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DIRAC FLOW ON THE 3-SPHERE

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Abstract: We illustrate some well-known facts about the evolution of the 3-sphere (S^3, g) generated by the Ricci flow. We define the Dirac flow and study the properties of the metric $\bar{g} = dt^2 + g(t)$, where $g(t)$ is a solution of the Dirac flow. In the case of a metric g conformally equivalent to the round metric on S^3 the metric $\bar{g}$ is of constant curvature. We study the properties of solutions in the case when g depends on two functional parameters. The flow on differential 1-forms whose solution generates the Eguchi–Hanson metric was written down. In particular cases we study the singularities developed by these flows.

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1. Introduction

In this article we define several evolution equations on geometric structures and study their relations to some classical four-dimensional metrics. We study the Dirac flow

$$\frac{\partial}{\partial t} g = \sqrt{\mathrm{Ric}(g) - 4Kg}$$

in the simplest situation: in the case of a metric g conformally equivalent to the standard round metric on the sphere this flow describes the space of constant curvature $K \in \{-1, 0, +1\}$. Namely, if $g(t) = f^2(t)\cdot ds_0^2$ is a metric conformally equivalent to the standard metric ds_0^2 on the 3-sphere of radius 1, which also is a solution to the Dirac flow, then the four-dimensional metric $dt^2 + g(t)$ is of constant curvature K. We study the properties of certain solutions to this flow also in the case that $g(t)$ depends on two functional parameters:

$$g(t) = A_1^2(t)(e^1)^2 + A_2(t)^2((e^2)^2 + (e^3)^2),$$

where e^i constitute Cartan's canonical basis for 1-forms on the 3-sphere.

The main problem in defining this flow consists in taking the square root of a bilinear form. Assume that $\sqrt{g} = g$; i.e., the original metric plays the role of a fixed element with respect to the square root operation. This agrees with the definition of square root for operators: to take the square root of a bilinear form $C = c_{ij}$, we should firstly convert it into an operator by raising one index, i.e. multiply it by $g^{-1} = g^{jk}$; then we take the root of the operator $c_i^k = c_{ij}g^{jk} : T_M \to TM$ and finally lowering one index down: $(\sqrt{C})_{il} = \sqrt{c_{ij}g^{jk}}g_{kl}$. Taking the metric g itself as the form C, we precisely obtain $\sqrt{g}_{il} = \sqrt{g_{ij}g^{jk}}g_{kl} = g_{il}$, naturally putting $\sqrt{\delta_i^j} = \delta_i^j$. Observe also that in this article the quadratic form $\mathrm{Ric}(g) - 4Kg$ is positive definite because we consider deformations of the metrics on the 3-sphere g.

To express and study the equations of the flow we use the basis $\{A_1(t)e^1, A_2(t)e^2, A_2(t)e^3\}$, in which $g(t)$ becomes the identity matrix. Therefore, we should understand the square root in the definition of flow to be the standard square root defined on the scalar functions that are components of the tensor $\mathrm{Ric}(g) - 4Kg$. Even in the simplest situation we may face the problem of correct choice of the branches of the square root and continuation of solutions to differential equations. In this article we prefer to

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skip the inspection of problems of the sort and focus on the construction and analysis of the concrete solutions, justifying our attention to geometric flows of the form.

We also define the flow on $\mathbb{R}P^3$ which yields the Eguchi–Hanson metric,

$$\frac{\partial}{\partial t}g_{ij} = \frac{1}{2}\sqrt{\det(\mathrm{Ric})}(\mathrm{Ric}^{-1})_{ij}, \quad i,j \in \{1,2,3\},$$

and reduces to the two nonlinear equations on A_1 and A_2. We can express this flow as the equation $(*\psi)' = d\psi$ on two 1-forms $\psi \in \{A_1(t)e^1, A_2(t)e^2\}$, where $d : \Lambda^i(S^3) \to \Lambda^{i+1}(S^3)$ is the standard differential on the forms on S^3.

Note that similar ideas are developed in [1]. In particular, it is stated in [1] that the Taubes-NUT metric can be described as the result of action of Ricci flow (or inverse Ricci flow). Considering Ricci flow on some three-dimensional Lie groups, [1] checks whether some classical metrics correspond to solutions to Ricci flow. Constructions of other flows are absent from [1].

2. The Round 3-Sphere in $\mathbb{R}^4$

Consider firstly the space $\mathbb{H} = \mathbb{R}^4$. The standard flat metric on $\mathbb{R}^4$ coincides with the cone metric over $S^3 = \mathrm{Sp}(1)$. It is known that the Ricci flow $\frac{\partial}{\partial t}g(t) = -2\,\mathrm{Ric}(t)$, when defined on an Einstein manifold M (the round sphere in particular), changes only the volume of the underlying manifold. Thus, we may regard $\mathbb{R}^4$ as the "configuration space" of the Ricci flow on round 3-spheres which only changes their radii. Roughly speaking, by using the temporal coordinate t of a flow changing only the radius of a sphere, we can find as the spatial coordinate τ the space $(\mathbb{R}_+ \times M(\tau))$ (in this example $M(t)$ is the radius t of the sphere) with restrictions on the curvature, in this case $R_{ijkl} \equiv 0$. Henceforth we take the 3-sphere S^3 (or the three-dimensional real projective space $\mathbb{R}P^3$) as M. Denote the metric and other tensors on S^3 by g, Ric, and so forth, while the tensors on the 4-manifold $\mathbb{R}_+ \times M(\tau)$ by $\bar{g}$, $\overline{\mathrm{Ric}}$, and so forth.

Consider S^3 as the Lie group $\mathrm{Sp}(1)$ of unit quaternions. In the tangent space $T_1\,\mathrm{Sp}(1)$ at 1 we can choose the basis (i,j,k) of three imaginary units. Using right multiplication on $\mathrm{Sp}(1)$, these three tangent vectors extend to three global tangent fields (ξ_1, ξ_2, ξ_3) on the whole sphere. The corresponding dual basis in $T^*S^3 = \Lambda^1(S^3)$ consists of the 1-forms $(e^1,\ e^2,\ e^3)$ with $e^i(\xi_j) = \delta^i_j$, usually called *Cartan's frame.* Consider the flat conical metric

$$\bar{g} = d\tau^2 + \tau^2((e^1)^2 + (e^2)^2 + (e^3)^2) = d\tau^2 + g(\tau), \tag{1}$$

where τ plays the role of the radius r of the sphere. Since $0 \in \mathbb{R}^4$ is a singularity of the coordinate system in which we express the metric, rather than the metric itself, assume that (1) is globally defined on $\mathbb{R}^4$.

