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ON PERIODIC GROUPS WITH NARROW SPECTRUM

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Abstract: We study groups with no elements of big orders. We prove that if the set of element orders of G is $\{1, 2, 3, 4, p, 9\}$, where $p \in \{7, 5\}$, then G is locally finite.

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1. Introduction

The *spectrum* of a periodic group G is the set $\omega(G)$ of its element orders. If $\omega(G)$ is finite, then $\mu(G)$ stands for the set of division maximal elements of $\omega(G)$. There is a survey of results on the structure of periodic groups with given spectrum in [1]. In particular, G with $\mu(G) = \{4, 9\}$ is locally finite [2]. The main result of this article is the following

Theorem. *Let G be a group with $\mu(G) = \{4, p, 9\}$, where $p \in \{7, 5\}$. Then $p = 7$ and G is an extension of a 2-group by $L_2(8)$. In particular, G is locally finite.*

Along the way we study groups whose element orders are not greater than 11.

Let $\Gamma_n = \Gamma_n(G)$ denote the set of elements of order n in G; $\Delta = \{x^2 | x \in \Gamma_4\}$. We write $x \sim y$ whenever x and y have the same order. Obviously, $uv \sim vu$. Speaking computations, we refer to using the coset enumeration algorithm in GAP [3].

2. Preliminary Results

Lemma 1 (Shunkov [4]). *If a periodic group G has an involution with finite centralizer, then G is locally finite.*

Lemma 2 (Shunkov [5, Theorem 2] and [6, Theorem 2.4]). *If G is an infinite 2-group of finite period and F is a finite subgroup of G, then $C_G(F)$ is infinite.*

Lemma 3 (Zhurtov [7]). *Let T be a periodic group acting freely on a nontrivial abelian group and $x \in \Gamma_3(T)$. Then either x is in the center of T or $\langle x^T \rangle$ is isomorphic to $SL_2(3)$ or $SL_2(5)$; in any case the center of T is nontrivial.*

Lemma 4 (Zhurtov and Mazurov [8, Theorem 2]). *Let T be a group with $\omega(T) = \{2, 3\} \bigcup \omega$, where each element in ω is either coprime to 6 or equals 9. Then T is locally finite.*

Lemma 5. *Let $x, y \in \Gamma_2$ and $xy \in \Gamma_n$. If $n = 2k$ is even, then $(xy)^k$ is in the center of $\langle x, y \rangle$. If $n = 2k + 1$ is odd and $z = (xy)^k$ then $y^z = x$.*

3. Groups Without Elements of Big Order

The main goal of this section is to prove the following statement which we will use later in the proof of the Theorem.

Let F be the group $\langle x, t | 1 = x^4 = t^2 = (x^2t)^3 = (xt)^4 = [x, t]^3 \rangle$. Note that F is a Frobenius group of order 36 with kernel $\langle a, a^x \rangle$ and complement $\langle x \rangle$, where $a = x^2t$. Indeed, the defining relations imply $[a, a^x] = (tx)^4 = 1$, while t is an involution inverting a and a^x; and computations show that $|F| = 36$.

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Proposition. *Assume that a periodic group G has no elements of order greater than* 11.

(a) *Let $x \in \Gamma_4(G)$, $t \in \Gamma_2(G)$ and $x^2t \in \Gamma_3(G)$. Then $K = \langle t, x\rangle$ is finite and isomorphic to one of the groups S_5, F, $L_2(7)$, and $(A_4 \times A_4) : C_4$.*

(b) *Assume that element orders of G are either odd or divide* 4. *Assume further that G has a subgroup H isomorphic to $L_2(7)$, and G is not locally finite. Then G has a subgroup isomorphic to $L_3(4)$.*

(c) *Assume that element orders of G are either odd or divide* 4. *Assume further that G has no subgroups isomorphic to $L_2(7)$. If there are $x \in \Gamma_4$ and $t \in \Gamma_2$ such that $x^2t \in \Gamma_3$, then G is locally finite.*

