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# FORMAL MATRIX RINGS AND THEIR ISOMORPHISMS

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**Abstract:** We study the isomorphism problem for formal matrix rings and obtain the description of semiartinian formal matrix rings and the max-rings of formal matrices.

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**Keywords:** formal matrix ring, semiartinian ring, perfect ring, max-ring

## § 1. Introduction

Formal matrix rings play a prominent role in ring and module theory. The Morita context rings form an important class of formal matrix rings (for example, see [1]). The triangular formal matrix rings appear frequently in the representation theory of artinian algebras and serve as a source of examples of rings with asymmetrical properties (for example, a right artinian but not left artinian ring and so on).

Each ring with nontrivial idempotents is isomorphic to a formal matrix ring. The endomorphism ring of a decomposable module is a formal matrix ring as well. This justifies the study of formal matrix rings.

In § 2 and § 3, we introduce the main definitions and facts on formal matrix rings. In § 4, we obtain description of right semiartinian formal matrix rings and the right max-rings of formal matrices. As a corollary we get description of right perfect formal matrix rings. In § 5, we study the isomorphism problem for the formal matrix rings $\mathbb{M}_{\beta_1,\dots,\beta_n}(R)$. Using the main theorems of § 4, we deduce the available results of [2–4] that are connected with the isomorphism problem for formal matrix rings over $R$.

## § 2. The Main Definitions

All rings are assumed associative and unital, while all modules and bimodules are assumed unital. Denote the Jacobson radical, the center, the set of zero divisors, and the group of invertible elements of a ring $R$ by $J(R)$, $C(R)$, $Z(R)$, and $U(R)$ respectively.

Let $R_1, R_2, \dots, R_n$ be some rings, and let $M_{ij}$ be some $(R_i, R_j)$-bimodules such that $M_{ii} = R_i$ for all $1 \le i, j \le n$. Let $\varphi_{ijk} : M_{ij} \otimes_{R_j} M_{jk} \to M_{ik}$ be the $(R_i, R_k)$-bimodule homomorphisms such that $\varphi_{iij}$ and $\varphi_{ijj}$ are canonical isomorphisms for all $1 \le i, j \le n$. Put $a \circ b = \varphi_{ijk}(a \otimes b)$ for $a \in M_{ij}$ and $b \in M_{jk}$. Denote by $K$ the set of all $(n \times n)$-matrices $(m_{ij})$ with $m_{ij} \in M_{ij}$ for all $1 \le i, j \le n$. Simple verification shows that $K$ is a ring under the usual operations of addition and multiplication if and only if $a \circ (b \circ c) = (a \circ b) \circ c$ for all $a \in M_{ik}$, $b \in M_{kl}$, and $c \in M_{lj}$, $1 \le i, k, l, j \le n$. The ring $K$ is a *formal matrix ring* of order $n$, and $K$ is denoted by $K(\{M_{ij}\} : \{\varphi_{ikj}\})$.

Inspecting modules over formal matrix rings, it suffices to consider the matrices of order 2, since a formal matrix ring of order $n$ reduces obviously to a formal matrix ring of order 2. Let $K = \begin{pmatrix} R & M \\ N & S \end{pmatrix}$ be a formal matrix ring of order 2. Let $X$ be a right $R$-module, let $Y$ be a right $S$-module, and assume that an $R$-module homomorphism $f : Y \otimes_S N \to X$ and an $S$-module homomorphism $g : X \otimes_R M \to Y$ are defined. Put $yn := f(y \otimes n)$, $xm := g(x \otimes m)$, and require the identities $(yn)m = y(nm)$ and $(xm)n = x(mn)$ for all $x \in X$, $y \in Y$, $m \in M$, and $n \in N$. In this case the group $(X, Y)$ of row vectors is naturally equipped with the structure of a right $K$-module. It is easy to show that each right $K$-module may be represented as a module of row vectors. Every homomorphism of $K$-modules may be represented as a pair of an $R$-homomorphism and an $S$-homomorphism. Namely, if $\Gamma : (X, Y) \to (X', Y')$

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is a homomorphism then there are an $R$-homomorphism $\alpha : X \to X'$ and an $S$-homomorphism $\beta : Y \to Y'$ such that $\Gamma(x, y) = (\alpha(x), \beta(y))$. Furthermore, $\alpha(yn) = \beta(y)n$ and $\beta(xm) = \alpha(x)m$ for all $x \in X$, $y \in Y$, $m \in M$, and $n \in N$.

DEFINITION 2.1. The formal matrix ring $K(\{M_{ij}\} : \{\varphi_{ikj}\})$ of order $n$ such that $M_{ij} = R$ for all $1 \le i, j \le n$ is a *formal matrix ring of order $n$ over $R$* denoted simply by $K_n(R)$ or $K_n(R : \{\varphi_{ikj}\})$.

Let $K_n(R : \{\varphi_{ijk}\})$ be the formal matrix ring of order $n$ over $R$. Put $\eta_{ijk} = \varphi_{ijk}(1 \otimes 1)$ for all $1 \le i, j, k \le n$. Then $a \circ b = \varphi_{ijk}(a \otimes b) = \eta_{ijk}ab$ for all $a, b \in R$. Given $a \in R$, we have $a\eta_{ijk} = \varphi_{ijk}(a \otimes 1) = \varphi_{ijk}(1 \otimes a) = \eta_{ijk}a$. Thus, $\eta_{ijk} \in C(R)$, and the conditions hold:

(1) $\eta_{iij} = \eta_{ijj} = 1$, $1 \le i, j \le n$,

(2) $\eta_{ijk}\eta_{ikl} = \eta_{ijl}\eta_{jkl}$, $1 \le i, j, k, l \le n$.

Condition (1) holds, since $\varphi_{iij}$ and $\varphi_{ijj}$ are canonical isomorphisms. By the associativity of $\circ$ we have $\eta_{ijk}\eta_{ikl}abc = \eta_{ijl}\eta_{jkl}abc$ for all $a, b, c \in R$. Putting $a = b = c = 1$ we get (2).

Simultaneously, for every set $\{\eta_{ijk} \mid 1 \le i, j, k \le n\}$ of central elements in $R$ satisfying conditions (1) and (2) we may put $\varphi_{ijk}(a \otimes b) = \eta_{ijk}ab$ for all $a, b \in R$. It is immediate that $K_n(R : \{\varphi_{ikj}\})$ is a formal matrix ring of order $n$ over $R$. Thus, the formal matrix ring $K_n(R : \{\varphi_{ikj}\})$ is uniquely defined by the set of central elements $\{\eta_{ijk} \mid 1 \le i, j, k \le n\}$. In this case, the formal matrix ring $K_n(R : \{\varphi_{ikj}\})$ is denoted by $K_n(R : \{\eta_{ikj}\})$.

REMARK 2.2. Given $\beta \in C(R)$, the ordered tetrad $\begin{pmatrix} R & R \\ R & R \end{pmatrix}$ is a ring if we imply the elementwise addition and the multiplication acting by the rule:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} ae + \beta bg & af + bh \\ ce + dg & \beta cf + dh \end{pmatrix}.$$

Krylov denoted the so-obtained ring by $K_\beta(R)$ in [3], and this ring was studied in [3–6].

However, even for the matrix ring $K_3(R : \{\eta_{ijk}\})$ of order 3 the problem of parametrization of multiplication becomes more difficult. It is immediate that

$$\eta_{123}\eta_{213} = \eta_{121} = \eta_{212} = \eta_{312}\eta_{321};$$
$$\eta_{321}\eta_{231} = \eta_{323} = \eta_{232} = \eta_{132}\eta_{123};$$
$$\eta_{132}\eta_{123} = \eta_{131} = \eta_{313} = \eta_{213}\eta_{231}.$$

But we may require presence of the equalities $\eta_{213} = \eta_{312}$, $\eta_{123} = \eta_{321}$, and $\eta_{132} = \eta_{231}$. In this case, all relations above may be written in simpler form.

Given $\beta_1, \dots, \beta_n \in C(R)$, we define $\eta_{ijk}$ for all $1 \le i, j, k \le n$ by the formula

$$\eta_{ijk} = \begin{cases} 1, & \text{if } i = j \text{ or } j = k, \\ \beta_j, & \text{if } i, j, \text{ and } k \text{ are distinct}, \\ \beta_i\beta_j, & \text{if } i = k \ne j. \end{cases}$$

It is immediate that $\{\eta_{ijk} \mid 1 \le i, j, k \le n\}$ satisfy conditions (1) and (2); therefore, it defines a formal matrix ring of order $n$ over $R$.

