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A WEAKLY PERIODIC GIBBS MEASURE FOR THE FERROMAGNETIC POTTS MODEL ON A CAYLEY TREE

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Abstract: We study the Potts model with q states in a Cayley tree of order $k \geq 2$. For the ferromagnetic Potts model, we prove the existence of a weakly periodic Gibbs measure that is not translation-invariant.

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1. Introduction

The notion of Gibbs measure for the Potts model on a Cayley tree is introduced in the usual manner (see [1–4]). In [5], the ferromagnetic Potts model with three states was studied and the existence was proved of a critical temperature T_c such that, for $T < T_c$, there exist three translation-invariant and countably many non-translation-invariant Gibbs measures. In [6], the results of [5] were generalized to the Potts model with finitely many states on a Cayley tree of arbitrary (finite) order.

It was proved in [7] that there is a unique translation-invariant Gibbs measure of the antiferromagnetic Potts measure with external field on a Cayley tree. The article [8] is devoted to the Potts model with countably many states and a nonzero external field on the Cayley tree. This model is proved to have the unique translation-invariant Gibbs measure.

In [9], all translation-invariant Gibbs measures were found and it was proved in particular that, for sufficiently low temperatures, their number equals $2^q - 1$. Also, it was demonstrated that there exist $[q/2]$ critical temperatures and the exact number of translation-invariant Gibbs measures for every intermediate temperature was given.

In [10], the periodic Gibbs measures were studied for the Potts model, and it was proved in particular that, for the ferromagnetic Potts model with three states, there are no periodic (non-translation-invariant) Gibbs measures. The periodic Gibbs measures for the Potts model with q states were considered in [11].

In [12], a weakly periodic Gibbs measure was introduced and some of these measures were found for the Ising model. In [13], the weakly periodic principal states and weakly periodic Gibbs measures were studied. The weakly periodic Gibbs measures in [13] are also translation-invariant.

In [14], the Potts model with q states on a Cayley tree of order $k \geq 2$ was studied, and, for the ferromagnetic Potts model, the sets of subgroups of index 2 for a group representation of a Cayley tree were distinguished for the ferromagnetic Potts model for which every weakly periodic Gibbs measure is translation-invariant, and for the antiferromagnetic Potts model for $k \geq 2$ and $q \geq 2$, the nonuniqueness of a weakly periodic Gibbs measure that is not translation-invariant was shown.

This article is devoted to weakly periodic (non-translation-invariant) Gibbs measures for the ferromagnetic Potts model on a Cayley tree. In Section 2, we give necessary definitions and familiar facts. Section 3 is devoted to the study of weakly periodic Gibbs measures corresponding to normal subgroups of index 2.

2. Definitions and Well-Known Facts

The Cayley tree $\Im^k$ of order $k \geq 1$ is an infinite tree; i.e., a graph without cycles from each of whose vertices there issue exactly $k+1$ edges. Let $\Im^k = (V, L, i)$, where V is the vertex set of $\Im^k$, while L

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is its edge set and i is the incidence function, assigning to each edge $l \in L$ its endpoints $x, y \in V$. If $i(l) = \{x, y\}$ then x and y are called *adjacent vertices* and this is denoted by $l = \langle x, y \rangle$. The distance $d(x, y)$, $x, y \in V$, on the Cayley tree is defined by the formula

$$d(x,y) = \min\{d \mid \exists x = x_0, x_1, \dots, x_{d-1}, x_d = y \in V \quad \text{such that } \langle x_0, x_1 \rangle, \dots, \langle x_{d-1}, x_d \rangle\}.$$

Given $x^0 \in V$, put $W_n = \{x \in V \mid (x, x^0) = n\}$,

$$V_n = \{x \in V \mid d(x, x^0) \le n\}, \quad L_n = \{l = \langle x, y \rangle \in L \mid x, y \in V_n\}. \tag{1}$$

Let G_k be the free product of $k+1$ cyclic groups $\{e, a_i\}$ of second order with generators $a_1, a_2, \dots, a_{k+1}$ respectively; i.e., $a_i^2 = e$ (see [15]).

There is a one-to-one correspondence between the vertex set of the Cayley tree of order k and the group G_k (see [4, 16]).