PROOF. (a) Let n be the order of t^xt. Assume first that $n = 2k+1$ is odd. By Lemma 5, $t^{(t^xt)^k} = t^x$, and so $(t^xt)^kx^{-1} \in C_G(t)$. Then K is a homomorphic image of $G(i,j,l,n,h) = \langle x, t | 1 = x^4 = t^2 = (x^2t)^3 = (xt)^i = ((xt)^3x^2t)^j = ((xt)^4x^3t)^l = (t^xt)^n = ((t^xt)^{k_n}x^{-1})^h\rangle$, where $n \in \{5,7,9,11\}$; $k_n = (n-1)/2$; $i,j,l \in \{6,7,8,9,10,11\}$; $h \in \{6,8,10\}$. Computations show that $G(6,8,6,*,*)$ is trivial or isomorphic to S_5, while $G(8,8,8,9,8)$ is isomorphic to F, and $G(i,j,l,n,h)$ has no elements of order 3 for other possible values of parameters.

Assume now that n is even. Then K is a homomorphic image of $G(h,i,j) = \langle x, t | 1 = x^4 = t^2 = (x^2t)^3 = (xt)^i = ((xt)^3x^2t)^j = (t^xt)^h\rangle$, where $i, j \in \{6,7,8,9,10,11\}$ and $h \in \{6,8,10\}$. Computations show that $G(8,9,9) \simeq L_2(17)$, which is impossible; $G(8,7,6) \simeq G(8,7,9) \simeq L_2(7)$; $G(10,6,8) \simeq S_5$; $G(6,8,8) \simeq (A_4 \times A_4) : C_4$, and $G(h,i,j)$ has no elements of order 3 for other possible values of parameters.

(b) Consider the matrices

$$\bar{x} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad \bar{y} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \tag{1}$$

from $L_3(2) \simeq L_2(7)$. Computations with matrices show that $\langle \overline{x}, \overline{y}\rangle = L_3(2)$; $\overline{x}^3 = \overline{y}^2 = (\overline{xy})^7 = \overline{z}^4 = 1$, where $\overline{z} = [\overline{x}, \overline{y}]$; $C_{L_3(2)}(\overline{z}^2) = \langle \overline{y}, \overline{y}^{\overline{x}}\rangle$; $\overline{x}^{\overline{yx}^2\overline{yx}^2} \in N_{L_3(2)}(\langle \overline{y}, \overline{z}^2\rangle)$; $\overline{y}^{\overline{x}^2\overline{yx}^2} \in C_{L_3(2)}(\overline{y}^{\overline{x}})$; $\overline{z}^2\overline{y}^{(\overline{xy})^2} \in \Gamma_3(L_3(2))$.

Let us identify H with $\langle x, y | 1 = x^3 = y^2 = (xy)^7 = z^4\rangle$, where $z = [x, y]$. By Lemma 1, $C = C_G(z^2)$ is an infinite 2-subgroup. Put $S = C_H(z^2)$. By Lemma 2, $C_C(S)$ is infinite. Let K be a group of order 16 from $C_C(S)$ which includes S. If K has the unique involution z^2, then K would be either cyclic or generalized quaternion [9, Theorem 12.5.2] and so K would contain an element of order 8, which is impossible. Let t be an involution in $C_C(S)$ different from z^2. Note that $S = \langle y, y^x\rangle$.

Let $a = x^{yx^2yx^2}$ be an element of order 3 from the normalizer $N_H(\langle y, z^2\rangle)$. Note that t centralizes the group $\langle y, z^2\rangle$, and so t^a and t^{a^2} should also centralize this group. Hence $\langle t^{\langle a\rangle}\rangle$ is a 2-group, where a acts fixed-point-freely. By [10, Lemma 2] $\langle t^{\langle a\rangle}\rangle = \langle t, t^a\rangle$ is abelian. Hence $(at)^3 = 1$.

