exists x) c_{j-1} < x < c_j$ or $I_j = \{x \in M : M \models c_{j-1} < x < c_j\}$ is a dense linear order without endpoints and there are $k_j \in \omega$ and $p_1^j, \dots, p_{k_j}^j \in S_1(\varnothing)$ such that $I_j = \bigcup_{s=1}^{k_j} p_s^j(M)$.*

(ii) *For each nonalgebraic type $p \in S_1(\varnothing)$ there is a positive integer $n_p \ge 1$ such that $RC(p) = n_p$; i.e., there are $\varnothing$-definable equivalences $E_1^p(x,y)$, $E_2^p(x,y)$, ..., $E_{n_p-1}^p(x,y)$ such that*

- *$E_{n_p-1}^p$ partitions $p(M)$ into infinitely many cosets of $E_{n_p-1}^p$, each of which is convex and open, so that the induced order on the cosets is a dense linear order without endpoints;*
- *for each $i \in \{1,\dots,n_p-2\}$ the relation E_i^p partitions each coset of E_{i+1}^p into infinitely many cosets of E_i^p, each of which is convex and open, so that the subcosets of E_i^p in each coset of E_{i+1}^p are densely ordered without endpoints.*

(iii) *For all nonalgebraic types $p,q \in S_1(\varnothing)$ with $p \not\perp^w q$ we have:*

(1) *if $\mathrm{dcl}(\{a\}) \cap q(M) \neq \varnothing$ for some $a \in p(M)$ then there is a unique $\varnothing$-definable function $f : p(M) \to q(M)$ such that*

—in the case $RC(p) = RC(q)$ the function f is a locally monotone bijection of depth k on $p(M)$ for some $0 \le k \le n_p - 1$,

—in the case $RC(p) > RC(q)$ the function f is locally constant of depth k on $p(M)$ for some $1 \le k \le n_q$, i.e., f is constant on each coset of $E_{n_p-n_q}^p$ and locally monotone on $p(M)/E_{n_p-n_q}^p$;

(2) *if $\mathrm{dcl}(\{a\}) \cap q(M) = \varnothing$ for all $a \in p(M)$ then*

—in the case $RC(p) = RC(q)$ there exist precisely $2n_p - 1$ (p,q)-splitting formulas $S_1(x,y)$, ..., $S_{2n_p-1}(x,y)$ such that $S_1(a,M) \subset \dots \subset S_{2n_p-1}(a,M)$ for all $a \in p(M)$, the function $f(x) := \sup S_{n_p}(x,M)$ is locally monotone of depth k on $p(M)$ for some $0 \le k \le n_p - 1$, and

$$S_i(x,y) \equiv \forall t\big[E_{n_p-i}^p(x,t) \to S_{n_p}(t,y)\big], \quad 1 \le i \le n_p - 1,$$

$$S_j(x,y) \equiv \exists t\big[E_{j-n_p}^p(x,t) \wedge S_{n_p}(t,y)\big], \quad n_p + 1 \le j \le 2n_p - 1;$$

—in the case $RC(p) > RC(q)$ there exist precisely $2n_q - 1$ (p,q)-splitting formulas $S_1(x,y)$, ..., $S_{2n_q-1}(x,y)$ such that $S_1(a,M) \subset \dots \subset S_{2n_q-1}(a,M)$ for all $a \in p(M)$, the function $f(x) := \sup S_{n_q}(x,M)$ is constant on each coset of $E_{n_p-n_q}^p$ and locally monotone on $p(M)/E_{n_p-n_q}^p$, and

$$S_i(x,y) \equiv \forall t\big[E_{n_p-i}^p(x,t) \to S_{n_q}(t,y)\big], \quad 1 \le i \le n_q - 1,$$

$$S_j(x,y) \equiv \exists t\big[E_{n_p-2n_q+j}^p(x,t) \wedge S_{n_q}(t,y)\big], \quad n_q + 1 \le j \le 2n_q - 1,$$