This correspondence is constructed as follows: we assign the unity e of G_k to an arbitrary fixed vertex $x_0 \in V$. Since the graph under consideration can be assumed planar without loss of generality, we assign to each vertex adjacent to x_0 (i.e., of e) the generator a_i, $i = 1, 2, \dots, k+1$, in the positive direction (Fig. 1).

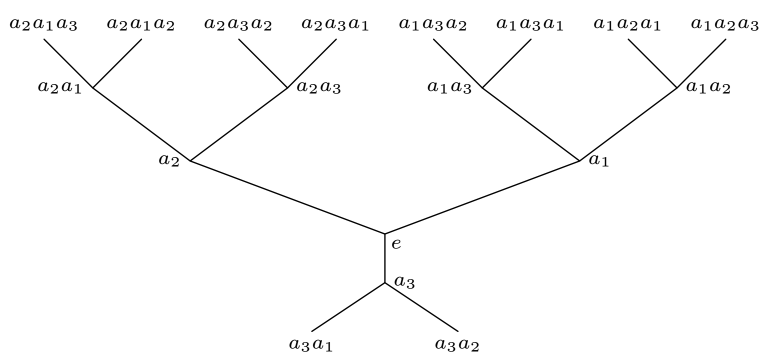

Fig. 1. The Cayley tree $\Im^2$ and the elements of the group representation of vertices

Considering each vertex a_i, define the word a_ia_j of length 2 for the vertices adjacent to a_i. Since e is one of the vertices adjacent to a_i, put $a_ia_i = e$, and then the numeration of the remaining neighbors of a_i is carried out uniquely by the above numeration rule. Further, for the vertices adjacent to a_ia_j, define a word of length 3 as follows: Since a_i is one of the adjacent vertices for a_ia_j, put $a_ia_ja_j = a_i$, and then the numeration of the adjacent vertices is carried out uniquely and has the form $a_ia_ja_l$, $i, j, l = 1, 2, \dots, k+1$. This correspondence agrees with the previous step because $a_ia_ja_j = a_ia_j^2 = a_i$. Thus, we can establish a one-to-one correspondence between the vertex set of the Cayley tree $\Im^k$ and the group G_k.

The above representation is called *right* since, in this case, if x and y are adjacent vertices and g and $h \in G_k$ are the corresponding elements of the group then either $g = ha_i$ or $h = ga_j$ for some i or j. The left representation is defined similarly.

In the group G_k (respectively, on the Cayley tree), consider the transformation of left (right) shift, defined as follows: given $g \in G_k$, put

$$T_g(h) = gh \quad (T_g(h) = hg) \quad \forall h \in G_k.$$

The set of all left (right) translations on G_k is isomorphic to G_k.

Each transformation S of G_k induces a transformation $\widehat{S}$ on the vertex set V of the Cayley tree $\Im^k$. Therefore, we identify V and G_k.

Theorem 1. *The group of left (right) shifts on the right (left) representation of a Cayley tree is a translation group (see [4, 16]).*

Consider the model where the spin variables take values in $\Phi = \{1, 2, \ldots, q\}$, $q \geq 2$, and are located at the vertices of the tree. Then a *configuration* σ on V is defined as a function $x \in V \to \sigma(x) \in \Phi$; the set of all configurations coincides with $\Omega = \Phi^V$.

The Hamiltonian of the Potts model is defined as

$$H(\sigma) = -J \sum_{\langle x,y\rangle \in L} \delta_{\sigma(x)\sigma(y)}, \tag{2}$$

where $J \in R$, $\langle x, y\rangle$ are adjacent vertices, and δ_{ij} is the Kronecker symbol:

$$\delta_{ij} = \begin{cases} 0 & \text{if } i \neq j, \\ 1 & \text{if } i = j. \end{cases}$$

Define the finite-dimensional distribution of a probability measure μ in a volume V_n as

$$\mu_n(\sigma_n) = Z_n^{-1} \exp\Big\{ -\beta H_n(\sigma_n) + \sum_{x \in W_n} h_{\sigma(x),x} \Big\}, \tag{3}$$

where $\beta = 1/T$, $T > 0$ is the temperature, Z_n^{-1} is a normalizing factor, $\{h_x = (h_{1,x}, \ldots, h_{q,x}) \in R^q, x \in V\}$ is a family of vectors, and

$$H_n(\sigma_n) = -J \sum_{\langle x,y\rangle \in L_n} \delta_{\sigma(x)\sigma(y)}.$$