DEFINITION 2.3. Let $R$ be a ring and $\beta_1, \dots, \beta_n \in C(R)$, $n \ge 2$. Assume that $\eta_{ijk}$ are defined as above. The formal matrix ring $K_n(R : \{\varphi_{ijk}\})$, defined by the set $\{\eta_{ijk}\}$, is a *formal matrix ring depending on* $\beta_1, \dots, \beta_n$; which is denoted by $\mathbb{M}_{\beta_1,\dots,\beta_n}(R)$.

Thus, $\mathbb{M}_{\beta_1,\dots,\beta_n}(R)$ is the set of all matrices of order $n$ over $R$ with the usual addition and multiplication as follows:

$$(a_{ij})(b_{ij}) = (c_{ij}), \quad c_{ij} = \sum_{k=1}^{n} \beta_i^{\delta_{ij}-\delta_{ik}} \beta_k^{1-\delta_{jk}} a_{ik}b_{kj},$$

where $(a_{ij})$ and $(b_{ij})$ are two matrices of order $n$ over $R$. For example,

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} a_{11}b_{11} + \beta_1\beta_2a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & \beta_1\beta_2b_{21}a_{12} + b_{22}a_{22} \end{pmatrix}$$

in $\mathbb{M}_{\beta_1,\beta_2}(R)$.

## § 3. Preliminaries

**Proposition 3.1.** *Let $R$ be a ring, $\beta_1, \dots, \beta_n \in C(R)$, and $n \geq 2$. Then $\Phi : \mathbb{M}_{\beta_1,\dots,\beta_n}(R) \to \mathbb{M}_n(R)$ is a ring homomorphism if $\Phi$ acts as $(a_{ij}) \mapsto \big(\beta_i^{1-\delta_{ij}} a_{ij}\big)$. Moreover,*

(1) $\Phi(1) = 1$;

(2) $(\operatorname{Ker}\Phi)^2 = 0$;

(3) $\Phi$ *is one-to-one if and only if $\beta_1, \dots, \beta_n$ are not zero divisors in $R$;*

(4) $\Phi$ *is surjective if and only if $\beta_1, \dots, \beta_n \in U(R)$, which is equivalent to the fact that $\Phi$ is a bijection.*

Proof. Given $A = (a_{ij})$ and $B = (b_{ij}) \in \mathbb{M}_{\beta_1,\dots,\beta_n}(R)$, we have $AB = (c_{ij})$, where

$$c_{ij} = \sum_{k=1}^{n} \eta_{ikj} a_{ik} b_{kj}.$$

Therefore, the $(i, j)$-element of $\Phi(AB)$ is

$$\beta_i^{1-\delta_{ij}} c_{ij} = \beta_i^{1-\delta_{ij}} \sum_{k=1}^{n} \beta_i^{\delta_{ij}-\delta_{ik}} \beta_k^{1-\delta_{kj}} a_{ik} b_{kj} = \sum_{k=1}^{n} \big(\beta_i^{1-\delta_{ik}} a_{ik}\big)\big(\beta_k^{1-\delta_{kj}} b_{kj}\big),$$

and it coincides with the $(i, j)$-element of $\Phi(A)\Phi(B)$. Thus, $\Phi$ preserves multiplication as well as addition. Hence, $\Phi$ is a ring homomorphism. It is easy to see that (1)–(4) hold. □

The two propositions below were presented in [2]. For the sake of completeness we provide them with proofs.

**Proposition 3.2** [2, Proposition 1]. *Let $K = K_n(R : \{\eta_{ikj}\})$ be a formal matrix ring of order $n$ over $R$, and let $I = (I_{ij})$ be an ideal of $K$. Then*

(1) *$K_n(\{R/I_{ij}\} : \{\psi_{ikj}\})$ is a formal matrix ring, where $\psi : R/I_{ij} \otimes R/I_{jk} \to R/I_{ik}$ is defined by $\psi_{ijk}(\bar{a} \otimes \bar{b}) = \eta_{ijk} ab + I_{ik}$, $1 \leq i, j, k \leq n$;*

(2) *$K/I \cong K_n(\{R/I_{ij}\} : \{\psi_{ijk}\})$.*

Proof. (1) is shown by immediate verification.

Let us prove (2). The mapping $\Phi : K \to K_n(\{R/I_{ij}\} : \{\psi_{ijk}\})$ is a ring epimorphism with the kernel $\operatorname{Ker}(\Phi) = I$ if $\Phi$ acts by the rule $\Phi((a_{ij})) = (a_{ij} + I_{ij})$. □

It is difficult to say something specific about the isomorphisms of arbitrary formal matrix rings over $R$. But if $R$ is commutative then we have

**Proposition 3.3.** *Let $K_n(R : \{\eta_{ijk}\})$ be a formal matrix ring of order $n$ over a commutative ring $R$, $\eta_{ijk} \in R$, and $1 \leq i, j, k, \leq n$. Then $K_n(R : \{\eta_{ijk}\}) \cong M_n(R)$ if and only if all $\eta_{ijk}$ belong to $U(R)$.*

Proof. (⇒): Define the mapping $\varphi : K_n(R : \{\eta_{ijk}\}) \to M_n(R)$ acting by the rule $\varphi((a_{ij})) = \eta_{1ij} a_{ij}$. It is immediate that $\varphi$ is a ring isomorphism.

(⇐): Denote the ring isomorphism by $\varphi : K_n(R : \{\eta_{ijk}\}) \to M_n(R)$. Assume that $R$ is a field. Show that in this case all coefficients $\eta_{ijk}$ are nonzero. Indeed, let $\eta_{lpq} = 0$. Define the mapping $\psi : K_n(R : \{\eta_{ijk}\}) \to M_n(R)$ by putting $\psi((a_{ij})) = (\eta_{lij} a_{ij})$. Since $\eta_{lij}\eta_{ljk} = \eta_{lik}\eta_{ijk}$, it is immediate that $\psi$ is a ring homomorphism. But $\operatorname{Ker}(\psi) \neq 0$; and since $\eta_{111} = 1$ we have $\varphi(\operatorname{Ker}(\psi)) \neq M_n(R)$. Hence, $\varphi(\operatorname{Ker}(\psi))$ is a nonzero two-sided ideal of $M_n(R)$. Since $R$ is a field; therefore, $M_n(R)$ is a simple ring. This contradiction shows that all coefficients $\eta_{ijk}$ are nonzero.

Consider the general case. Put $K = K_n(R : \{\eta_{ijk}\})$ and $M = M_n(R)$. Assume that there is $\eta_{abc} \notin U(R)$. Then $\eta_{abc}R \neq R$ is an ideal of $R$, and so it lies in some maximal ideal $I$ of $R$. Depending on situation, the unity of $K$ or $M$ will be denoted by $E$. Since $C(K) = RE$ and $C(M) = RE$, there exists $\alpha \in \operatorname{Aut}(R)$ such that $\varphi(rE) = \alpha(r)E$ for all $r \in R$. We have $K/IK \cong M_n(R)/\alpha(I)M$. By Proposition 3.2 $K_n(R/I : \{\overline{\eta_{ijk}}\}) \cong M_n(R/\alpha(I))$, where $\overline{\eta_{ijk}} = \eta_{ijk} + I$. But $R/\alpha(I)$ is a field, and $\overline{\eta_{abc}} = \overline{0}$. This contradiction shows that $\eta_{ijk} \in U(R)$, $1 \leq i, j, k \leq n$. □

The ideal structure of $\mathbb{M}_{\beta_1,\dots,\beta_n}(R)$ gives

**Proposition 3.4.** *Let $R$ be a ring, $n \geq 2$, and $\beta_1, \dots, \beta_n \in C(R)$. Then $I \subseteq \mathbb{M}_{\beta_1,\dots,\beta_n}(R)$ is an ideal if and only if $I = (I_{ij})$ and the following hold*:

(1) $I_{ii} \subseteq \bigcap_{k\neq i}(I_{ik} \cap I_{ki})$, $1 \leq i \leq n$;
(2) $\beta_i\beta_j(I_{ij} + I_{ji}) \subseteq I_{ii} \cap I_{jj}$, $i \neq j$;
(3) $\beta_i I_{ij} \subseteq \bigcap_{k\neq j} I_{kj}$;
(4) $\beta_j I_{ij} \subseteq \bigcap_{k\neq i} I_{ik}$.