Consider the case that the sphere is embedded into $\mathbb{R}^4$ in the standard way, and show that the standard increase of its radius is described as the flow of mean curvature. Firstly calculate the second quadratic form of the hypersurface $S^3 \subset \mathbb{R}^4$. Recall that the second quadratic form B of the embedding $r : S^3 \to \mathbb{R}^4$ is defined from the equalities $r_{ij} = b_{ij}\mathbf{m} + \Gamma^k_{ij}r_k$ for $i,j,k = 1,2,3$, where $r_i = \frac{\partial r}{\partial u^i}$ and $r_{ij} = \frac{\partial^2 r}{\partial u^i \partial u^j}$ are the first and second partial derivatives of the embedding r, and $\mathbf{m}$ is the unit normal, while Γ^k_{ij} are Christoffel symbols; then $b_{ij} = \langle r_{ij}, \mathbf{m}\rangle$. We can specify a sufficiently small neighborhood of an arbitrary point of $r(S^3)$ to be the set of zeros of some function $F : \mathbb{R}^4 \to \mathbb{R}$. Then its gradient (with respect to the Euclidean metric) coincides with $\mathbf{m}$ up to multiplication by a scalar, i.e., there is an open domain of $\mathbb{R}^4$ diffeomorphic to $r(S^3) \times (-\varepsilon, \varepsilon)$. Since we consider the 3-sphere, $F(x) = |x|^2$ for $x \in \mathbb{R}^4$, while the sphere of unit radius is $S^3 = \{x \mid F(x) - 1 = 0\}$. Assume that $\tau = u^0$ and r_τ is a variation of the embedding r such that $\frac{\partial}{\partial \tau}r_\tau = r_0 = \mathbf{m}$; in this case $b_{ij} = \Gamma^0_{ij}$.

It is known [2, (IV–2)] that the Levi-Civita connection is determined by the equality

$$2g(\nabla_X Y, Z) = Xg(Y,Z) + Yg(X,Z) - Zg(X,Y) + g([X,Y],Z) + g([Z,X],Y) + g([Z,Y],X).$$

In our case, by putting $Z = \mathbf{m} = \frac{\partial}{\partial\tau} = \xi_0$, we obtain $2B(X,Y)$ on the left-hand side. Firstly calculate b_{11} for the flat metric (1)

$$2b_{11} = 2g(\nabla_{\xi_1}\xi_1, \xi_0) = -\xi_0 g(\xi_1, \xi_1) = -2\tau$$

because only one of the terms on the right-hand side does not vanish: The Lie brackets of fields on S^3 are also fields on S^3 and they remain orthogonal to the radial direction ξ_0; the fields ξ_i are independent of the radial coordinate τ, and so commute with ξ_0. Similarly we calculate the two remaining diagonal terms of B. The off-diagonal terms vanish identically because $g(\xi_i, \xi_j) = \delta_{ij}\tau^2$. Thus, we have shown that the metric $g(t)$ satisfies the flow of mean curvature:

$$(g_{ij})'_\tau = \begin{pmatrix} 2\tau & 0 & 0 \\ 0 & 2\tau & 0 \\ 0 & 0 & 2\tau \end{pmatrix} = -2b_{ij}, \quad i,j = 1,2,3.$$

Calculate the second quadratic form by Cartan's method. Recall that the connection form of a Riemannian manifold M is a skew matrix ω^i_j consisting of 1-forms such that

$$d\varepsilon^i = -\omega^i_j \wedge \varepsilon^j,$$

where $\{\varepsilon^1, \dots, \varepsilon^n\}$ is Cartan's orthonormal (co)frame, i.e., the basis of 1-forms in $T^*(M)$ dual to the basis of orthonormal tangent vector fields. In our case $\{\xi_0, \tau^{-1}\xi_1, \tau^{-1}\xi_2, \tau^{-1}\xi_3\}$ constitute an orthonormal frame with respect to $\bar{g}$. Differentiate the corresponding 1-forms:

$$d\varepsilon^0 = d(d\tau) = 0, \quad d\varepsilon^1 = d(\tau e^1) = \frac{1}{\tau}\varepsilon^0 \wedge \varepsilon^1 + \frac{2}{\tau}\varepsilon^2 \wedge \varepsilon^3,$$

$$d\varepsilon^2 = d(\tau e^2) = \frac{1}{\tau}\varepsilon^0 \wedge \varepsilon^2 + \frac{2}{\tau}\varepsilon^3 \wedge \varepsilon^1, \quad d\varepsilon^3 = d(\tau e^3) = \frac{1}{\tau}\varepsilon^0 \wedge \varepsilon^3 + \frac{2}{\tau}\varepsilon^1 \wedge \varepsilon^2.$$

Since ω^i_j is a skew matrix, we find easily that

$$-(\overline{\omega}^i_j)|_{i,j=0,\dots,3} = \frac{1}{\tau}\begin{pmatrix} 0 & \varepsilon^1 & \varepsilon^2 & \varepsilon^3 \\ -\varepsilon^1 & 0 & -\varepsilon^3 & \varepsilon^2 \\ -\varepsilon^2 & \varepsilon^3 & 0 & -\varepsilon^1 \\ -\varepsilon^3 & -\varepsilon^2 & \varepsilon^1 & 0 \end{pmatrix}.$$

Remark. We could calculate the connection form in the old basis $\{e^0, e^1, e^2, e^3\}$, but then would have to require that ω^i_j, instead of being a skew matrix in the algebra $\mathfrak{so}(n)$, must belong to the matrix algebra

$$\{A \mid AG + GA^{\mathrm{T}} = 0\},$$

where G is the matrix of $\bar{g}$ in the old basis which is not equal to the identity matrix. In other words, working with Cartan's structure equations forces us to use orthonormal frames and coframes.