Note that $i = y^{x^2yx^2} \in C_H(y^x)$. Also $[t, y^x] = 1$ and $(it)^4 = 1$. Let $j = y^{(xy)^2}$. Then $z^2j \in \Gamma_3$. Observe that zt is an element of order 4 such that $(zt)^2 = z^2$. By item (a) $zjt = (xy)^3t \in \Gamma_4 \bigcup \Gamma_7$.

Therefore $\langle x, y, t\rangle$ is a homomorphic image of $G(r_1, r_2, r_3, r) = \langle x, y, t | 1 = x^3 = y^2 = (xy)^7 = z^4 = t^2 = [t, z] = [y, t] = (at)^3 = (ti)^4 = (xt)^{r_1} = (t^xt)^{r_2} = (yti)^{r_3} = ((xy)^3t)^r\rangle$, where $r_1, r_2, r_3 \in \{4,5,7,9,11\}$ and $r \in \{4, 7\}$. Computations show that the index of $\langle x, y\rangle$ in $G(r_1, r_2, r_3, r)$ is finite; and either it is trivial, or $G(7,4,9,4) \simeq L_3(4)$, or $|G(9,*,9,7) : \langle x, y\rangle| = 56$. In the last case $56 \cdot |\langle x, y\rangle|$ is not divisible by 9, and so we may assume that $r_1 = 3$. Then computations show that $|G(3,*,9,7)| = 168$; a contradiction.

(c) Assume that the statement is false. By item (a), $\langle t, x\rangle \simeq F$. Hence $(xt)^4 = (t^xt)^3 = (x^2t)^3$. By Lemmas 1 and 2 there is an involution y that centralizes x and is not in $\langle t, x\rangle$. Obviously, $xy \in \Gamma_4$ and $(xy)^2 = x^2$. By item (a), $\langle t, xy\rangle \simeq F$, and hence $(xyt)^4 = (t^{xy}t)^3 = 1$. Therefore $\langle x, y, t\rangle$ is a homomorphic image of $G(r_1, r_2) = \langle x, y, t | 1 = x^4 = y^2 = t^2 = (x^2t)^3 = y^xy = (xt)^4 = (t^xt)^3 = (x^2t)^3 = (xyt)^4 = (t^{xy}t)^3 = (yt)^{r_1} = (x^2yt)^{r_2}\rangle$. Computations show that for any $r_1, r_2 \in \{4,5,7,9,11\}$ the order of $G(r_1, r_2)$ either divides 36, or does not divide 3. A contradiction.

The proof of the Proposition is complete.

To prove the Theorem we also need the following

Lemma 6. *Let $\omega(G) \subseteq \{1,2,3,4,5,7,9\}$, $x \in \Gamma_3$, and $t \in \Delta$. Assume further that G is not locally finite. Then either G has a subgroup isomorphic to $L_3(4)$, or $\langle x,t\rangle \simeq A_4$.*

Proof. If the order of $t^x t$ divides 9, then by the properties of dihedral groups there is an involution t' such that $tt' \in \Gamma_3(G)$, and by the Proposition either G is locally finite or G contains a subgroup isomorphic to $L_3(4)$. Therefore we may assume that the order of $t^x t$ does not divide 9. Let us proceed case by case.

1. Assume first that $xt \in \Gamma_7$. If the order of tt^x is even, then $\langle x,t\rangle$ is a homomorphic image of the simple group $L_2(7)$. If the order of tt^x equals $2k+1$, then $(t^x t)^k x^{-1} \in C_G(t)$. Therefore $\langle x,t\rangle$ is a homomorphic image of $G(k) = \langle x,t | 1 = x^3 = t^2 = (xt)^7 = (tt^x)^{2k+1} = ((t^x t)^k x^{-1})^4\rangle$. Computations show that $G(k)$ is trivial when $k \in \{1,2,3\}$.