so that T admits quantifier elimination to the language

$$\{=,<\}\cup\{c_i : i\le n\}\cup\left\{U_s(x) : s\le r=\sum_{j=1}^{n}k_j\right\}$$
$$\cup\big\{E_l^{p_s}(x,y) : RC(p_s)=n_{p_s},\ 1\le l\le n_{p_s}-1 \text{ and } s\le r\big\}$$
$$\cup\{f_{i,j} : \mathrm{dcl}(\{a\})\cap p_j(M)\ne\varnothing \text{ for some } a\in p_i(M), RC(p_i)\ge RC(p_j)\}$$
$$\cup\{S_{i,j}(x,y) : p_i\not\perp^w p_j, \mathrm{dcl}(\{a\})\cap p_j(M)=\varnothing$$
$$\text{for all } a\in p_i(M), RC(p_i)\ge RC(p_j),$$
$$S_{i,j}(x,y) \text{ is a basis } (p_i,p_j)\text{-splitting formula}\},$$

where $U_s(x)$ isolates the type p_s for each $s\le r$.

Moreover, to each ordering with selected elements as in (i)–(iii) *there corresponds a countably categorical weakly o-minimal theory of finite convexity rank as above.*

Proof of Theorem 2.8. (i) Put $C=\{c\in M : c \text{ is } \varnothing\text{-definable in } M\}$.

Since T is countably categorical, C must be finite. Enumerate C, or $C\cup\{-\infty,+\infty\}$ in the case that M lacks the first or last elements, as $\{c_0,\dots,c_n\}$ and suppose that

$$I_j=\{x\in M : M\models c_{j-1}<x<c_j\}\ne\varnothing.$$

Then I_j must be dense without endpoints. If I_j is 1-indiscernible over $\varnothing$ then there exist $p^j\in S_1(\varnothing)$ with $I_j=p^j(M)$, i.e., $k_j=1$. If I_j is not 1-indiscernible over $\varnothing$ then by the countable categoricity there exist $k_j\in\omega$ and $p_1^j,\dots,p_{k_j}^j\in S_1(\varnothing)$ such that $I_j=\bigcup_{s=1}^{k_j}p_s^j(M)$.

(ii) Since T is of finite convexity rank, for each nonalgebraic 1-type $p\in S_1(\varnothing)$ there exists an integer $n_p\ge 1$ with $RC(p)=n_p$. Hence, by Corollary 1.16 there exist $\varnothing$-definable equivalences $E_1^p(x,y),\dots,E_{n_p-1}^p(x,y)$ with the required properties. Moreover, since T is binary by Theorem 1.17, for every $\alpha\in p(M)$ the sets $E_1^p(M,\alpha)$ and $E_{j+1}^p(M,\alpha)\setminus E_j^p(M,\alpha)$ for each $1\le j\le n_p-2$ are indiscernible over $\varnothing$.

(iii) This follows from Corollary 2.3 and Lemma 2.7.

Consider arbitrary nonalgebraic types $p_i,p_j,p_k\in S_1(\varnothing)$ with $p_i\not\perp^w p_j$, $p_j\not\perp^w p_k$, and $RC(p_i)\ge RC(p_j)\ge RC(p_k)$. Since $p_i\not\perp^w p_k$ by Lemma 1.3, let us verify explicitly that an $\varnothing$-definable function mapping $p_i(M)$ onto $p_k(M)$ or a basis (p_i,p_k)-splitting formula relating the types p_i and p_k are uniquely defined.

The following cases are possible.

Case 1. $RC(p_i)=RC(p_j)=RC(p_k)$.

Case 2. $RC(p_i)=RC(p_j)>RC(p_k)$.

Case 3. $RC(p_i)>RC(p_j)=RC(p_k)$.

Case 4. $RC(p_i)>RC(p_j)>RC(p_k)$.