Since the connection form amounts to a generalization of the Christoffel symbols, $\omega^i_j = \Gamma^i_{jk}\varepsilon^k$; we already have all we need to calculate the second quadratic form

$$b_{jk} = \Gamma^0_{jk} = \omega^0_j(\tau^{-1}\xi_k) = -\frac{1}{\tau}\delta_{jk}, \quad j,k = 1,2,3.$$

Verify again that the standard expanding sphere satisfies the flow of mean curvature in the basis $\{\varepsilon_1, \varepsilon_2, \varepsilon_3\}$:

$$(\bar{g})'_\tau = \left(\varepsilon_1^2 + \varepsilon_2^2 + \varepsilon_3^2\right)'_\tau = \left(\tau^2\left(e_1^2 + e_2^2 + e_3^2\right)\right)'_\tau = \frac{2}{\tau}\left(\varepsilon_1^2 + \varepsilon_2^2 + \varepsilon_3^2\right) = -2b.$$

Below we do not assume that the sphere is embedded into $\mathbb{R}^4$, but study the metric properties of the space looking locally as $(0,1) \times S^3$. In particular, we obtain the metric of constant curvature on $(0,1) \times S^3$ which extends uniquely to the metric on S^4.

Recall [2, (II-5)] that the curvature form Ω is the matrix consisting of the 2-forms

$$\Omega^i_j = d\omega^i_j + \omega^i_k \wedge \omega^k_j$$

and generalizing the curvature tensor; namely, $\Omega^i_j = \frac{1}{2} R^i_{jkl} \varepsilon^k \wedge \varepsilon^l$. It is not difficult to verify that for the metric $\bar{g}$ the curvature form vanishes identically; the space $\mathbb{R}^4$ under consideration is flat. Let us calculate the Ricci tensor Ric of g. The symmetries of the curvature tensor yield

$$\mathrm{Ric}_{11} = R^1_{111} + R^2_{121} + R^3_{131} = R^1_{212} + R^1_{313}.$$

To calculate the last two terms, we should calculate the curvature form Ω of g:

$$\Omega^1_2 = d\omega^1_2 + \omega^1_k \wedge \omega^k_2 = d\left(\frac{1}{\tau}\varepsilon^3\right) + \left(-\frac{1}{\tau}\varepsilon^2\right) \wedge \left(-\frac{1}{\tau}\varepsilon^1\right) = -\frac{1}{\tau^2}\varepsilon^0 \wedge \varepsilon^3$$
$$+\frac{1}{\tau}\left(\frac{1}{\tau}\varepsilon^0 \wedge \varepsilon^3 + \frac{2}{\tau}\varepsilon^1 \wedge \varepsilon^2\right) + \frac{1}{\tau^2}\varepsilon^2 \wedge \varepsilon^1 = \frac{1}{\tau^2}\varepsilon^1 \wedge \varepsilon^2 = \frac{1}{2} R^1_{212}\varepsilon^1 \wedge \varepsilon^2,$$

where $k \in \{1,2,3\}$ is the summation index. Then $\mathrm{Ric}_{11} = \frac{4}{\tau^2} = \mathrm{Ric}_{22} = \mathrm{Ric}_{33}$. It may seem here that we arrive at a contradiction because we know (see [3, 1.159] for instance) that, when the metric is multiplied by a constant λ, so does the Riemann (4,0)-tensor, the scalar curvature is multiplied by λ^{-1}, while the Ricci tensor is unchanged. We obtain all spheres from the radius 1 sphere by simple homothety, and $\lambda \equiv \tau$. In this case we infer that the Ricci tensor depends on τ. The thing is that the basis $\{\varepsilon_1, \varepsilon_2, \varepsilon_3\}$ we used to calculated the components of Ric_{ij} depends on τ too. In the equation $g'_\tau = -2\,\mathrm{Ric}$ for the Ricci flow in the chosen coordinates the left-hand side is a positive definite form, while the right-hand side is a negative definite form; i.e., as spatial coordinate τ increases, the metric must "decrease," and the corresponding sphere collapses. Thus, we see that the standard collapse of the sphere described by the spherical coordinate system is not a solution to the Ricci flow.

3. The Case of Metrics with One Functional Parameter

Consider the metric

$$\bar{g} = dt^2 + f(t)^2\left(\sum_{i=1}^{3}(e^i)^2\right) = dt^2 + g(t) = \left(\sum_{i=0}^{3}(\varepsilon^i)^2\right). \tag{2}$$

Although the 1-forms $\varepsilon^i = f(t)e^i$, for $i = 1,2,3$, depend on f, we use the previous notation. Then in the basis $\{\varepsilon^0, \varepsilon^1, \varepsilon^2, \varepsilon^3\}$ the connection form of $\bar{g}$ becomes

$$-\overline{\omega} = \frac{1}{f}\begin{pmatrix} 0 & f'\varepsilon^1 & f'\varepsilon^2 & f'\varepsilon^3 \\ -f'\varepsilon^1 & 0 & -\varepsilon^3 & \varepsilon^2 \\ -f'\varepsilon^2 & \varepsilon^3 & 0 & -\varepsilon^1 \\ -f'\varepsilon^3 & -\varepsilon^2 & \varepsilon^1 & 0 \end{pmatrix}.$$

An analogous elementary calculation of the curvature form Ω shows that $\mathrm{Ric}_{ij} = \frac{4}{f^2}\delta_{ij}$ for $i,j = 1,2,3$. In order to satisfy the Ricci flow equation, we should require that

$$(g_{ij})'_t = \frac{2f'}{f}\delta_{ij} = -\frac{8}{f^2}\delta_{ij} = -2\,\mathrm{Ric}_{ij}, \quad i,j = 1,2,3; \tag{3}$$

i.e., $f(t) = \sqrt{8(t_0 - t)}$. Here we take it into account that the basis $\{\varepsilon^1, \varepsilon^2, \varepsilon^3\}$ is orthonormal for g, and the metric is represented in this basis by the identity matrix δ_{ij}. The radius of the sphere f must depend on time t as the square root, and under the action of the Ricci flow the sphere collapses in finite time, which is a well-known property. Note also that in the chosen basis the flow equation for mean curvature is satisfied automatically: it is the first equality in (3). Therefore, the flow of mean curvature essentially becomes a tautology when we work with an invariant basis.