Proof. Let $I \subseteq \mathbb{M}_{\beta_1,\dots,\beta_n}(R)$ be an ideal. Denote the matrix units of $\mathbb{M}_{\beta_1,\dots,\beta_n}(R)$ by $E_{ij}$. Since $E_{ii}$ are orthogonal idempotents with sum equal to the unity, $I = \sum_{i,j} E_{ii} I E_{jj}$. Put $I_{ij} = E_{ii} I E_{jj}$. Then $I = (I_{ij})$. The remaining properties and the converse statement are immediate from the ideal properties. $\square$

Take $s \in C(R)$. Put $J_s(R) = (s : J(R)) = \{x \in R \mid sx \in J(R)\}$. By [7] we have

**Theorem 3.5.** *Let $K = K_n(R : \{\eta_{ikj}\})$ be a formal matrix ring of order $n$ over $R$. Then*

$$J(K) = \begin{pmatrix} J(R) & J_{\eta_{121}}(R) & \dots & J_{\eta_{1n1}}(R) \\ J_{\eta_{212}}(R) & J(R) & \dots & J_{\eta_{2n2}}(R) \\ \dots & \dots & \dots & \dots \\ J_{\eta_{n1n}}(R) & J_{\eta_{n2n}}(R) & \dots & J(R) \end{pmatrix}.$$

Through this section we assume that $R$ is a unital commutative ring, and $\beta_1, \dots, \beta_n$ are arbitrary fixed elements in $R$. Show that the formal matrix rings $\mathbb{M}_{\beta_1,\dots,\beta_n}(R)$ may be considered as some subrings of the classical matrix rings over a new ring that is an extension of $R$. Consider the free $R$-module $\widehat{R}$ over $R$ with the basis $\{e_A\}_{A\in 2^{\{1,\dots,n\}}}$. The multiplication operation in $\widehat{R}$ we define by the rule

$$\Big(\sum r'_A e_A\Big)\Big(\sum r''_B e_B\Big) = \sum r'_A r''_B e_A e_B,$$

where $e_A e_B = \beta_{A\cap B} e_{A\triangle B}$, $\beta_C = \prod_{k\in C} \beta_k$, and $\beta_\varnothing = 1$.

**Lemma 3.6.** (1) *$\widehat{R}$ is a commutative ring under the above-introduced operations. The initial ring $R$ may be considered as a subring of $\widehat{R}$ under the embedding $r \mapsto re_\varnothing$, $r \in R$.*

(2) *The mapping $\pi : \mathbb{M}_{\beta_1,\dots,\beta_n}(R) \to M_n(\widehat{R})$ such that $\pi((a_{ij})) = (a_{ij} e_{\{i\}\triangle\{j\}})$ is an injective ring homomorphism.*

Proof. By the definition of multiplication $e_A e_B = e_B e_A$ for all $A, B \in 2^{\{1,\dots,n\}}$. It remains to prove associativity. Take $A, B, C \in 2^{\{1,\dots,n\}}$. We have

$$\begin{aligned}(e_A e_B) e_C &= \beta_{A\cap B} e_{A\triangle B} e_C = \beta_{A\cap B} \beta_{(A\triangle B)\cap C} e_{(A\triangle B)\triangle C} \\ &= \beta_{(A\cap B)\cup(A\cap C)\cup(B\cap C)} e_{(A\cup B\cup C)\setminus((A\cap B)\cup(A\cap C)\cup(B\cap C))},\end{aligned}$$

whence $(e_A e_B) e_C = (e_B e_C) e_A = e_A (e_B e_C)$. This yields (1).

(2) is proved by immediate verification. $\square$

Introduce the determinant and the characteristic polynomial for $A \in \mathbb{M}_{\beta_1,\dots,\beta_n}(R)$, putting

$$\det_{\beta_1,\dots,\beta_n}(A) = \det_{\widehat{R}}(\pi(A)), \quad \chi_{\beta_1,\dots,\beta_n;A}(\lambda) = \chi_{\pi(A)}(\lambda).$$

Show that $\det_{\beta_1,\dots,\beta_n}(A) \in R$. By definition,

$$\det_{\beta_1,\dots,\beta_n}(A) = \det_{\widehat{R}}(\pi(A)) = \sum_{\tau\in S_n} \operatorname{sgn}(\tau) \prod_{i=1}^n a_{i,\tau(i)} e_{\{i\}\triangle\{\tau(i)\}}.$$

Fix a permutation $\tau_0 \in S_n$. We have

$$\prod_{i=1}^n a_{i,\tau_0(i)} e_{\{i\}\triangle\{\tau_0(i)\}} = \Big(\prod_{i=1}^n a_{i,\tau_0(i)}\Big)\Big(\prod_{i=1}^n e_{\{i\}\triangle\{\tau_0(i)\}}\Big).$$

Write $\tau_0$ as a product of disjoint cycles. Let $(i_1, \dots, i_k)$ be one of these cycles. Then by the commutativity for the corresponding factor from the product above we have

$$e_{\{i_1\}\triangle\{i_2\}} \cdot e_{\{i_2\}\triangle\{i_3\}} \cdot \dots \cdot e_{\{i_k\}\triangle\{i_1\}}$$
$$= \beta_{\{\{i_1\}\triangle\{i_2\}\}\cap\{\{i_2\}\triangle\{i_3\}\}} e_{\{i_1\}\triangle\{i_3\}} \cdot \dots \cdot e_{\{i_k\}\triangle\{i_1\}} = \dots = \beta_C e_\varnothing \in R$$

for some $C \in 2^{\{1,\dots,n\}}$.

From here, $\det_{\beta_1,\dots,\beta_n}(A) \in R$, whence $\chi_{\beta_1,\dots,\beta_n;A}(\lambda) \in R[\lambda]$. Thus, as a corollary we obtain

**Theorem 3.7.** *Let $A, B \in \mathbb{M}_{\beta_1,\dots,\beta_n}(R)$. Then*
(1) $\det_{\beta_1,\dots,\beta_n}(AB) = \det_{\beta_1,\dots,\beta_n}(A)\det_{\beta_1,\dots,\beta_n}(B)$;
(2) *$A$ is invertible in $\mathbb{M}_{\beta_1,\dots,\beta_n}(R)$ if and only if $\det_{\beta_1,\dots,\beta_n}(A) \in U(R)$;*
(3) $\chi_{\beta_1,\dots,\beta_n;A}(A) = 0$.

Proof. Embed $\mathbb{M}_{\beta_1,\dots,\beta_n}(R)$ into $M_n(\widehat{R})$ and use the facts that $\det_{\beta_1,\dots,\beta_n}(A) \in R$ and $\chi_{\beta_1,\dots,\beta_n;A}(\lambda) \in R[\lambda]$ for $A \in \mathbb{M}_{\beta_1,\dots,\beta_n}(R)$. □

## § 4. Ring-Theoretic Properties of Formal Matrix Rings

In this section we describe right semiartinian formal matrix rings and right formal matrix max-rings.

**Lemma 4.1.** *Let $T = \begin{pmatrix} R & M \\ N & S \end{pmatrix}$ be a formal matrix ring, and let $(A, B)$ be a right $T$-module.*

(1) *If $\mathrm{Soc}(A)$ is essential in $A$ and $\mathrm{Soc}(B)$ is essential in $B$ then $\mathrm{Soc}((A, B))$ is essential in $(A, B)$.*

(2) *If nonzero quotient modules of the modules $A$ and $B$ contain maximal submodules then every nonzero quotient module of the module $(A, B)$ contains a maximal submodule.*

Proof. (1): Let $(A, B)$ be a right $T$-module, and let $(A_0, B_0)$ be its nonzero submodule. Without less of generality we may assume that $A_0 \neq 0$. Since $\mathrm{Soc}(A)$ is essential in $A$; therefore, $A_0$ has a simple submodule $aR$, where $a \in A_0$. If $aRM = aM = 0$ then $(aR, 0)$ is a simple submodule of the $T$-module $(A_0, B_0)$. Since $\mathrm{Soc}(B)$ is essential in $B$; therefore, if $aM \neq 0$ then the $S$-module $aM$ has a simple submodule $bS$, where $b \in B_0$. Clearly, $b$ is of the form $b = am$, where $m \in M$. If $bN = 0$ then $(0, bS)$ is a simple submodule of the $T$-module $(A_0, B_0)$. If $bN \neq 0$ then $bN = amN \subset aMN$ is a nonzero submodule of the simple module $aR$. Hence, $bN = aR$. The simplicity of $bS$ and $aRM = bNM$ imply that $aRM = bS$. Since $aR$ is a simple $R$-module, $bS$ is a simple $S$-module, $aRM = bS$, and $bSN = aR$; therefore, $(aR, bS)$ is a simple submodule of the $T$-module $(A_0, B_0)$.