The probability distribution (3) is said to be *compatible* if

$$\sum_{\omega_n \in \Phi^{W_n}} \mu_n(\sigma_{n-1} \vee \omega_n) = \mu_{n-1}(\sigma_{n-1}) \tag{4}$$

for all $n \geq 1$ and $\sigma_{n-1} \in \Phi^{V_{n-1}}$.

Here $\sigma_{n-1} \vee \omega_n$ is the union of the configurations. In this case, there exists a unique measure μ on Φ^V such that

$$\mu(\{\sigma|_{V_n} = \sigma_n\}) = \mu_n(\sigma_n)$$

for all n and $\sigma_n \in \Phi^{V_n}$. This measure is called the *split Gibbs measure* corresponding to Hamiltonian (2) and the vector-valued function h_x, $x \in V$.

The following assertion describes a condition on h_x guaranteeing the compatibility of $\mu_n(\sigma_n)$.

Theorem 2 [7]. *The probability distribution $\mu_n(\sigma_n)$, $n = 1, 2, \ldots$, in (3) is compatible if and only if*

$$h_x = \sum_{y \in S(x)} F(h_y, \theta) \tag{5}$$

for every $x \in V$. Here $F : h = (h_1, \ldots, h_{q-1}) \in R^{q-1} \to F(h, \theta) = (F_1, \ldots, F_{q-1}) \in R^{q-1}$ is defined as

$$F_i = \log \left(\frac{(\theta - 1)e^{h_i} + \sum\limits_{j=1}^{q-1} e^{h_j} + 1}{\theta + \sum\limits_{j=1}^{q-1} e^{h_j}} \right)$$

and $\theta = \exp(J\beta)$, $S(x)$ is the set of direct successors of x.

Let $G_k/G_k^* = \{H_1, \ldots, H_r\}$ be the quotient group, where G_k^* is a normal subgroup of index $r \geq 1$.

Definition 1. A set of vectors $h = \{h_x,\ x \in G_k\}$ is G_k^*-*periodic* if $h_{yx} = h_x$ for all $x \in G_k$ and $y \in G_k^*$.

The G_k-periodic sets of vectors are called *translation-invariant*.

DEFINITION 2. A set of vectors $h = \{h_x,\ x \in G_k\}$ is G_k^*-*weakly periodic* if $h_x = h_{ij}$ for $x \in H_i$ and $x_\downarrow \in H_j$ for every $x \in G_k$.

DEFINITION 3. A measure μ is G_k^*-*periodic* (*weakly periodic*) if μ corresponds to a G_k^*-periodic (weakly periodic) set of vectors h.

REMARK 1. Note that every periodic (translation-invariant) Gibbs measure is also weakly periodic but the converse fails.

3. Weakly Periodic Measures

Let q be arbitrary, i.e., $\sigma : V \to \Phi = \{1, 2, 3, \dots, q\}$. In this article we consider $q \geq 2$. Let $A \subset \{1, 2, \dots, k+1\}$. Note that, in the case of $|A| = k+1$ (where $|A|$ is the size of A), i.e., for $A = N_k$, the notion of weak periodicity coincides with the usual periodicity. Therefore, consider $A \subset N_k$ such that $A \neq N_k$, and suppose that $H_A = \{x \in G_k \mid \sum_{j\in A} w_j(x) \text{ is even}\}$, where $w_j(x)$ is the number of a_j's in the word x, and $G_k/H_A = \{H_A, G_k \setminus H_A\}$ is a quotient group. For simplicity, put $H_0 = H_A$, $H_1 = G_k \setminus H_A$, H_A is a weakly periodic family of vectors, $h = \{h_x \in R^{q-1} \mid x \in G_k\}$ has the form:

$$h_x = \begin{cases} h_1 & \text{if } x_\downarrow \in H_0,\ x \in H_0, \\ h_2 & \text{if } x_\downarrow \in H_0,\ x \in H_1, \\ h_3 & \text{if } x_\downarrow \in H_1,\ x \in H_0, \\ h_4 & \text{if } x_\downarrow \in H_1,\ x \in H_1. \end{cases}$$