Calculate the curvature form of (2) for the ambient space. Now t plays the role of a spatial coordinate:

$$\overline{\Omega}^0_1 = -\frac{f''}{f}\varepsilon^0 \wedge \varepsilon^1 = \frac{1}{2}\overline{R}^0_{101}\varepsilon^0 \wedge \varepsilon^1, \quad \overline{\Omega}^1_2 = \frac{1-f'^2}{f^2}\varepsilon^1 \wedge \varepsilon^2 = \frac{1}{2}\overline{R}^1_{212}\varepsilon^1 \wedge \varepsilon^2.$$

Accordingly, the Ricci tensor becomes

$$\overline{\mathrm{Ric}}_{00} = -6\frac{f''}{f}, \quad \overline{\mathrm{Ric}}_{11} = \frac{1}{f^2}(4 - 4f'^2 - 2f''f) = \overline{\mathrm{Ric}}_{22} = \overline{\mathrm{Ric}}_{33},$$

while the scalar curvature is as follows:

$$\overline{R} = \frac{3}{f^2}(4 - 4f'^2 - 4f''f).$$

If $f(t) = f_0(t) = t$ then we obtain the flat space $\mathbb{R}^4$; if $f(t) = f_{+1}(t) = \sin(t)$ then we obtain the round sphere S^4 of radius 1; and if $f(t) = f_{-1}(t) = \sinh(t)$ then we obtain the hyperbolic space H^4.

If we consider the Ricci flow with respect to time τ then, for the appropriately chosen dependence between the coordinate t and time τ, under the action of this flow the sphere S^3 sweeps a space of constant curvature. For the flat space $\mathbb{R}^4$ we should put $\frac{dt}{d\tau} = -\frac{4}{t}$ or $\tau = h_0 = \mathrm{const} - \frac{t^2}{8}$. For the sphere S^4 we have $\tau = h_{+1} = \mathrm{const} + \frac{1}{8}\sin^2 t$. For the hyperbolic space we have $\tau = h_{-1} = \mathrm{const} - \frac{1}{8}\sinh^2 t$. This yields the following rather obvious assertion.

Proposition 1. *Suppose that the round sphere $(S^3, g(\tau))$ satisfies the Ricci flow*

$$\frac{\partial}{\partial\tau}g(\tau) = -2\,\mathrm{Ric}\,. \tag{4}$$

If $\tau = h_K(t)$, where the function h_K is described above, then

$$\bar{g} = dt^2 + g(\tau(t)) = dt^2 + f^2(t)\bigg(\sum_{i=1}^{3}(e^i)^2\bigg)$$

is a metric of constant curvature $K \in \{-1, 0, +1\}$.

This is obvious indeed: Consider the Ricci flow as a procedure changing in time the radius of the sphere S^3. Varying the rate of change, we can make the space swept by the sphere to be of constant curvature. Express the left-hand side of the Ricci flow in the basis $\{\varepsilon^1, \varepsilon^2, \varepsilon^3\}$:

$$\frac{\partial}{\partial\tau}g(\tau(t)) = \frac{\partial}{\partial\tau}\bigg(\sum_{i=1}^{3}(\varepsilon^i)^2\bigg) = \frac{\partial}{\partial t}\bigg[f^2(t)\bigg(\sum_{i=1}^{3}(e^i)^2\bigg)\bigg]\frac{\partial t}{\partial\tau} = 2ff'_t\bigg(\sum_{i=1}^{3}(e^i)^2\bigg)\frac{\partial t}{\partial\tau} = \frac{2f'_t}{f}\bigg(\sum_{i=1}^{3}(\varepsilon^i)^2\bigg)\frac{\partial t}{\partial\tau}.$$

As calculated above, the right-hand side equals

$$\mathrm{Ric}_{ij} = \frac{4}{f^2}\delta_{ij} = \frac{4}{f^2}\bigg(\sum_{i=1}^{3}(\varepsilon^i)^2\bigg).$$

Thus, we arrive at the scalar equation

$$\frac{2f'_t}{f}\frac{\partial t}{\partial\tau} = -\frac{8}{f^2}.$$

In the case $K = 0$ the hypotheses yield $\frac{\partial t}{\partial\tau} = -\frac{4}{t}$, and this equation reduces to $ff'_t = t$, whose solution is the affine function $f_0(t) = t + \mathrm{const}$. We can take the integration constant to be zero, and $\bar{g}$ is the metric of the flat $\mathbb{R}^4$. Similar arguments apply to $K \in \{-1, +1\}$. The proof of Proposition 1 is complete.

On the other hand, the following theorem holds:

Theorem 1. *Suppose that the round sphere $(S^3, g(t))$ of radius $f(t)$ changes under the action of the flow*

$$\frac{\partial}{\partial t} g = \sqrt{\operatorname{Ric} - 4Kg}. \tag{5}$$

Then the metric $\bar{g} = dt^2 + g(t)$ is isometric to the metric of a space of constant curvature $K \in \{-1, 0, +1\}$.

Proof. Demonstration reduces to expressing (5) in our particular case of a metric conformally equivalent to the standard round metric on the sphere. In the chosen orthonormal basis

$$g = (\varepsilon^1)^2 + (\varepsilon^2)^2 + (\varepsilon^3)^2 = f(t)^2((e^1)^2 + (e^2)^2 + (e^3)^2).$$

Then

$$\frac{\partial g}{\partial t} = 2f(t)f'(t)((e^1)^2 + (e^2)^2 + (e^3)^2) = \frac{2f'(t)}{f(t)}((\varepsilon^1)^2 + (\varepsilon^2)^2 + (\varepsilon^3)^2).$$

Similarly, (3) yields $\operatorname{Ric} = \frac{4}{f(t)^2}((\varepsilon^1)^2 + (\varepsilon^2)^2 + (\varepsilon^3)^2)$. Thus, (5) reduces to the simple scalar equation

$$2\frac{f'}{f} = \sqrt{\frac{4}{f^2} - 4K}.$$

With the suitable initial data $f(0) = 0$ and $f'(0) = 1$ this equation obviously has the solutions $f_- = \sinh(t)$ for $K = -1$, while $f_0 = t$ for $K = 0$ and $f_+ = \sin(t)$ for $K = +1$. The corresponding metrics g have curvature K. The proof of the theorem is complete.