(2): Let $(X, Y)$ be a proper submodule of $(A, B)$. By hypothesis, if $(A/X)M \neq B/Y$ then $B$ has a maximal submodule $Y'$ such that $(A/X)M \subset Y'/Y$. In this case, it is easy to see that $(A/X, Y'/Y)$ is a maximal submodule of $(A/X, B/Y)$. Analogously, if $(B/Y)M \neq A/X$ then $(A/X, B/Y)$ has a maximal submodule. Assume that $(A/X)M = B/Y$ and $(B/Y)N = A/X$. By hypothesis, $A$ has a maximal submodule $A_0$ such that $X \subset A_0$. Consider a submodule $B_0$ of $B$ such that $B_0/Y = \{\bar{b} \in B/Y \mid \bar{b}N \subset A_0/X\}$. Clearly, $B_0/Y \neq B/Y$ and $(A_0/X)M \subset B_0/Y$. Show that $B_0$ is a maximal submodule of the $S$-module $B$. Let $\bar{b} \notin B_0/Y$. Then $\bar{b}N \not\subseteq A_0/X$, and so $\bar{b}N + A_0/X = A/X$, $\bar{b}NM + (A_0/X)M = (A/X)M = B/Y$. Thus, $\bar{b}S + B_0/Y = B/Y$ for every $\bar{b} \in (B/Y) \setminus (B_0/Y)$. Hence, $(B/Y)/(B_0/Y)$ is a simple $S$-module. Since $(A_0/X)M \subset B_0/Y$ and $(B_0/Y)N \subset A_0/X$; therefore, $(A_0/X, B_0/Y)$ is a submodule of the $T$-module $(A/X, B/Y)$. It is easy to note that the right $T$-module $((A/X)/(A_0/X), (B/Y)/(B_0/Y))$ is of the length at most 2. Hence, $(X, Y)$ lies in a maximal submodule of $(A, B)$. □

A ring $R$ is *right semiartinian* provided that each nonzero right $R$-module has a simple submodule. If every nonzero right module over $R$ has a maximal submodule then $R$ is a *right* max-*ring.*

**Theorem 4.2.** *For a formal matrix ring* $T = \begin{pmatrix} R & M \\ N & S \end{pmatrix}$ *the following are equivalent*:

(1) $T$ *is a right semiartinian ring*;

(2) $R$ *and* $S$ *are right semiartinian rings.*

PROOF. (1) $\Rightarrow$ (2) Let $A$ be a nonzero right $R$-module. Use a construction from [5]. Consider a right $T$-module $(A, \mathrm{Hom}_R(N, A))$ such that the mappings $A \otimes M \to \mathrm{Hom}_R(N, A)$, $a \otimes m \mapsto (n \mapsto a(mn))$, and $\mathrm{Hom}_R(N, A) \otimes N \to A$, $f \otimes n \mapsto f(n)$, are homomorphisms of module multiplication.

By hypothesis, the right $T$-module $(A, \mathrm{Hom}_R(N, A))$ has a simple submodule $(X, Y)$. If $X = 0$ then for every $f \in Y$ we have $f(N) = fN = 0$. Hence, $Y = 0$, which is impossible. Thus, $X \neq 0$, and the simplicity of $(X, Y)$ implies that $X$ is a simple submodule of the $R$-module $A$. By the above, each nonzero right $R$-module has a simple submodule. Therefore, $R$ is a right semiartinian ring. Analogously we can show that $S$ is a right semiartinian ring.

The implication (2) $\Rightarrow$ (1) follows from Lemma 4.1. $\square$

**Theorem 4.3.** *Given a formal matrix ring* $T = \begin{pmatrix} R & M \\ N & S \end{pmatrix}$, *the following are equivalent*:

(1) $T$ *is a right* Max-*ring*;

(2) $R$ *and* $S$ *are right* Max-*rings.*

PROOF. (1) $\Rightarrow$ (2): Let $A$ be a nonzero right $R$-module. Use a construction from [5]. Consider a right $T$-module $(A, A \otimes M)$ such that the identical automorphism $A \otimes M \to A \otimes M$ and the mapping $(A \otimes M) \otimes N \to A$, $(a \otimes m)n = amn$, are homomorphisms of module multiplication.

By hypothesis, the right $T$-module $(A, A \otimes M)$ has a maximal submodule $(X, Y)$. If $X = A$ then obviously $Y = A \otimes M$, which is impossible. Thus, $X \neq A$, and the maximality of $(X, Y)$ implies that $X$ is a maximal submodule of the $R$-module $A$. By the above, each nonzero right $R$-module has a maximal submodule. Hence, $R$ is a right Max-ring. Analogously we can show that $S$ is a right Max-ring.

The implication (2) $\Rightarrow$ (1) follows from Lemma 4.1. $\square$

Description of right perfect formal matrix rings $\begin{pmatrix} R & M \\ N & S \end{pmatrix}$ was obtained in [5] in the case that $MN = 0$ and $NM = 0$, and it was done in [1] in the case that the right modules $M_S$ and $N_R$ are finitely generated.

**Corollary 4.4.** *For a formal matrix ring* $T = \begin{pmatrix} R & M \\ N & S \end{pmatrix}$ *the following are equivalent*:

(1) $T$ *is a left perfect ring*;

(2) $R$ *and* $S$ *are left perfect rings.*

PROOF. (1) $\Rightarrow$ (2): If $T$ is a left perfect ring then $T$ is a semiperfect right semiartinian ring by [8, 6.48]. Then $R$ and $S$ are left perfect rings by Theorem 4.2.

(2) $\Rightarrow$ (1): If $R$ and $S$ are left perfect rings then $T$ is a semiperfect right semiartinian ring by Theorem 4.2. Then $T$ is a left perfect ring by [8, 6.48]. $\square$

A ring $R$ is *right quasi-invariant* provided that every maximal right ideal of $R$ is an ideal.

**Corollary 4.5.** *If a formal matrix ring* $T = \begin{pmatrix} R & M \\ N & S \end{pmatrix}$ *has the rings* $R$ *and* $S$ *quasi-invariant then the following are equivalent*:

(1) $T$ *is a right* max-*ring*;

(2) *the quotient rings* $R/J(R)$ *and* $S/J(S)$ *are strictly regular and the ideals* $J(R)$ *and* $J(S)$ *are right t-nilpotent.*

PROOF. The equivalence (1) $\Leftrightarrow$ (2) follows from Theorem 4.3 and [9, Theorem 26.8]. $\square$

Let $T = \begin{pmatrix} R & M \\ N & S \end{pmatrix}$ be a formal matrix ring. A bimodule $M$ is $N$-*regular* ($N$-*fully right idempotent*) provided that $m \in mNm$ for every $m \in M$ ($m \in mNmS$). Analogously, the notions of $M$-regularity and $M$-full right idempotency of a bimodule $N$ are defined.

A right semiartinian ring is a right *SV-ring* if every simple right module over this ring is injective.

**Theorem 4.6.** *Let $T = \begin{pmatrix} R & M \\ N & S \end{pmatrix}$ be a formal matrix ring, let $M$ be a finitely generated right $S$-module, and let $N$ be a finitely generated right $R$-module. Then the following are equivalent*:

(1) *$T$ is a right $SV$-ring*;

(2) *$R$ and $S$ are right $SV$-rings, $M$ is $N$-regular, and $N$ is $M$-regular*;

(3) *$R$ and $S$ are right $SV$-rings, $M$ is $N$-fully idempotent, and $N$ is $M$-fully idempotent.*

Proof. The implication (1) $\Rightarrow$ (2) follows from [10, Theorems 2.7 and 2.9].

The implication (2) $\Rightarrow$ (3) is obvious.