2. Assume now that $xt \in \Gamma_9$. Put $y = (xt)^3$ and note that $yt = xtxtx \sim t^x t$. So by item 1 the order of $t^x t$ does not divide 7. Hence $\langle x,t\rangle$ is a homomorphic image of the group $H(k) = \langle x,t | 1 = x^3 = t^2 = (xt)^9 = (tt^x)^k\rangle$, where $k \in \{3,4,5\}$. Computations show that $H(3)$ is a cyclic group of order 3, $H(4) \simeq A_4$, and $H(5) \simeq H(3) \times L_2(19)$.

3. Finally, assume that the order of xt is at most 7. Then $\langle x,t\rangle$ is a homomorphic image of $K(r_1,r_2) = \langle x,t | 1 = x^3 = t^2 = (xt)^{r_1} = (tt^x)^{r_2}\rangle$, where $r_1 \in \{3,4,5\}$ and $r_2 \in \{3,4,5,7\}$. Computations show that $K(4,3) \simeq S_4$ and $K(5,5) \simeq A_5$ has a subgroup isomorphic to S_3, which is impossible; $K(3,4) \simeq A_4$; and $K(r_1,r_2)$ is trivial for other values of the parameters.

The lemma is proved.

4. Proof of the Theorem

Lemma 7. *Let G be a locally finite group with $\mu(G) = \{4,p,9\}$, where $p \in \{7,5\}$. Then $p = 7$ and G is an extension of a 2-group by $L_2(8)$.*

Proof. Let $x,y \in \Gamma_4(G)$, $z \in \Gamma_9(G)$, and $t \in \Gamma_p(G)$. Then $K = \langle x,y,z,t\rangle$ is a finite group and $\omega(K) = \omega(G)$. A group K is 3-primary and has no elements of composite orders, and so it cannot be a Frobenius or a double-Frobenius group [11, Lemma 1.1].

Using the classification of 3-primary groups [12], we see that $p = 7$ and $\overline{K} = K/O_2(K) \simeq L_2(8)$, where $O_2(K)$ is a direct product of minimal normal subgroups of orders 2^6 in K, which are isomorphic as $\overline{K}$-modules to the natural $GF(2^n)SL_2(2^n)$-modules. Therefore, x^2 and y^2 generate 2-subgroup. Hence, $\langle\Delta\rangle$ is a normal 2-subgroup in G and $G/\langle\Delta\rangle \simeq L_2(8)$. The lemma is proved.

Lemma 8. *Let $x \in \Gamma_3(G)$ and $4 \in \omega(G)$. If $ax \in \Gamma_3$ for all $a \in \Delta$, then $\langle\Delta\rangle$ is locally finite.*

Proof. By induction $\langle\Delta\rangle\Gamma_3 = \Gamma_3$.

If $h = x_1 \dots x_n$ is an element of order 3, where $x_1,\dots,x_n \in \Delta$, then $n > 2$ and $x_n = x_{n-1} \dots x_2 x_1 h \in \Gamma_3$, which is impossible. Hence, $3 \notin \omega(\langle\Delta\rangle)$. We have $\langle\Delta,x\rangle = \langle\Delta\rangle \rtimes \langle x\rangle$ and by [10, Lemma 7] $\langle\Delta\rangle$ is nilpotent of degree at most 2. Hence $\langle\Delta\rangle$ is locally finite.

The lemma is proved.

Proof of the Theorem. Let $\mu(G) = \{4,p,9\}$, where $p \in \{7,5\}$. By Lemma 7 it is enough to show that G is locally finite. Assume not. By the Proposition G has no subgroups isomorphic to $L_2(7)$. By the Proposition and Lemma 8 $\langle\Delta\rangle$ is a locally finite 2-group.

If $T = G/\langle\Delta\rangle$ has no involutions, then T acts freely on the center of $\langle\Delta\rangle$, which is impossible by Lemma 3. Therefore, T has an involution, and an element of order 3, and no elements of order 4; so T is locally finite by Lemma 4. Hence, G is locally finite by Schmidt's Theorem; a contradiction. The theorem is proved.

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