Let us explain our notation. Since the Ricci tensor is the Laplacian of the metric modulo lower order terms, the flow under consideration is in fact the Dirac equation on the space of metrics. Recall that the Dirac equation is

$$\frac{\partial}{\partial t}\Psi = \sqrt{\Delta - U}\Psi,$$

where Ψ is a section of the spinor bundle. Consider the right-hand side of (5) as a pseudodifferential operator of order 1 on the space of metrics. We can see formal analogies between the heat equation and Ricci flow, the Schrödinger equation and normalized Ricci flow, and so we claim that it is rather natural to study (5). Since (5) is a junction of the theory of geometric flows and the theory of Dirac operators, we call it the *Dirac flow*.

We express (5) in the coordinate-free form for the simple reason: It is known from linear algebra that each symmetric matrix (for instance, the matrix $\operatorname{Ric}_{ij} - Kg_{ij}$) has real eigenvalues and a basis of pairwise orthogonal eigenvectors. Thus, Jordan's theory implies that the matrix square root defined on the space of symmetric matrices can always be reduced to the square root on the space of diagonal matrices. Consequently, fixing an orthonormal basis, we can always pass from the tensor form of (5) to coordinates. The questions remain of global definition of the right-hand side, the choice of branch for the square root, and extension to the complex domain, but we will avoid them here.

4. The Case of Metrics with Two Functional Parameters

Consider now the metric

$$\bar{g} = dt^2 + A_1^2(t)(e^1)^2 + A_2^2(t)((e^2)^2 + (e^3)^2) = dt^2 + g(t). \tag{6}$$

As above, introduce the coframe

$$\varepsilon^1 = A_1 e^1, \quad \varepsilon^2 = A_2 e^2, \quad \varepsilon^3 = A_2 e^3$$

orthonormal with respect to g. The connection form becomes

$$-\overline{\omega} = \begin{pmatrix} 0 & \frac{A_1'}{A_1}\varepsilon^1 & \frac{A_2'}{A_2}\varepsilon^2 & \frac{A_2'}{A_2}\varepsilon^3 \\ -\frac{A_1'}{A_1}\varepsilon^1 & 0 & \frac{-A_1}{A_2^2}\varepsilon^3 & \frac{A_1}{A_2^2}\varepsilon^2 \\ -\frac{A_2'}{A_2}\varepsilon^2 & \frac{A_1}{A_2^2}\varepsilon^3 & 0 & \frac{A_1^2 - 2A_2^2}{A_1 A_2^2}\varepsilon^1 \\ -\frac{A_2'}{A_2}\varepsilon^3 & \frac{-A_1}{A_2^2}\varepsilon^2 & -\frac{A_1^2 - 2A_2^2}{A_1 A_2^2}\varepsilon^1 & 0 \end{pmatrix}.$$

The curvature form is

$$\overline{\Omega}_1^0 = \varepsilon^0 \wedge \varepsilon^1 \left[-\frac{A_1''}{A_1} \right] + \varepsilon^2 \wedge \varepsilon^3 \left[-\frac{2A_1'}{A_2^2} + \frac{2A_1A_2'}{A_2^3} \right],$$

$$\overline{\Omega}_2^0 = \varepsilon^0 \wedge \varepsilon^2 \left[-\frac{A_2''}{A_2} \right] + \varepsilon^3 \wedge \varepsilon^1 \left[\frac{A_1'}{A_2^2} - \frac{A_1A_2'}{A_2^3} \right],$$

$$\overline{\Omega}_2^1 = \varepsilon^0 \wedge \varepsilon^3 \left[\frac{A_1'}{A_2^2} - \frac{A_1A_2'}{A_2^3} \right] + \varepsilon^1 \wedge \varepsilon^2 \left[\frac{A_1^2}{A_2^4} - \frac{A_1'A_2'}{A_1A_2} \right],$$

$$\overline{\Omega}_3^2 = \varepsilon^0 \wedge \varepsilon^1 \left[\frac{2A_1A_2'}{A_2^3} - \frac{2A_1'}{A_2^2} \right] + \varepsilon^2 \wedge \varepsilon^3 \left[\frac{4}{A_2^2} - \frac{3A_1^2}{A_2^4} - \frac{A_2'^2}{A_2^2} \right],$$

and we can calculate the remaining components similarly. Observe that for (6) the curvature tensor $\overline{R}^i_{jkl}$ is not diagonal; by a diagonal tensor we understand a curvature tensor with only the sectional curvatures $\overline{R}^i_{jij}$ not vanishing. Then the Ricci tensor of g has two nontrivial components: $\mathrm{Ric}_{11} = R^1_{212} + R^1_{313} = 4\frac{A_1^2}{A_2^4}$ and $\mathrm{Ric}_{22} = \mathrm{Ric}_{33} = R^1_{212} + R^3_{232} = \frac{4}{A_2^2}\big(2 - \frac{A_1^2}{A_2^2}\big)$. Assume moreover that at each fixed time t the metric g depends on A_1 and A_2 as constants; therefore, the components R^i_{jkl} of the curvature tensor are independent of the derivatives of A_1 and A_2.

The Ricci flow reduces to the system

$$\begin{cases} \frac{A_1'}{A_1} = -4\frac{A_1^2}{A_2^4}, \\ \frac{A_2'}{A_2} = -\frac{4}{A_2^2}\Big(2 - \frac{A_1^2}{A_2^2}\Big). \end{cases} \tag{7}$$

Put $\alpha = A_1^2$ and $\beta = A_2^2$. The system becomes

$$\begin{cases} \alpha' = -8\frac{\alpha^2}{\beta^2}, \\ \beta' + 16 = \frac{8\alpha}{\beta}. \end{cases} \tag{8}$$

It has two obvious solutions. The first corresponds to the case

$$\alpha = \beta = 8(t_0 - t) \tag{9}$$

considered already, and the second solution

$$\alpha = 0, \quad \beta = 16(t_0 - t), \tag{10}$$

to the collapse of the 2-sphere $S^2 = S^3/S^1$ when the fiber S^1 of the Hopf fibration collapses identically. In fact, the second solution determines the metric (6) on the three-dimensional space.