(3) $\Rightarrow$ (1): By hypothesis, the ideal $I = MN$ of $R$ is finitely generated as a right ideal. By [10, Theorem 2.7] $R$ is a regular ring. Hence, $I = eR$, where $e$ is an idempotent in $R$. Since $(1-e)Re = 0$ and $J(R) = 0$, $e$ is a central idempotent. Analogously, the ideal $J = NM$ of $S$ is of the form $J = fS$, where $f$ is a central idempotent in $S$. By hypothesis $M = (MN)M = IM = eM$ and $M = M(NM) = MJ = Mf$, whence $M$ may be considered as an $eRe$-$fRf$-bimodule. Analogously, $N$ may be considered as an $fRf$-$eRe$-bimodule. Then we have the isomorphism

$$\begin{pmatrix} R & M \\ N & S \end{pmatrix} \cong (1-e)R \times (1-f)S \times \begin{pmatrix} eRe & {}_{eRe}M_{fRf} \\ {}_{fRf}N_{eRe} & fRf \end{pmatrix}.$$

Since $R$ and $S$ are right $SV$-rings, $(1-e)R$ and $(1-f)S$ are right $SV$-rings as well. Since $MN = eRe$ and $NM = fRf$; therefore by [5, Theorem 8.5] the Morita ring $T' = \begin{pmatrix} eRe & {}_{eRe}M_{fRf} \\ {}_{fRf}N_{eRe} & fRf \end{pmatrix}$ is equivalent to $eRe$. Hence, $T'$ is a right $SV$-ring. Thus, $T$ is a right $SV$-ring. $\square$

**Theorem 4.7.** *Let $K = K_n(R : \{\eta_{ijk}\})$ be a formal matrix ring with values in some ring $R$. Then*

(1) *$K$ is a right semiartinian ring if and only if $R$ is a right semiartinian ring*;

(2) *$K$ is a right* max*-ring if and only if $R$ is a right* max*-ring*;

(3) *$K$ is a right $SV$-ring if and only if $R$ is a right $SV$-ring and $\{\eta_{ijk}\} \subset U(R)$.*

Proof. The assertions (1) and (2) follow from Theorems 4.2 and 4.3 respectively.

(3): Let $K$ be a right $SV$-ring. Then $R$ is a right $SV$-ring by Theorem 4.6. By [10, Theorem 2.7], $K$ is a regular ring. Hence, by [11, Theorem 29] for arbitrary $1 \leqslant i, j \leqslant n$ there is $r \in R$ such that $1 = \varphi_{iji}(1 \otimes r) = \eta_{iji}r$. Thus, $\eta_{iji}$ are invertible. Since $\eta_{iji} = \eta_{ijk}\eta_{jik}$ for every $1 \le k \le n$, $\{\eta_{ijk}\} \subset U(R)$. If $R$ is a right $SV$-ring and $\{\eta_{ijk}\} \subset U(R)$ then $K \cong M_n(R)$ by Proposition 3.3, and $K$ is a right $SV$-ring. $\square$

## § 5. The Isomorphism Problem for the Rings $\mathbb{M}_{\beta_1,\dots,\beta_n}(R)$

Since $\mathbb{M}_{\beta_1,\beta_2}(R) = \mathbb{M}_{1,\beta_1\beta_2}(R) = K_{\beta_1\beta_2}(R)$, as a corollary we have

**Theorem 5.1** [1, Corollary 4.11]. *Let $R$ be a commutative ring such that $Z(R) \subseteq J(R)$, and let $\beta, \gamma \in R$. Then $\mathbb{M}_{1,\beta}(R) \cong \mathbb{M}_{1,\gamma}(R)$ if and only if $\gamma = v\alpha(\beta)$, where $v \in U(R)$ and $\alpha \in \operatorname{Aut}(R)$.*

Thus, it suffices to consider the case $n \ge 3$. Start with the simplest facts and gradually proceed to the main result of this section. The following proposition was given in [2] without proof.

**Proposition 5.2.** *Let $R$ be a ring, $n \ge 3$, and $\beta_1, \beta_2, \dots, \beta_n \in C(R)$.*

(1) *If $v_1, \dots, v_n \in U(R) \cap C(R)$ and $\alpha \in \operatorname{Aut}(R)$ then $\mathbb{M}_{\beta_1,\dots,\beta_n}(R) \cong \mathbb{M}_{v_1\alpha(\beta_1),\dots,v_n\alpha(\beta_n)}(R)$.*

(2) *If $\pi \in S_n$ then $\mathbb{M}_{\beta_1,\dots,\beta_n}(R) \cong \mathbb{M}_{\beta_{\pi(1)},\dots,\beta_{\pi(n)}}(R)$.*

(3) *If $\pi_1, \dots, \pi_n \in S_n$ and the decomposition of the unity $1 = a_1 + a_2 + \dots + a_n$ into the sum of orthogonal idempotents $a_i$ is given then $M_{\beta_1,\dots,\beta_n}(R) \cong M_{\gamma_1,\dots,\gamma_n}(R)$, where $\gamma_i = \sum_{j=1}^n \beta_{\pi_j(i)}a_j$, $1 \le i \le n$.*

Proof. (1) Define the mapping

$$\Theta : \mathbb{M}_{\beta_1,\dots,\beta_n}(R) \to \mathbb{M}_{v_1\beta_1,\dots,v_n\beta_n}(R)$$

by the rule $\Theta((a_{ij})) = \big((v_i^{\delta_{ij}-1} a_{ij})\big)$. It is easy to see that $\Theta$ is a bijection, which preserves addition. Moreover, for $A = (a_{ij})$ and $B = (b_{ij})$ in $\mathbb{M}_{\beta_1,\dots,\beta_n}(R)$ we have $AB = (c_{ij})$, where $c_{ij} = \sum_{k=1}^n \beta_i^{\delta_{ij}-\delta_{ik}} \beta_k^{1-\delta_{jk}} a_{ik} b_{kj}$, and the $(i,j)$-element of $\Theta(AB)$ is equal to

$$v_i^{\delta_{ij}-1} c_{ij} = \sum_{k=1}^n (v_i\beta_i)^{\delta_{ij}-\delta_{ik}} (v_k\beta_k)^{1-\delta_{jk}} \big(v_i^{\delta_{ik}-1} a_{ik}\big)\big(v_k^{\delta_{kj}-1} b_{kj}\big),$$

which coincides with the $(i,j)$-element of $\Theta(A)\Theta(B)$. So, $\Theta$ preserves multiplication, whence $\mathbb{M}_{\beta_1,\dots,\beta_n}(R) \cong \mathbb{M}_{v_1\beta_1,\dots,v_n\beta_n}(R)$. Therefore, $\mathbb{M}_{\alpha(\beta_1),\dots,\alpha(\beta_n)}(R) \cong \mathbb{M}_{v_1\alpha(\beta_1),\dots,v_n\alpha(\beta_n)}(R)$. However, it is easy to see that $(a_{ij}) \mapsto (\alpha(a_{ij}))$ gives an isomorphism between $\mathbb{M}_{\beta_1,\dots,\beta_n}(R)$ and $\mathbb{M}_{\alpha(\beta_1),\dots,\alpha(\beta_n)}(R)$. Thus, $\mathbb{M}_{\beta_1,\dots,\beta_n}(R) \cong \mathbb{M}_{v_1\alpha(\beta_1),\dots,v_n\alpha(\beta_n)}(R)$.

(2) Define the mapping $\Theta : \mathbb{M}_{\beta_1,\dots,\beta_n}(R) \to \mathbb{M}_{v_1\alpha(\beta_1),\dots,v_n\alpha(\beta_n)}(R)$ by the rule $\Theta((a_{ij})) = ((a_{\pi(i)\pi(j)}))$. It is easy to see that $\Theta$ is a bijection, which preserves addition. Moreover, for $A = (a_{ij})$ and $B = (b_{ij})$ in $\mathbb{M}_{\beta_1,\dots,\beta_n}(R)$ we have $AB = (c_{ij})$, where

$$c_{ij} = \sum_{k=1}^n \beta_i^{\delta_{ij}-\delta_{ik}} \beta_k^{1-\delta_{jk}} a_{ik} b_{kj},$$

and the $(\pi(i),\pi(j))$-element of $\Theta(AB)$ is equal to

$$c_{\pi(i)\pi(j)} = \sum_{\pi(k)=1}^n \beta_{\pi(i)}^{\delta_{\pi(i)\pi(j)}-\delta_{\pi(i)\pi(k)}} \beta_{\pi(k)}^{1-\delta_{\pi(j)\pi(k)}} a_{\pi(i)\pi(k)} b_{\pi(k)\pi(j)},$$

which coincides with the $(\pi(i),\pi(j))$-element of $\Theta(A)\Theta(B)$.

(3) We have

$$\begin{aligned} M_{\beta_1,\beta_2,\dots,\beta_n}(R) &\cong M_{a_1\beta_1,a_2\beta_2,\dots,a_1\beta_n}(a_1R) \times \cdots \times M_{a_n\beta_1,a_n\beta_2,\dots,a_n\beta_n}(a_nR) \\ &\cong M_{a_1\beta_{\pi_1(1)},a_1\beta_{\pi_1(2)},\dots,a_1\beta_{\pi_1(n)}}(a_1R) \times \cdots \times M_{a_n\beta_{\pi_n(1)},a_n\beta_{\pi_3(2)},\dots,a_n\beta_{\pi_n(n)}}(a_nR) \\ &\cong M_{\beta_{\pi_1(1)}a_1+\cdots+\beta_{\pi_n(1)}a_n,\beta_{\pi_1(2)}a_1+\cdots+\beta_{\pi_n(2)}a_n,\beta_{\pi_1(n)}a_1+\cdots+\beta_{\pi_n(n)}a_n}(R). \quad \square \end{aligned}$$

A ring is *normal* provided that its all idempotents are central.