Here $h_i = (h_{i1}, h_{i2}, \dots, h_{iq-1})$, $i = 1, 2, 3, 4$. Then, by (5),

$$\begin{cases} h_1 = (k - |A|)F(h_1, \theta) + |A|F(h_2, \theta), \\ h_2 = (|A| - 1)F(h_3, \theta) + (k + 1 - |A|)F(h_4, \theta), \\ h_3 = (|A| - 1)F(h_2, \theta) + (k + 1 - |A|)F(h_1, \theta), \\ h_4 = (k - |A|)F(h_4, \theta) + |A|F(h_3, \theta). \end{cases} \tag{6}$$

Introduce the notations: $e^{h_{ij}} = z_{ij}$, $i = 1, 2, 3, 4$. Then the last system can be rewritten as

$$\begin{aligned}
z_{1j} &= \left(\frac{(\theta-1)z_{1j} + \sum\limits_{i=1}^{q-1} z_{1i} + 1}{\sum\limits_{i=1}^{q-1} z_{1i} + \theta}\right)^{k-|A|} \left(\frac{(\theta-1)z_{2j} + \sum\limits_{i=1}^{q-1} z_{2i} + 1}{\sum\limits_{i=1}^{q-1} z_{2i} + \theta}\right)^{|A|}, \\
z_{2j} &= \left(\frac{(\theta-1)z_{3j} + \sum\limits_{i=1}^{q-1} z_{3i} + 1}{\sum\limits_{i=1}^{q-1} z_{3i} + \theta}\right)^{|A|-1} \left(\frac{(\theta-1)z_{4j} + \sum\limits_{i=1}^{q-1} z_{4i} + 1}{\sum\limits_{i=1}^{q-1} z_{4i} + \theta}\right)^{k+1-|A|}, \\
z_{3j} &= \left(\frac{(\theta-1)z_{2j} + \sum\limits_{i=1}^{q-1} z_{2i} + 1}{\sum\limits_{i=1}^{q-1} z_{2i} + \theta}\right)^{|A|-1} \left(\frac{(\theta-1)z_{1j} + \sum\limits_{i=1}^{q-1} z_{1i} + 1}{\sum\limits_{i=1}^{q-1} z_{1i} + \theta}\right)^{k+1-|A|}, \\
z_{4j} &= \left(\frac{(\theta-1)z_{4j} + \sum\limits_{i=1}^{q-1} z_{4i} + 1}{\sum\limits_{i=1}^{q-1} z_{4i} + \theta}\right)^{k-|A|} \left(\frac{(\theta-1)z_{3j} + \sum\limits_{i=1}^{q-1} z_{3i} + 1}{\sum\limits_{i=1}^{q-1} z_{3i} + \theta}\right)^{|A|}.
\end{aligned} \tag{7}$$

Here $j = 1, 2, 3, \dots, q-1$.

Put

$$I = \{\mathbf{z} = (z_1, z_2, \dots, z_{q-1}) \in R^{q-1} \mid z_1 = z_2 = \dots = z_{q-1}\}. \tag{8}$$

Let $\mathbf{z}_i = (z_{i1}, \dots, z_{iq-1}) \in I$ for $i = 1, 2, 3, 4$. Take $z_i = z_{i1} = \dots = z_{iq-1}$. Then the system of equations (7) is reduced to the following' system of equations:

$$\begin{aligned}
z_1 &= \left(\frac{(\theta+q-2)z_1+1}{(q-1)z_1+\theta}\right)^{k-|A|}\left(\frac{(\theta+q-2)z_2+1}{(q-1)z_2+\theta}\right)^{|A|},\\
z_2 &= \left(\frac{(\theta+q-2)z_3+1}{(q-1)z_3+\theta}\right)^{|A|-1}\left(\frac{(\theta+q-2)z_4+1}{(q-1)z_4+\theta}\right)^{k+1-|A|},\\
z_3 &= \left(\frac{(\theta+q-2)z_2+1}{(q-1)z_2+\theta}\right)^{|A|-1}\left(\frac{(\theta+q-2)z_1+1}{(q-1)z_1+\theta}\right)^{k+1-|A|},\\
z_4 &= \left(\frac{(\theta+q-2)z_4+1}{(q-1)z_4+\theta}\right)^{k-|A|}\left(\frac{(\theta+q-2)z_3+1}{(q-1)z_3+\theta}\right)^{|A|}.
\end{aligned} \tag{9}$$

It is proved in [14] that, for the Potts model for $\theta > 1$ and $|A| > \frac{k}{2}$, all H_A-weakly periodic Gibbs measures are translation-invariant.