Remark. Solution (9) corresponds to a collapsing round 3-sphere. Gibbons and Hawking called [4] a solution with this singularity a “nut,” since in this case the whole 3-sphere collapses. Another resolution of the cone singularity was called a “bolt.” The bolt reminds us clearly of the cylinder $S^1 \times \mathbb{R}$ one of whose coordinates is defined on the circle S^1 of small radius, while the second coordinate is extended, i.e., defined on $\mathbb{R}$. The second resolution of singularity amounts to collapsing the circle S^1 in the Hopf fibration multiplied by the 2-sphere of radius bounded below. In other words, the 2-sphere hosts the extended coordinate, while the one-dimensional circle S^1 (section of the bolt) collapses. In our situation solution (10) is degenerate because the one-dimensional circle collapses identically, $\alpha \equiv 0$, and the 2-sphere also collapses eventually, $\beta = 16(t_0 - t)$.

System (8) is completely integrable. From the second equation we have $\alpha = \frac{1}{8}\beta(\beta' + 16)$; inserting this into the first equation yields

$$\beta\beta'' + 2\beta'^2 + 48\beta' + 256 = 0.$$

The general solution to this differential equation is

$$-\frac{1}{16}\beta - \frac{\sqrt{2}}{128}c_1 \operatorname{arctg}\left(\frac{4\sqrt{2}\beta}{\sqrt{c_1^2 - 32\beta^2}}\right) = t + c_2, \tag{11}$$

where c_1 and c_2 are constants of integration. When the right-hand side of (11) tends to 0 from below, the left-hand side tends to 0 from above. Furthermore, we have

$$-\frac{c_1 + \sqrt{c_1^2 - 32\beta^2}}{\sqrt{c_1^2 - 32\beta^2}} \cdot \frac{\beta'}{16} = 1,$$

i.e., $\beta' \to -8$ and $\alpha' = \frac{\beta'}{8}(\beta' + 16) + \frac{\beta}{8}\beta'' = \frac{\beta'}{8}(\beta' + 16) + \frac{\beta}{8}(\frac{-1}{\beta})(256 + 48\beta' + 2\beta'^2) \to -8$ as $t \downarrow -c_2 = t_0$. In other words, we recover the well-known property: Under the action of Ricci flow, a not round 3-sphere collapses to a point in finite time (as a round sphere of infinitesimal radius). We may assert that the solution (10) is "nonperturbative"; i.e.,

$$\lim_{t \to t_0} \alpha'(t) = \lim_{t \to t_0} \beta'(t) = -8$$

independently of how small $\alpha(0) \neq 0$ is. This agrees with the general theory: When initial data are not too bad, the Ricci flow makes curvature "uniformly distributed" over all points on the manifold and all tangent area elements; i.e., under the action of Ricci flow the sphere collapses completely, the Ricci flow cannot decrease only the one-dimensional fiber of the Hopf fibration, keeping the two-dimensional base of radius bounded below. Hence, we conclude that using a flow differing slightly from the Ricci flow, we are unlikely to describe metrics with resolution of a conical singularity of the type "bolt."

Let us study the qualitative behavior of solutions. If $\beta(0) > \alpha(0)$ then at the start the right-hand side of (7) is sufficiently small, and α behaves as a constant, while β decreases as $-16t$. When the radius $A_2(t)$ of the sphere becomes sufficiently close to the radius $A_1(t)$ of the circle, they merge in one solution (9). If $\alpha(0) > \beta(0)$ then, since the right-hand side of the first equation is a negative number of large absolute value, rather quickly α becomes equal to β and they merge. It is not difficult to deduce this behavior directly from (11).

Remark. In simulations, due to calculation errors the solution to (7) sometimes (i.e., not in all simulations) extends beyond the time t_0 of singularity. The function α extends identically by zero, while the function β becomes equal to $16(t_0 - t)$, i.e., it enters the negative domain, and (6) ceases to be a Riemannian metric; moreover, β ceases to be a metric on a 4-manifold. Nevertheless, we may assert that (9) becomes (10) passing through the singularity. It is unclear whether we can find a meaningful interpretation for this effect or it is just an error of simulations.

Observe also that the above-mentioned transition from (9) to (10) has nothing to do with the resolution of singularity using the normalization of Ricci flow, at least because in this transition the dimension of the manifold drops. Recall that the normalized Ricci flow is of the form

$$\frac{\partial}{\partial t} g_{ij} = -2\operatorname{Ric}_{ij} + \frac{2}{n}\mathfrak{R} g_{ij},$$

where n is the dimension of the manifold, and $\mathfrak{R}$ is the average scalar curvature. In our case the normalized flow becomes

$$A_1' = -\frac{16}{3}\frac{A_1}{A_2^4}(A_1^2 - A_2^2), \quad A_2' = \frac{8}{3} \cdot \frac{A_1^2 - A_2^2}{A_2^3}$$

because the functions A_1 and A_2 depend only on time, and it is unnecessary to average curvature over the sphere. We can easily solve these equations, especially if we recall that the volume $\sqrt{\det(g)} = A_1 A_2^2$ is preserved under the action of the normalized Ricci flow. It is clear from the right-hand sides of this system that both functions tend to the same constant as $t \to \infty$; i.e., at infinite time we again obtain the round sphere.

Recall that a metric is called *anti-self-dual* whenever its connection form satisfies $\omega^i_j = -\frac{1}{2}\varepsilon_{ijkl}\omega^k_l$, where ε_{ijkl} is the antisymmetric Levi-Civita symbol. For anti-self-dual metrics the Ricci tensor vanishes automatically, as for self-dual metrics. In the four-dimensional case, anti-self-duality reduces to the pair of equations $\omega^0_1 = -\omega^2_3$ and $\omega^0_2 = \omega^1_3$, or in our case

$$\frac{A_1'}{A_1} = -\frac{A_1^2 - 2A_2^2}{A_1 A_2^2}, \quad \frac{A_2'}{A_2} = \frac{A_1}{A_2^2}. \tag{12}$$

It is well-known that (12) can be integrated in the general case: This is how the classical Eguchi–Hanson metric

$$ds^2 = [1-(a/r)^4]^{-1}dr^2 + r^2((e^2)^2 + (e^3)^2) + r^2[1-(a/r)^4](e^1)^2$$

was found [5]. It was the first metric with the holonomy group $SU(2)$ expressed in elementary functions. As r tends to a, the Eguchi–Hanson metric has singularity of the type "bolt," while for sufficiently large R the set $\{r = R\}$ is homeomorphic to $\mathbb{R}P^3$.