**Lemma 5.3** [2, Theorem 20]. *Let $R$ be a normal ring, let $T$ be a ring, $n \geq 3$, and $\beta_1,\dots,\beta_n \in C(R)$. Then if $\mathbb{M}_{\underbrace{0,\dots,0}_{n}}(R) \cong \mathbb{M}_{\beta_1,\dots,\beta_n}(T)$ then $\beta_1 = \cdots = \beta_n = 0$.*

We have a stronger result:

**Lemma 5.4** [5, Lemma 9.2]. *Let $R$ be a normal ring, let $T$ be a ring, and $n \geq 3$. Then if $\mathbb{M}_{\underbrace{0,\dots,0}_{n}}(R) \cong K_n(T : \{\eta_{ikj}\})$ then all $\eta_{ijk}$ are zero, except for the cases when either $i = j$ or $j = k$.*

Let $R$ be a ring, $n \geq 3$, $\beta_i, \gamma_i \in C(R)$, $1 \leq i \leq n$, and let $S_1 = \mathbb{M}_{\beta_1,\dots,\beta_n}(R)$, $S_2 = \mathbb{M}_{\gamma_1,\dots,\gamma n}(R)$. Let $E_{ij}$ be the matrix units of $S_1$, let $F_{ij}$ be the matrix units of $S_2$, let $I \subseteq S_1$ be the ideal generated by $\{E_{ij} \mid i \neq j\}$, and let $J \subseteq S_2$ be the ideal generated by $\{F_{ij} \mid i \neq j\}$.

**Lemma 5.5.** *Let $R$ be a commutative ring, and let $\Theta : S_1 \to S_2$ be an isomorphism. Then $\Theta(I) = J$.*

Proof. Assume that there are $i \neq j$ such that $F_{ij} \notin \Theta(I)$. Then $F_{ii}(F_{ii} + F_{ij}) - (F_{ii} + F_{ij})F_{ii} = F_{ij} \notin \Theta(I)$. It shows that $(F_{ii} + F_{ij})$ modulo the ideal $\Theta(I)$ is not central in $S_2/\Theta(I)$. Denote $\beta_{[l]}(R) = \sum_{k=1,\, k\neq l}^{n} \beta_k R$. It is easy to see that

$$I = \begin{pmatrix} \beta_1\beta_{[1]}(R) & R & \dots & R \\ R & \beta_2\beta_{[2]}(R) & \dots & R \\ \vdots & \vdots & \ddots & \vdots \\ R & R & \dots & \beta_n\beta_{[n]}(R) \end{pmatrix}.$$

But $S_2/\Theta(I) \cong S_1/I \cong \bigoplus_{k=1}^{n} R/\beta_k\beta_{[k]}(R)$, and the last ring is commutative; a contradiction. Therefore, $J \subseteq \Theta(I)$. The proof of the fact that $I \subseteq \Theta^{-1}(J)$ is analogous. We have $J \subseteq \Theta(I)$ and $J \supseteq \Theta(I)$, whence $\Theta(I) = J$. $\square$

By Proposition 5.2 it is immediate that $\mathbb{M}_{\underbrace{\beta,0,\dots,0}_{n}}(R)$ and $\mathbb{M}_{\beta a_1, \beta a_2, \dots, \beta a_n}(R)$ are isomorphic for every decomposition of the unity $1 = a_1 + a_2 + \dots + a_n$, $n \geq 3$, into the sum of orthogonal idempotents $a_i$. Moreover, for a commutative ring $R$ we have

**Theorem 5.6.** *Let $R$ be a commutative ring, $n \geq 3$, $\beta, \gamma_1, \dots, \gamma_n \in R$, and $\operatorname{ann}_R(\beta) \subseteq J(R)$. Then $\mathbb{M}_{\underbrace{\beta,0,\dots,0}_{n}}(R) \cong \mathbb{M}_{\gamma_1,\gamma_2,\dots,\gamma_n}(R)$ if and only if $\gamma_i = \alpha(\beta)v_i a_i$ for all $i = \overline{1,n}$, where $\alpha \in \operatorname{Aut}(R)$, $v_i \in U(R)$, and $1 = a_1 + a_2 + \dots + a_n$ is a decomposition of the unity into the sum of orthogonal idempotents $a_i$.*

Proof. ($\Leftarrow$): Sufficiency follows from Proposition 5.2.

($\Rightarrow$): The argument is analogous to the proof of Theorem 16 from [2]. Denote $K_1 = \mathbb{M}_{\underbrace{\beta,0,\dots,0}_{n}}(R)$ and $K_2 = \mathbb{M}_{\gamma_1,\gamma_2,\dots,\gamma_n}(R)$. Let $\Theta : K_1 \to K_2$ be an isomorphism. The matrix units of $K_1$ denote by $E_{ij}$, and the matrix units of $K_2$ denote by $F_{ij}$. For every $r \in R$ we have $rE \in C(K_1)$ and $\Theta(rE) \in C(K_2)$. Then there is $s \in R$ such that $\Theta(rE) = sE$. Define a mapping $\alpha : R \to R$ and put $\alpha(r) = s$. The so-obtained mapping is an automorphism of $R$.

Let $I = \beta K_1$ be an ideal of $K_1$, and let $\Theta(I) = \alpha(\beta)K_2$ be an ideal of $K_2$. Then

$$\mathbb{M}_{\underbrace{\bar\beta,0,\dots,0}_{n}}(R/\beta R) \cong K_1/I \cong K_2/\Theta(I) \cong \mathbb{M}_{\bar\gamma_1,\bar\gamma_2,\dots,\bar\gamma_n}(R/\alpha(\beta)R).$$

By Lemma 5.3 there are $y_1, y_2, \dots, y_n \in R$ such that

$$\gamma_1 = \alpha(\beta)y_1, \gamma_2 = \alpha(\beta)y_2, \dots, \gamma_n = \alpha(\beta)y_n.$$

Put $x_i = \alpha^{-1}(y_i)$, $i = \overline{1,n}$. Then there exists an isomorphism $\Psi : K_2 \to \mathbb{M}_{\beta x_1, \beta x_2, \dots, \beta x_n}(R)$. The automorphism of $R$ corresponding to the isomorphism $\Psi \circ \Theta$ is identical. Now, denote the isomorphism $\Psi \circ \Theta$ by $\Theta$, and $\mathbb{M}_{\beta x_1, \beta x_2, \dots, \beta x_n}(R)$ by $K_2$.

Simultaneously, we can consider the ideal $I = \beta x_1 K_2 + \beta x_2 K_2 + \dots + \beta x_n K_2$ in $K_2$; $\Theta^{-1}(I) = \beta x_1 K_1 + \beta x_2 K_1 + \dots + \beta x_n K_1$ is an ideal of $K_1$, and $K_2/I \cong K_1/\Theta^{-1}(I)$. Again by Lemma 5.3 there are $\eta_1, \eta_2, \dots, \eta_n \in R$ such that $\beta = \beta x_1\eta_1 + \beta x_2\eta_2 + \dots + \beta x_n\eta_n$.

Consider the ideal

$$I = \begin{pmatrix} 0 & R & \dots & R \\ R & 0 & \dots & R \\ \vdots & \vdots & \ddots & \vdots \\ R & R & \dots & 0 \end{pmatrix}.$$

Find $\Theta(I)$. By Lemma 5.5

$$J = \begin{pmatrix} 0 & R & \dots & R \\ R & 0 & \dots & R \\ \vdots & \vdots & \ddots & \vdots \\ R & R & \dots & 0 \end{pmatrix} \subseteq \Theta(I).$$

Indeed, we claim that the inclusion is, in fact, equality. Let $C = \mathrm{diag}(c_1, c_2, \dots, c_n) \in \Theta(I)$. As $\Theta(I)$ is an ideal of $K_2$, $CF_{11} = c_1F_{11} \in \Theta(I)$. Denote $A = (a_{ij}) = \Theta^{-1}(F_{11})$. Since $(c_1F_{11})F_{11} = c_1F_{11}$, $(c_1A)A = c_1A$ in $K_1$. By $c_1A \in I$ we have $c_1a_{11} = c_1a_{22} = \dots = c_1a_{nn} = 0$. Denote by $\widehat{A}$ the matrix obtained from $A$ by zeroing the main diagonal. Then $c_1A = c_1\widehat{A}$. We have $c_1A = c_1A^3 = c_1(\widehat{A})^3 = 0$. Indeed, the components $c_1(\widehat{A})^3$ are some sums of the summands of the form $pa_{i_0i_1}a_{i_1i_2}a_{i_2i_3}$. Let $a_{i_0i_1}$, $a_{i_1i_2}$, and $a_{i_2i_3}$ be nonzero. Then one of the numbers $i_1$ and $i_2$ is different from the unity; for example, $i_1$. By $\eta_{i_0i_1i_2} = 0$, $pa_{i_0i_1}a_{i_1i_2}a_{i_2i_3}$ is zero. Thus, $c_1 = 0$. Analogously, we infer that $C$ is zero.