Furthermore, each case splits into the subcases:

(a) $\mathrm{dcl}(\{a\})\cap p_j(M)\ne\varnothing$ and $\mathrm{dcl}(\{b\})\cap p_k(M)\ne\varnothing$ for some $a\in p_i(M)$ and $b\in p_j(M)$;

(b) $\mathrm{dcl}(\{a\})\cap p_j(M)\ne\varnothing$ and $\mathrm{dcl}(\{b\})\cap p_k(M)=\varnothing$ for some $a\in p_i(M)$ and $b\in p_j(M)$;

(c) $\mathrm{dcl}(\{a\})\cap p_j(M)=\varnothing$ and $\mathrm{dcl}(\{b\})\cap p_k(M)\ne\varnothing$ for some $a\in p_i(M)$ and $b\in p_j(M)$;

(d) $\mathrm{dcl}(\{a\})\cap p_j(M)=\varnothing$ and $\mathrm{dcl}(\{b\})\cap p_k(M)=\varnothing$ for some $a\in p_i(M)$ and $b\in p_j(M)$;

Case 1. (a) There are $\varnothing$-definable locally monotone bijections $f_{i,j}:p_i(M)\to p_j(M)$ and $f_{j,k}:p_j(M)\to p_k(M)$. It is obvious that $f_{i,k}:=f_{j,k}\circ f_{i,j}$ is a locally monotone bijection of $p_i(M)$ onto $p_k(M)$.

(b) There are an $\varnothing$-definable locally monotone bijection $f_{i,j}:p_i(M)\to p_j(M)$ and a (p_j,p_k)-splitting formula $S_{j,k}(x,y)$ such that $f_{j,k}(x):=\sup S_{j,k}(x,M)$ is locally monotone of depth $s\le n_{p_i}-1$ on $p_j(M)$. It is obvious that $S_{i,k}(x,z):=\exists y[y=f_{i,j}(x)\land S_{j,k}(y,z)]$ is the required (p_i,p_k)-splitting formula.

(c) There are a (p_i,p_j)-splitting formula $S_{i,j}(x,y)$ such that $f_{i,j}(x):=\sup S_{i,j}(x,M)$ is locally monotone on $p_i(M)$ and an $\varnothing$-definable locally monotone bijection $f_{j,k}:p_j(M)\to p_k(M)$. If $f_{j,k}$ is strictly increasing on each coset of $E_1^{p_j}$ then

$$S_{i,k}(x,z):=\exists y\exists y_1\big[S_{i,j}(x,y)\land\neg S_{i,j}(x,y_1)\land E_1^{p_j}(y,y_1)\land z\le f_{j,k}(y)\big].$$

If $f_{j,k}$ is strictly decreasing on each coset of $E_1^{p_j}$ then

$$S_{i,k}(x,z) := \exists y \exists y_1 \big[\neg S_{i,j}(x,y) \wedge S_{i,j}(x,y_1) \wedge E_1^{p_j}(y,y_1) \wedge z \leq f_{j,k}(y)\big].$$

(d) There are a (p_i,p_j)-splitting formula $S_{i,j}(x,y)$ and a (p_j,p_k)-splitting formula $S_{j,k}(x,y)$ such that the functions $f_{i,j}(x) := \sup S_{i,j}(x,M)$ and $f_{j,k}(x) := \sup S_{j,k}(x,M)$ are locally monotone on $p_i(M)$ and $p_j(M)$ respectively. If $f_{j,k}$ is strictly increasing on each coset of $E_1^{p_j}$ then

$$S_{i,k}(x,z) := \exists y \exists y_1 \big[S_{i,j}(x,y) \wedge \neg S_{i,j}(x,y_1) \wedge E_1^{p_j}(y,y_1) \wedge S_{j,k}(y,z)\big].$$

If $f_{j,k}$ is strictly decreasing on each coset of $E_1^{p_j}$ then

$$S_{i,k}(x,z) := \exists y \exists y_1 \big[\neg S_{i,j}(x,y) \wedge S_{i,j}(x,y_1) \wedge E_1^{p_j}(y,y_1) \wedge S_{j,k}(y,z)\big].$$

Case 2. (a) There are $\varnothing$-definable functions $f_{i,j}$ and $f_{j,k}$ such that $f_{i,j} : p_i(M) \to p_j(M)$ is a locally monotone bijection and $f_{j,k} : p_j(M) \to p_k(M)$ is locally constant on $p_j(M)$. It is obvious that $f_{i,k} := f_{j,k} \circ f_{i,j}$ is locally constant on $p_i(M)$.