Even though the right-hand sides of (12) are expressed in terms of the components of the connection form, we can also express them in terms of the components of the Ricci tensor:

$$-\frac{A_1^2 - 2A_2^2}{A_1 A_2^2} = \frac{1}{2}\mathrm{Ric}_{22}(\mathrm{Ric}_{11})^{-1/2}, \quad \frac{A_1}{A_2^2} = \frac{1}{2}(\mathrm{Ric}_{11})^{1/2}.$$

Thus, (12) is equivalent to the flow

$$\frac{\partial}{\partial t} g = \frac{1}{2}\sqrt{\det(\mathrm{Ric})}\mathrm{Ric}^{-1}. \tag{13}$$

This flow is defined on the 3-sphere, and there is no need to specify it on the fibers S^1 and base S^2 of the Hopf fibration. Unfortunately, the resulting flow has an extremely unpleasant right-hand side. This confirms our guess that metrics with singularity of type "bolt" cannot be described by geometric flows with right-hand sides depending on the Ricci tensor in a certain "good" fashion. For $A_1 = A_2$ the function $f(t)$ above is a solution, while the corresponding metric g is the metric of a flat space. This is straightforward from the anti-self-duality of the metric of flat space.

Theorem 2. *Suppose that the projective space $\mathbb{R}P^3 = S^3/\mathbb{Z}_2$ equipped with the metric*

$$A_1^2(t)(e^1)^2 + A_2^2(t)((e^2)^2 + (e^3)^2)$$

changes under the action of (13)*. If $A_1(0) = 0$ and $A_2(0) = a$ then the corresponding metric* (6) *is isometric to the Eguchi–Hanson metric.*

Proof. We showed above that (12) is equivalent to (13). The Eguchi–Hanson metric was found [5] by integrating (12). It remains to understand that this system does not have solutions other than the Eguchi–Hanson metric. This follows from counting the free parameters. The general theory of ordinary differential equations implies that the solution set to (12) is parameterized by two quantities, namely $A_1(0)$ and $A_2(0)$. Fixing initial data as in the statement of the theorem, require that for $t = 0$ the circle in the Hopf fibration collapse, while the 2-sphere be of radius a. The variables r and t are related as

$$\frac{r^2 dr}{\sqrt{r^4 - a^4}} = dt.$$

Assume that the constant of integration vanishes. Obviously, the variable r is then defined on the segment $[a, \infty)$; while the variable t, on the segment $[0, \infty)$. It remains to observe from the form of the metric that $A_2(r) = r$ and $A_1(r) = r^2[1 - (a/r)^4]$, and for $r = a$ the initial conditions are satisfied.

The proof of the theorem is complete.

Since (13) has a highly nonlinear right-hand side, it would be natural to consider an evolution equation defined on some structures agreeing with the Hopf fibration. Recall that a 1-form on a manifold of dimension $2n+1$ is called *contact* whenever $\psi \wedge (d\psi)^n \neq 0$. We can verify that

$$\varepsilon^1 \wedge d\varepsilon^1 = 2\frac{A_1}{A_2^2}\varepsilon^1 \wedge \varepsilon^2 \wedge \varepsilon^3,$$

where $d : \Lambda^i(S^3) \to \Lambda^{i+1}(S^3)$. Similarly,

$$\varepsilon^2 \wedge d\varepsilon^2 = 2\frac{1}{A_1}\varepsilon^1 \wedge \varepsilon^2 \wedge \varepsilon^3.$$

Considering the equation

$$(*\psi)' = d\psi, \tag{14}$$

where $*$ is the Hodge star operator with respect to the orthonormal frame $\{\varepsilon^1, \varepsilon^2, \varepsilon^3\}$, we precisely obtain for $\psi = \varepsilon^1$ the second equation of (12), and for $\psi = \varepsilon^2$

$$\left(\frac{A_1'}{A_1} + \frac{A_2'}{A_2}\right)\varepsilon^3 \wedge \varepsilon^1 = \frac{2}{A_1}\varepsilon^3 \wedge \varepsilon^1.$$

Taking it into account the equation for $\psi = \varepsilon^1$, we deduce from this the first equation of (12). It is obvious that inserting $\psi = \varepsilon^3$ into (14) yields the same equation. Thus, we can state the following assertion.

Theorem 3. *Consider the projective space $\mathbb{R}P^3 = S^3/\mathbb{Z}_2$ with the metric*

$$A_1^2(t)(e^1)^2 + A_2^2(t)((e^2)^2 + (e^3)^2) = (\varepsilon^1)^2 + (\varepsilon^2)^2 + (\varepsilon^3)^2$$

generated by three contact 1-forms $\varepsilon^1 = A_1(t)e^1$, $\varepsilon^2 = A_2(t)e^2$, and $\varepsilon^3 = A_2(t)e^3$ satisfying the flow (14). Then for suitable initial conditions the metric $dt^2 + A_1^2(t)(e^1)^2 + A_2^2(t)((e^2)^2 + (e^3)^2)$ is isometric to the Eguchi–Hanson metric.

Proof. Demonstration of this theorem is similar to the proof of the previous theorem, with (14) replacing (13).

Remark on a relation to the holonomy group G_2. We must mention that the flow (14) was introduced by Hitchin (see [6] for instance) with the goal of obtaining an evolution equation on the 3-form ϕ_0 determining a G_2-structure on a 7-manifold. He showed that for sufficiently small time t the solution ϕ_t to (14) determines a G_2-structure.

Observe also that trivial changes of variables reduce (12) to a system written down in [7], where some metrics with holonomy group G_2 were constructed on deformations of cones over the twistor space of a 3-Sasakian 7-manifold.

Verify now that the Dirac flow has meaningful solutions not only in the case of metrics depending only on one conformal factor $f(t)$. In the case (6), when g depends on the two functional parameters A_1 and A_2, the Dirac flow equations become

$$A_1' = \frac{A_1}{A_2^2}\sqrt{A_1^2 - KA_2^4}, \quad A_2' = \sqrt{2 - \frac{A_1^2}{A_2^2} - KA_2^2}. \tag{15}$$

Choose the standard branches of the square root function and study (15) for $K = 0$:

$$A_1' = \frac{A_1^2}{A_2^2}, \quad A_2' = \sqrt{2 - \frac{A_1^2}{A_2^2}}. \tag{16}$$

Let us show that as $T \to \infty$ the sectional curvature of $\bar{g}$ vanishes in (T, ∞) for $K = 0$. This would follow from the property that the metric $\bar{g}$ is asymptotically conical. For $K = \pm 1$, the technique below fails to help us analyze the behavior of solutions of the flow exactly.