Since $\Theta(I)$ is an ideal of $S_2$; therefore, $(\beta x_i)(\beta x_j) = 0$ for all $i \neq j$. Let $U_{ij} = \Theta(E_{ij})$ for all $1 \le i, j \le n$, and let

$$U_{ii} = \begin{pmatrix} f_1^{(i)} & * & \dots & * \\ * & f_2^{(i)} & \dots & * \\ \vdots & \vdots & \ddots & \vdots \\ * & * & \dots & f_n^{(i)} \end{pmatrix}.$$

The equalities hold in $S_1$: $E_{ii}^2 = E_{ii}$, $E_{ii}E_{jj} = 0$ for all $i \neq j$, and $1 = E_{11} + E_{22} + \dots + E_{nn}$. The analogous equalities for $U_{ii}$ give $1 = f_k^{(1)} + f_k^{(2)} + \dots + f_k^{(n)}$ for all $1 \le k \le n$, where $f_k^{(i)}$ are some orthogonal idempotents.

Note that $S_2 = \bigoplus_{i,j} U_{ij}R$. Since $E_{ij} \in I$ for all $i \neq j$; therefore, $U_{ij} \in J$ for all $i \neq j$. Hence,

$$\begin{pmatrix} R & 0 & \dots & 0 \\ 0 & R & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & R \end{pmatrix} = \widehat{U_{11}}R \oplus \widehat{U_{22}}R \oplus \dots \oplus \widehat{U_{nn}}R,$$

where $\widehat{U_{ii}} = \mathrm{diag}\big(f_1^{(i)}, f_2^{(i)}, \dots, f_n^{(i)}\big)$.

In particular, we have $F_{11} = w_1\widehat{U}_{11} + w_2\widehat{U}_{22} + \dots + w_n\widehat{U}_{nn}$ for some $w_1, \dots, w_n \in R$. Thus, we obtain $f_1^{(1)} = w_1f_1^{(1)}$ and $w_1f_k^{(1)} = 0$ for $k \ge 2$. Hence, $f_1^{(1)}f_k^{(1)} = w_1f_1^{(1)}f_k^{(1)} = 0$. An analogous argument shows that $f_i^{(k)}f_j^{(k)} = 0$ for all $k$ and $i \neq j$. Denote $f^{(k)} = f_1^{(k)} + f_2^{(k)} + \dots + f_n^{(k)}$. Then $U_{kk}(1 - f^{(k)}) = \widehat{U}_{kk}(1 - f^{(k)}) + (U_{kk} - \widehat{U}_{kk})(1 - f^{(k)}) = (U_{kk} - \widehat{U}_{kk})(1 - f^{(k)}) \in J$.

Since by the corresponding equality for the preimages $U_{kk}R \cap J = 0$; therefore, $U_{kk}(1 - f^{(k)}) = 0$, and $f^{(k)} = 1$.

The equality $\Theta(I) = J$ implies $\Theta(I^2) = J^2$, and $I^2 = \begin{pmatrix} 0 & 0 \\ 0 & I_1 \end{pmatrix}$, where $I_1$ is a square matrix of order $n - 1$ with the zero diagonal elements and the off-diagonal elements equal to $\beta R$, $J^2 = \sum_{\substack{i,j=1 \\ i\neq j}}^n \sum_{\substack{k=1 \\ k\neq i,j}}^n \beta x_kF_{ij}R$.

In particular, $\beta E_{23} \in I^2$, whence $\beta U_{23} \in J^2$. Let $\beta U_{23} = (a_{ij})$, where $a_{ii} = 0$ for all $1 \le i \le n$, and $a_{ij} = \sum_{\substack{k=1 \\ k\neq i,j}}^n \beta x_k a_{ij}^{(k)}$ for all $i \neq j$. Since $\beta U_{23} = \beta(U_{22}U_{23}) = U_{22}(\beta U_{23})$ and $\beta U_{23} = (\beta U_{23})U_{33}$; therefore, $\beta U_{23} = (f_i^{(2)}f_j^{(3)}a_{ij})$. An analogous argument shows that for all $k \neq l$, $k, l \ge 2$, the $(i, j)$-element of $\beta U_{kl}$ belongs to the ideal $f_i^{(k)}f_j^{(l)}\beta R$.

Since $\beta x_1 F_{23} \in J^2 = \Theta(I^2)$, we have

$$\beta x_1 F_{23} = \Theta\Big(\sum_{i,j=2,\, i\neq j}^{n} \mu_{ij}\beta E_{ij}\Big) = \sum_{i,j=2,\, i\neq j}^{n} \mu_{ij}\beta U_{ij}$$

for some $\mu_{ij} \in R$.

For the $(2,3)$-element of $\beta x_1 F_{23}$ we obtain

$$\beta x_1 = \beta\Big(\sum_{i,j=2,\, i\neq j}^{n} f_2^{(i)} f_3^{(j)} \lambda_{ij}\Big)$$

for some $\lambda_{ij} \in R$.

It is easy to see that $\big(f_2^{(i_1)} f_3^{(j_1)}\big)\big(f_2^{(i_2)} f_3^{(j_2)}\big) = 0$ for $(i_1, j_1) \neq (i_2, j_2)$; therefore,

$$\Big(\sum_{i,j=2,\, i\neq j}^{n} f_2^{(i)} f_3^{(j)} \lambda_{ij}\Big)\Big(\sum_{i,j=2,\, i\neq j}^{n} f_2^{(i)} f_3^{(j)}\Big) = \Big(\sum_{i,j=2,\, i\neq j}^{n} f_2^{(i)} f_3^{(j)} \lambda_{ij}\Big).$$

We have shown that

$$\beta x_1 = \beta x_1\Big(\sum_{i,j=2,\, i\neq j}^{n} f_2^{(i)} f_3^{(j)}\Big) = \beta x_1\Big(\sum_{i=2}^{n} f_2^{(i)}\Big(\sum_{j=2,\, j\neq i}^{n} f_3^{(j)}\Big)\Big) =: \beta x_1 f_{23}.$$

An analogous argument gives that for every $k \geq 3$

$$\beta x_1 = \beta x_1\Big(\sum_{i,j=2,\, i\neq j}^{n} f_2^{(i)} f_k^{(j)}\Big) = \beta x_1\Big(\sum_{i=2}^{n} f_2^{(i)}\Big(\sum_{j=2,\, j\neq i}^{n} f_k^{(j)}\Big)\Big) =: \beta x_1 f_{2k},$$

whence $\beta x_1 = \beta x_1 * \big(\prod_{k=3}^{n} f_{2k}\big)$. Immediate verification shows that

$$\prod_{k=3}^{n} f_{2k} = \sum_{\pi\in S_{n-1}} f_2^{(1+\pi(1))} f_3^{(1+\pi(2))} \dots f_n^{(1+\pi(n-1))} = \sum_{\pi\in S_{n-1}} f_1^{(\pi)},$$

where $f_1^{(\pi)} = f_2^{(1+\pi(1))} f_3^{(1+\pi(2))} \dots f_n^{(1+\pi(n-1))}$. We have $f_1^{(\pi_1)} f_1^{(\pi_2)} = 0$ for $\pi_1 \neq \pi_2$ as well. Moreover, if we introduce the analogous idempotents

$$f_k^{(\pi)} = \Big(\prod_{i=1}^{i=k-1} f_i^{(1+\pi(i))}\Big)\Big(\prod_{i=k+1}^{i=n} f_i^{(1+\pi(i-1))}\Big)$$

for $\beta x_k$ then $f_k^{(\pi)} f_l^{(\sigma)} = 0$ for all $k \neq l$ and $\pi, \sigma \in S_{n-1}$. Also,

$$1 = \big(f_1^{(2)} + f_2^{(2)} + \dots + f_n^{(2)}\big) \dots \big(f_1^{(n)} + f_2^{(n)} + \dots + f_n^{(n)}\big) = \sum_{k=1}^{n}\Big(\sum_{\pi\in S_{n-1}} f_k^{(\pi)}\Big).$$

Put $a_k = \sum_{\pi\in S_{n-1}} f_k^{(\pi)}$. Then $\beta x_k = \beta x_k a_k$ for all $1 \leq k \leq n$ and $1 = a_1 + \dots + a_n$, where $a_k$ are some orthogonal idempotents. In the beginning of the proof we demonstrated that $\beta = \beta x_1\eta_1 + \beta x_2\eta_2 + \dots + \beta x_n\eta_n$ for some $\eta_1, \dots, \eta_n \in R$, whence $\beta x_k \eta_k a_k = \beta a_k$ for all $1 \leq k \leq n$.