(b) There are an $\varnothing$-definable locally monotone bijection $f_{i,j} : p_i(M) \to p_j(M)$ and a (p_j,p_k)-splitting formula $S_{j,k}(x,y)$ such that $f_{j,k}(x) := \sup S_{j,k}(x,M)$ is locally constant on $p_j(M)$. In this event $S_{i,k}(x,z) := \exists y[y = f_{i,j}(x) \wedge S_{j,k}(y,z)]$.

(c) There are a (p_i,p_j)-splitting formula $S_{i,j}(x,y)$ such that $f_{i,j}(x) := \sup S_{i,j}(x,M)$ is locally monotone on $p_i(M)$ and the $\varnothing$-definable function $f_{j,k} : p_j(M) \to p_k(M)$ is locally constant on $p_j(M)$. In this event

$$S_{i,k}(x,z) := \exists y \exists y_1 \big[S_{i,j}(x,y) \wedge \neg S_{i,j}(x,y_1) \wedge E_1^{p_j}(y,y_1) \wedge z \leq f_{j,k}(y)\big].$$

(d) There are a (p_i,p_j)-splitting formula $S_{i,j}(x,y)$ and a (p_j,p_k)-splitting formula $S_{j,k}(x,y)$ such that $f_{i,j}(x) := \sup S_{i,j}(x,M)$ is locally monotone on $p_i(M)$ and $f_{j,k}(x) := \sup S_{j,k}(x,M)$ is locally constant on $p_j(M)$. In this event

$$S_{i,k}(x,z) := \exists y \exists y_1 \big[S_{i,j}(x,y) \wedge \neg S_{i,j}(x,y_1) \wedge E_1^{p_j}(y,y_1) \wedge S_{j,k}(y,z)\big].$$

Case 3. (a) There are $\varnothing$-definable functions $f_{i,j}$ and $f_{j,k}$ such that $f_{i,j} : p_i(M) \to p_j(M)$ is locally constant on $p_i(M)$ and $f_{j,k} : p_j(M) \to p_k(M)$ is a locally monotone bijection. It is obvious that $f_{i,k} := f_{j,k} \circ f_{i,j}$ is locally constant on $p_i(M)$.

(b) There are an $\varnothing$-definable function $f_{i,j} : p_i(M) \to p_j(M)$ locally constant on $p_i(M)$ and a (p_j,p_k)-splitting formula $S_{j,k}(x,y)$ such that $f_{j,k}(x) := \sup S_{j,k}(x,M)$ is locally monotone on $p_j(M)$. In this event $S_{i,k}(x,z) := \exists y[y = f_{i,j}(x) \wedge S_{j,k}(y,z)]$.

(c) There are a (p_i,p_j)-splitting formula $S_{i,j}(x,y)$ such that $f_{i,j}(x) := \sup S_{i,j}(x,M)$ is locally constant on $p_i(M)$ and an $\varnothing$-definable locally monotone bijection $f_{j,k} : p_j(M) \to p_k(M)$. If $f_{j,k}$ is strictly increasing on each coset of $E_1^{p_j}$ then

$$S_{i,k}(x,z) := \exists y \exists y_1 \big[S_{i,j}(x,y) \wedge \neg S_{i,j}(x,y_1) \wedge E_1^{p_j}(y,y_1) \wedge z \leq f_{j,k}(y)\big].$$

If $f_{j,k}$ is strictly decreasing on each coset of $E_1^{p_j}$ then

$$S_{i,k}(x,z) := \exists y \exists y_1 \big[\neg S_{i,j}(x,y) \wedge S_{i,j}(x,y_1) \wedge E_1^{p_j}(y,y_1) \wedge z \leq f_{j,k}(y)\big].$$