Observe that if $K = 0$ then (16) is invariant under the substitution

$$A_1 \to \lambda A_1, \quad A_2 \to \lambda A_2, \quad t \to \tau,$$

where τ is the new time related to the old time by $dt = \lambda d\tau$. With this remark, we can split our system into the two parts: normal and tangential. This method was used substantially in [7, 8]. Rearrange the system using the notation

$$\frac{dR}{dt} = V(R),$$

where $R = (A_1, A_2)$ and $V(R) = \big(\frac{A_1^2}{A_2^2}, \sqrt{2 - \frac{A_1^2}{A_2^2}}\big)$. Then the tangential part becomes

$$\frac{dS}{d\tau} = V(S) - \langle V(S), S\rangle S = W(S),$$

where $S = (\alpha_1(\tau), \alpha_2(\tau)) = \frac{1}{\lambda}R$ is the projection of $R = (A_1(t), A_2(t))$ to the unit circle. The normal part becomes

$$\frac{1}{\lambda}\frac{d\lambda}{d\tau} = \langle V(S), S\rangle\, dt = \lambda\, d\tau.$$

The zeros of $W(S)$ determine the asymptotics at infinity of the original system. Assume firstly that $\alpha_2 \neq 0$. Formally, the right-hand side of (16) is undefined for $A_2 = 0$; nevertheless, we can choose the convergence of A_1 and A_2 to a point p so that the right-hand side vanishes. These points p for more complicated systems were called *conditionally stationary* [7]. The remaining zeros of $W(S)$ are called *stationary points.*

Lemma. *All stationary points of* (16) *are* $\big(\pm\frac{\sqrt{2}}{2}, \pm\frac{\sqrt{2}}{2}\big)$ *and* $\pm(0, 1)$.

Proof. Demonstration relies on easy algebraic calculations. Express $W(S)$ in coordinates:

$$W(\alpha_1, \alpha_2) = \left(\frac{\alpha_1^2}{\alpha_2^2}, \sqrt{2 - \frac{\alpha_1^2}{\alpha_2^2}}\right) - \left(\frac{\alpha_1^2}{\alpha_2^2}\alpha_1 + \sqrt{2 - \frac{\alpha_1^2}{\alpha_2^2}}\alpha_2\right)(\alpha_1, \alpha_2).$$

Under the condition $\alpha_2 \neq 0$, the zeros of $W(S)$ satisfy

$$\begin{cases} \alpha_1\big(\alpha_1 - \alpha_1^3 - \alpha_2^2\sqrt{2\alpha_2^2 - \alpha_1^2}\big) = 0, \\ \sqrt{2\alpha_2^2 - \alpha_1^2}\big(1 - \alpha_2^2\big) - \alpha_1^3 = 0. \end{cases}$$

It is not difficult to observe that the zeros of this system are listed in the statement.

Theorem 4. *On the 3-sphere consider the metric*

$$\bar{g} = dt^2 + A_1^2(t)(e^1)^2 + A_2^2(t)((e^2)^2 + (e^3)^2) = dt^2 + g(t)$$

such that the metric $g(t)$ is a solution to the Dirac flow with $K = 0$ and initial data $A_2(0) > A_1(0) > 0$. Then $\bar{g}$ is a complete asymptotically conical metric.

Proof. Demonstration begins with taking the sector $\{(A_1, A_2) \in \mathbb{R}^2 \mid A_2 > A_1 > 0\}$. Its boundary consists clearly of two invariant trajectories of the system. The first trajectory $A_1(t) = A_2(t) = t$ corresponds to the case of flat metric on $\mathbb{R}^4$ considered above. The second trajectory $A_1(t) \equiv 0$ is the degenerate three-dimensional flat metric $dt^2 + 2t^2((e^2)^2 + (e^3)^2)$ with singularity. The structure of trajectories of (16) in the sector is as follows: The point with coordinates $(A_1(t), A_2(t))$ moves away from the origin because the right-hand side of (16) is strictly positive in the sector. The vector field W on the circular arc is directed from $\big(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\big)$ to $(0, 1)$. This is easy to verify, for instance, by inserting the values $\alpha_1 = \frac{3}{5}$ and $\alpha_2 = \frac{4}{5}$ into the first coordinate of $W(S)$:

$$W_1\left(\frac{3}{5}, \frac{4}{5}\right) = \frac{9}{25} - \frac{3}{25}\sqrt{23} < 0.$$

Therefore, the projection (α_1, α_2) of (A_1, A_2) moves from $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$ to $(0, 1)$. Therefore, the trajectory (A_1, A_2) itself starts at the origin with the initial velocity vector $(1, 1)$. This ensures the absence of singularity at the start and completeness of the metric. Furthermore, at infinity we have

$$A_1' = \left(\frac{A_1}{A_2}\right)^2 = \left(\frac{\alpha_1}{\alpha_2}\right)^2 \to 0, \quad A_2' \to \sqrt{2}.$$

Consequently, at infinity we can approximate the function A_1 by a constant and the function A_2, by a linear function:

$$A_1(t) = \mathrm{const} + B_1(t), \quad A_2(t) = \sqrt{2}t + B_2(t),$$

where $\frac{B_1(t)}{t} \to 0$ and $\frac{B_2(t)}{t} \to 0$ as $t \to \infty$. Thus, the metric is asymptotically conical.

The proof of the theorem is complete.

It is not difficult to see that all coefficients of the components $\overline{\Omega}_j^i$ of the curvature form decrease at least as fast as $\frac{1}{t^2}$. Hence, for $K = 0$ the Dirac flow defined above deforms the metric depending on two functional parameters on the 3-sphere so that the part of the four-dimensional space swept by this sphere has no singularities and for sufficiently large $t \in (T, \infty)$ has curvature at most $\frac{1}{T^2}$.

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