For the ferromagnetic Potts model, consider the case of $|A| = 1$. Introduce the notation:

$$f(z) = \frac{(\theta+q-2)z+1}{(q-1)z+\theta}.$$

Then (9) takes the form

$$\begin{cases}
z_1 = (f(z_1))^{k-1}\cdot(f(z_2)),\\
z_2 = (f(z_4))^{k},\\
z_3 = (f(z_1))^{k},\\
z_4 = (f(z_4))^{k-1}\cdot(f(z_3)).
\end{cases} \tag{10}$$

Proposition 1. *Let $\mathbf{z} = \{z_1, z_2, z_3, z_4\}$ be a solution to the system of equations* (10). *If $z_i = z_j$ for $i \neq j$, where $i, j = \overline{1,4}$, then $z_1 = z_2 = z_3 = z_4$.*

Proof. For $\theta > 1$, the function $f(z)$ is strictly increasing since

$$f'(z) = \frac{(\theta-1)(\theta+q-1)}{((q-1)z+\theta)^2}.$$

Let $z_1 = z_2$. Then the first and second equations of (10) yield $(f(z_1))^k = (f(z_4))^k$. Since $f(z)$ is strictly increasing, we have $z_1 = z_4$. The second and third equations of (10) imply that $z_2 = z_3$. Consequently, $z_1 = z_2 = z_3 = z_4$.

Suppose that $z_1 = z_4$. Then the second and third equations of (10) give $z_2 = z_3$, and the first and second equations of (10) yield

$$\frac{z_1}{z_2} = \frac{f(z_2)}{f(z_1)}.$$

Since $f(z)$ is strictly increasing, the last equality holds only for $z_1 = z_2$. Therefore, $z_1 = z_2 = z_3 = z_4$.

The remaining cases are settled analogously. The assertion is proved.

Theorem 3. *Suppose that $|A| = 1$, $k \geq 6$, and $q \geq 3$. Then the ferromagnetic Potts model has critical values θ_1 and θ_2 such that, for $\theta \in (\theta_1, \theta_2)$, there exist at least two H_A-weakly periodic* (*non-translation-invariant*) *Gibbs measures, where $\theta_1 = \frac{4-3q+qk-q\sqrt{k^2-6k+1}}{4}$ and $\theta_2 = \frac{4-3q+qk+q\sqrt{k^2-6k+1}}{4}$.*

Proof. The system of equations (10) leads easily to the system of equations

$$\frac{z_1}{(f(z_1))^{k-1}} = f((f(z_4))^k), \quad \frac{z_4}{(f(z_4))^{k-1}} = f((f(z_1))^k). \tag{11}$$

The domain of $f(z)$ is $D_f = (0, +\infty)$, and its range E_f is $\left(\frac{1}{\theta}, 1 + \frac{\theta-1}{q-1}\right)$. For simplicity, put $r = \frac{1}{\theta}$ and $t = 1 + \frac{\theta-1}{q-1}$. If (11) has a solution then

$$r < \frac{z_i}{(f(z_i))^{k-1}} < t, \tag{12}$$

where $i = 1, 4$. Put $\varphi_1(z) = \frac{z}{(f(z))^{k-1}}$. It is easy to check that the function $\varphi_1(z)$ is continuous on D_f and $\varphi_1(1) = 1$. Then there exist r_1 and t_1 such that $r_1 < 1, t_1 > 1$ and (12) holds for $z_i \in (r_1, t_1)$, $i = 1, 4$. Then (11) reduces to the following system:

$$\begin{cases} f^{-1}\left(\frac{z_1}{(f(z_1))^{k-1}}\right) = (f(z_4))^k, \\ f^{-1}\left(\frac{z_4}{(f(z_4))^{k-1}}\right) = (f(z_1))^k. \end{cases} \tag{13}$$