(d) There are a (p_i,p_j)-splitting formula $S_{i,j}(x,y)$ and a (p_j,p_k)-splitting formula $S_{j,k}(x,y)$ such that the function $f_{i,j}(x) := \sup S_{i,j}(x,M)$ is locally constant on $p_i(M)$ and $f_{j,k}(x) := \sup S_{j,k}(x,M)$ is locally monotone on $p_j(M)$. If $f_{j,k}$ is strictly increasing on each coset of $E_1^{p_j}$ then

$$S_{i,k}(x,z) := \exists y \exists y_1 \big[S_{i,j}(x,y) \wedge \neg S_{i,j}(x,y_1) \wedge E_1^{p_j}(y,y_1) \wedge S_{j,k}(y,z)\big].$$

If $f_{j,k}$ is strictly decreasing on each coset of $E_1^{p_j}$ then

$$S_{i,k}(x,z) := \exists y \exists y_1 \big[\neg S_{i,j}(x,y) \wedge S_{i,j}(x,y_1) \wedge E_1^{p_j}(y,y_1) \wedge S_{j,k}(y,z)\big].$$

CASE 4. (a) There are $\varnothing$-definable functions $f_{i,j} : p_i(M) \to p_j(M)$ and $f_{j,k} : p_j(M) \to p_k(M)$ locally constant on $p_i(M)$ and $p_j(M)$ respectively. Obviously, $f_{i,k} := f_{j,k} \circ f_{i,j}$ is locally constant on $p_i(M)$.

(b) There are an $\varnothing$-definable function $f_{i,j} : p_i(M) \to p_j(M)$, locally constant on $p_i(M)$, and a (p_j, p_k)-splitting formula $S_{j,k}(x, y)$ such that $f_{j,k}(x) := \sup S_{j,k}(x, M)$ is locally constant on $p_j(M)$. In this event $S_{i,k}(x, z) := \exists y[y = f_{i,j}(x) \wedge S_{j,k}(y, z)]$.

(c) There are a (p_i, p_j)-splitting formula $S_{i,j}(x, y)$ such that $f_{i,j}(x) := \sup S_{i,j}(x, M)$ is locally constant on $p_i(M)$ and an $\varnothing$-definable function $f_{j,k} : p_j(M) \to p_k(M)$ locally constant on $p_j(M)$. In this event

$$S_{i,k}(x, z) := \exists y \exists y_1 \left[S_{i,j}(x, y) \wedge \neg S_{i,j}(x, y_1) \wedge E_1^{p_j}(y, y_1) \wedge z \leq f_{j,k}(y)\right].$$

(d) There are a (p_i, p_j)-splitting formula $S_{i,j}(x, y)$ and a (p_j, p_k)-splitting formula $S_{j,k}(x, y)$ such that the function $f_{i,j}(x) := \sup S_{i,j}(x, M)$ and $f_{j,k}(x) := \sup S_{j,k}(x, M)$ are locally constant on $p_i(M)$ and $p_j(M)$ respectively. In this event

$$S_{i,k}(x, z) := \exists y \exists y_1 \left[S_{i,j}(x, y) \wedge \neg S_{i,j}(x, y_1) \wedge E_1^{p_j}(y, y_1) \wedge S_{j,k}(y, z)\right].$$

Finally, since T is binary, the complete type of every m-tuple $\langle a_1, \dots, a_m\rangle$ in M is defined by some formula Ψ that is a conjunction of formulas of the form $x = y$, $x < y$, $c_i < x$, $x < c_i$, $U_s(x)$, $y = f_{i,j}(x)$, $y < f_{i,j}(x)$, $f_{i,j}(x) < y$, $S_{i,j}(x, y)$, and $E^s(x, y)$ and their negations, which hold on the coordinates of the tuple $\langle a_1, \dots, a_m\rangle$. This implies the claimed quantifier elimination. □

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