Put $v_k = x_k a_k + (1 - a_k)$ and $\xi_k = \eta_k a_k + (1 - a_k)$. We have $\beta v_k \xi_k = \beta x_k \eta_k a_k + \beta(1 - a_k) = \beta a_k + \beta(1 - a_k) = \beta$ $\beta(1 - v_k\xi_k) = 0$.

Since $\mathrm{ann}_R(\beta) \subseteq J(R)$; therefore, $(1 - v_k\xi_k) \in J(R)$, whence $1 - (1 - v_k\xi_k) = v_k\xi_k \in U(R)$ and $v_k \in U(R)$. It is easy to see that $\beta x_k = \beta v_k a_k$. $\square$

**Corollary 5.7.** *Let $R$ be a commutative ring, $n \geq 3$, $\beta, \gamma_1, \ldots, \gamma_n \in R$, and*

$$\mathbb{M}_{\underbrace{\beta,0,\ldots,0}_{n}}(R) \cong \mathbb{M}_{\gamma_1,\gamma_2,\ldots,\gamma_n}(R).$$

*Then the following hold:*

*(1) If $\beta$ is not a zero divisor in $R$ then there are $\alpha \in \mathrm{Aut}(R)$, $v_1, \ldots, v_n \in U(R)$, and a decomposition of the unity $1 = a_1 + a_2 + \cdots + a_n$ such that $\gamma_i = \alpha(\beta)v_i a_i$, $i = \overline{1, n}$.*

*(2) If $R$ is an integral ring then there are $\alpha \in \mathrm{Aut}(R)$ and $v \in U(R)$ such that $\gamma_i = \alpha(\beta)v$ and $\gamma_j = 0$ for some $1 \leq i \leq n$ and $i \neq j$.*

The following results were given in [2] without proofs.

**Theorem 5.8** [2, Theorem 21]. *Let $R$ be a commutative ring, $n \geq 3$, $\beta, \gamma_1, \ldots, \gamma_n \in R$, and $\mathrm{ann}_R(\beta^2) \subseteq J(R)$. Then $\mathbb{M}_{\underbrace{\beta,\beta,\ldots,\beta}_{n}}(R) \cong \mathbb{M}_{\gamma_1,\gamma_2,\ldots,\gamma_n}(R)$ if and only if $\gamma_i = \alpha(\beta)v_i$ for all $i = \overline{1, n}$, where $\alpha \in \mathrm{Aut}(R)$ and $v_i \in U(R)$.*

Proof. ($\Leftarrow$): The sufficiency follows from Proposition 5.2.

($\Rightarrow$): Let $S_1 = \mathbb{M}_{\underbrace{\beta,\beta,\ldots,\beta}_{n}}(R)$, $S_2 = \mathbb{M}_{\gamma_1,\gamma_2,\ldots,\gamma_n}(R)$, let $E_{ij}$ be the matrix units of $S_1$, let $F_{ij}$ be the matrix units of $S_2$, and let $\Theta : S_1 \to S_2$ be some given isomorphism. The argument analogous to the argument from the proof of Theorem 5.6 shows that we may assume that $S_2 = \mathbb{M}_{\beta x_1,\ldots,\beta x_n}(R)$ for some $x_1, \ldots, x_n \in R$.

Let $I \subseteq S_1$ be the ideal generated by $\{E_{ij} \mid i \neq j\}$, and let $J \subseteq S_2$ be the ideal generated by $\{F_{ij} \mid i \neq j\}$. By Lemma 5.5 we have $\Theta(I) = J$. Since $\beta^2 E \in I$, $\beta^2 E \in J$ as well. For the $(1, 1)$-element of $\beta^2 E$ considered as an element in $J$, we have $\beta^2 = \beta^2 x_1 \mu_1$ or $\beta^2(1 - x_1\mu_1) = 0$ for some $\mu_1 \in R$. It is known that $\mathrm{ann}_R(\beta^2) \subseteq J(R)$, whence $(1 - x_1\mu_1) \in J(R)$, $1 - (1 - x_1\mu_1) = x_1\mu_1 \in U(R)$, and $x_1 \in U(R)$. The proof of the fact that $x_2, \ldots, x_n \in U(R)$ is analogous. □

**Theorem 5.9.** *Let $R$ be a commutative ring such that $Z(R) \subseteq J(R)$. Let $n \geq 3$ and $\beta, \gamma_1, \ldots, \gamma_n \in R$. Then*

$$\mathbb{M}_{\underbrace{\beta,\beta,\ldots,\beta}_{n}}(R) \cong \mathbb{M}_{\gamma_1,\gamma_2,\ldots,\gamma_n}(R)$$

*if and only if $\gamma_i = \alpha(\beta)v_i$ for all $i = \overline{1, n}$, where $\alpha \in \mathrm{Aut}(R)$ and $v_i \in U(R)$.*

Proof. ($\Leftarrow$): Sufficiency follows from Proposition 5.2.

($\Rightarrow$): If $\beta^2 \neq 0$ then $\mathrm{ann}_R(\beta^2) \subseteq J(R)$, and the theorem holds by Theorem 5.8. Let $\beta^2 = 0$. Use the same notation as under the proof of Theorem 5.8. We may assume that $S_2 = \mathbb{M}_{\beta x_1,\ldots,\beta x_n}(R)$ for some $x_1, \ldots, x_n \in R$.

Show that $x_1 \in U(R)$. The proof of invertibility of the remaining $x_i$ is analogous. Since $\beta^2 = 0$; therefore, $I \cap \mathrm{diag}(R, \ldots, R) = \mathrm{diag}(0, \ldots, 0)$ and $J \cap \mathrm{diag}(R, \ldots, R) = \mathrm{diag}(0, \ldots, 0)$, $J \ni F_{21}J = \beta x_1 F_{23}R \oplus \cdots \oplus \beta x_1 F_{2n}R = \beta x_1 T$.

Put $I \ni A = (a_{ij}) = \Theta^{-1}(F_{21})$. We have $\beta x_1 \Theta^{-1}(T) = \Theta^{-1}(F_{21}J) = AI$.

For $i \neq j$ we get

$$\beta x_1 \Theta^{-1}(T) = AI \ni AE_{ij} = \beta \sum_{k=1}^{n} a_{ki} E_{kj},$$

since $a_{kk} = 0$ for all $1 \leq k \leq n$.

Hence, $\beta a_{ki} = \beta x_1 u_{ki}$ for some $u_{ki} \in R$. Thus, $\beta A = x_1 U$, whence $\beta F_{21} = \beta\Theta(A) = \Theta(\beta A) = \Theta(\beta x_1 U) = \beta x_1 \Theta(U)$, and we have $\beta = \beta x_1 y_1$ for some $y_1 \in R$. It is easy to verify that $x_1 \in U(R)$. □

The class of formal matrix rings with $\eta_{ijk} = s^{1+\delta_{ik}-\delta_{ij}-\delta_{jk}}$ and $s \in C(R)$ was considered in [4]. This formal matrix ring is denoted by $\mathbb{M}_n(R; s)$. Note that $\mathbb{M}_n(R; s) = \mathbb{M}_{\underbrace{s,\ldots,s}_{n}}(R)$.

**Corollary 5.10** [4, Theorem 18]. *Let $R$ be a commutative ring such that $Z(R) \subseteq J(R)$. Let $s, t \in R$ and $n \geq 3$. Then $\mathbb{M}_n(R; s) \cong \mathbb{M}_n(R; t)$ if and only if $t = v\alpha(s)$, where $v \in U(R)$ and $\alpha \in \mathrm{Aut}(R)$.*

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