Since $D_{f^{-1}} = (r, t), E_{f^{-1}} = (0, +\infty)$, this system of equations can be rewritten as

$$\begin{cases} \sqrt[k]{f^{-1}\left(\frac{z_1}{(f(z_1))^{k-1}}\right)} = f(z_4), \\ \sqrt[k]{f^{-1}\left(\frac{z_4}{(f(z_4))^{k-1}}\right)} = f(z_1). \end{cases} \tag{14}$$

If (14) has a solution then

$$r^k < f^{-1}\left(\frac{z_i}{(f(z_i))^{k-1}}\right) < t^k, \quad i = 1, 4. \tag{15}$$

Put $\varphi_2(z) = f^{-1}\left(\frac{z}{(f(z))^{k-1}}\right)$. It is easy to check that, for $z \in (r_1, t_1)$, the function $\varphi_2(z)$ is continuous and $\varphi_2(1) = 1$. Then there exist r_2 and t_2 such that $r_2 < 1, t_2 > 1$ and (15) is fulfilled for $z_i \in (r_2, t_2)$, $i = 1, 4$. Put $P = \max\{r_1, r_2\}$ and $Q = \min\{t_1, t_2\}$. Clearly, $P < 1$, $Q > 1$, and (12) and (15) hold for $z_i \in (P, Q)$, $i = 1, 4$. Then (11) reduces to the system of equations

$$z_1 = \psi(z_4), \quad z_4 = \psi(z_1), \tag{16}$$

where $\psi(z) = f^{-1}\left(\sqrt[k]{f^{-1}\left(\frac{z}{(f(z))^{k-1}}\right)}\right)$. Clearly, (16) has as many solutions as the equation $\psi(\psi(z)) = z$. For studying (16), we use the following well-known lemma:

Lemma 1. *Let $\gamma : [0, 1] \to [0, 1]$ be a continuous function with a fixed point $\xi \in (0, 1)$. Suppose that γ is differentiable at $\xi \in (0, 1)$ and $\gamma'(\xi) < -1$. Then there exist x_0 and x_1 such that $0 \le x_0 < \xi < x_1 \le 1$, while $\gamma(x_0) = x_1$ and $\gamma(x_1) = x_0$ (see [17, p. 70]).*

It is easy to see that the following hold for the function $\psi(z)$:
(1) $\psi(1) = 1$;
(2) $\psi(z)$ is defined on $[P_1; Q_1]$, where $P < P_1 < 1 < Q_1 < Q$;
(3) $\psi(z)$ is bounded and differentiable at $\xi = 1$.

Then, by Lemma 1, for $\psi'(1) < -1$, (16) has three solutions of the form $(1, 1)$, (x_0, x_1), (x_1, x_0), where $\psi(x_0) = x_1$ and $\psi(x_1) = x_0$. The inequality $\psi'(1) < -1$ is equivalent to the inequality

$$\frac{(\theta + q - 1)(2\theta + q - 2 - k\theta + k)}{(\theta - 1)^2 k} < -1,$$

from which we obtain

$$(\theta - \theta_1)(\theta - \theta_2) < 0,$$

where $\theta_{1,2} = \frac{4-3q+qk\pm q\sqrt{k^2-6k+1}}{4}$. Proposition 1 implies that, for $\theta \in (\theta_1, \theta_2)$, the system of equations (10) has al least two solutions of the form $\mathbf{z} = (z_1, z_2, z_3, z_4)$, where $z_i \ne z_j$, $i \ne j$, $i, j = \overline{1, 4}$; i.e., there exist at least two H_A-weakly periodic (non-translation-invariant) Gibbs measures. The theorem is proved.

Thus, for the ferromagnetic Potts model with q states, there exist H_A-weakly periodic (non-translation-invariant) Gibbs measures, whereas there are no periodic Gibbs measures for this model with three states (see [10]).

REMARK 2. Note that the H_A-weakly periodic measures in Theorem 3 are new, and they make it possible to describe continuum many non-translation-invariant Gibbs measures different from those previously known.

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