

ZETA-INVARIANTS OF THE STEKLOV SPECTRUM OF A PLANAR DOMAIN

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Abstract: The classical inverse problem of the determination of a smooth simply-connected planar domain by its Steklov spectrum [1] is equivalent to the problem of the reconstruction, up to conformal equivalence, a positive function $a \in C^\infty(\mathbb{S})$ on the unit circle $\mathbb{S} = \{e^{i\theta}\}$ from the spectrum of the operator $a\Lambda_e$, where $\Lambda_e = (-d^2/d\theta^2)^{1/2}$. We introduce $2k$ -forms $Z_k(a)$ ($k = 1, 2, \dots$) of the Fourier coefficients of a , called the *zeta-invariants*. These invariants are determined by the eigenvalues of $a\Lambda_e$. We study some properties of the forms $Z_k(a)$; in particular, their invariance under the conformal group. A few open questions about zeta-invariants is posed at the end of the article.

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1. Introduction. Three Forms of the Inverse Problem for the Steklov Spectrum

Let $\mathbb{D} = \{(x, y) \mid x^2 + y^2 \leq 1\} \subset \mathbb{R}^2 = \mathbb{C}$ be the unit disk and let $\mathbb{S} = \partial\mathbb{D} = \{e^{i\theta} \mid \theta \in \mathbb{R}\}$ be the unit circle. Define the first-order pseudodifferential operator

$$\Lambda_e = \sqrt{-d^2/d\theta^2} : C^\infty(\mathbb{S}) \rightarrow C^\infty(\mathbb{S}). \quad (1.1)$$

The equivalent definition of this operator consists in defining its values at the elements of the trigonometric basis: $\Lambda_e e^{in\theta} = |n| e^{in\theta}$. For a reason explained below, Λ_e is called the *Dirichlet-to-Neumann operator of the Euclidean metric e* (briefly, the *DN-operator*). The spectrum of this operator is as follows:

$$\text{Sp}(\Lambda_e) = \{0, 1, 1, 2, 2, \dots\},$$

where each eigenvalue is repeated according to its multiplicity.

For a positive function $a \in C^\infty(\mathbb{S})$, the operator $a\Lambda_e$ also has discrete nonnegative spectrum

$$\text{Sp}(a\Lambda_e) = \{0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \dots\},$$

which is called the *Steklov spectrum* of $a\Lambda_e$. This article addresses the question: To what extent is a function $0 < a \in C^\infty(\mathbb{S})$ determined by the Steklov spectrum $\text{Sp}(a\Lambda_e)$? This problem admits a natural gauge group, consisting of all conformal and anticonformal transformations of the disk \mathbb{D} . Let us give the definition.

The derivative $d\varphi/d\theta \in C^\infty(\mathbb{S})$ of a smooth mapping $\varphi : \mathbb{S} \rightarrow \mathbb{S}$ is defined by the equality $\varphi^*(d\theta) = (d\varphi/d\theta) d\theta$.

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DEFINITION 1.1. Two functions $a, b \in C^\infty(\mathbb{S})$ are called *conformally equivalent* if there exists a conformal or anticonformal transformation $\Phi : \mathbb{D} \rightarrow \mathbb{D}$ such that

$$b = a \circ \varphi |d\varphi/d\theta|^{-1}, \quad \text{where } \varphi = \Phi|_{\mathbb{S}}. \quad (1.2)$$

If a and b do not vanish then (1.2) can also be rewritten as

$$d\theta/b(\theta) = \pm \varphi^*(d\theta/a(\theta)).$$

REMARK. We stress the difference of the given definition from the corresponding definition in [2, § 3]: Two positive functions a and b are conformally equivalent in the sense of our definition if and only if the functions $1/a$ and $1/b$ are e -conformally equivalent in the sense of [2]. This difference arose from our desire to simplify the notation $a^{-1}\Lambda_e$ to $a\Lambda_e$. Formally speaking, the operator $a\Lambda_e$ is defined for an arbitrary (complex-valued) function $a \in C^\infty(\mathbb{S})$ and some of our results hold in this generality but the spectrum $\text{Sp}(a\Lambda_e)$ will be discussed only for a positive function a .

It is not hard to prove that $\text{Sp}(a\Lambda_e) = \text{Sp}(b\Lambda_e)$ for conformally equivalent positive functions $a, b \in C^\infty(\mathbb{S})$ (see [2]). The question of the validity of the converse assertion remains open.

Conjecture 1.2. *Given two positive functions $a, b \in C^\infty(\mathbb{S})$, we have*

$$\text{Sp}(a\Lambda_e) = \text{Sp}(b\Lambda_e) \quad (1.3)$$

if and only if these functions are conformally equivalent.

Frankly speaking, we are rather pessimistic about the validity of this conjecture in the general case. Nevertheless, this problem has many variations that are worth studying even if the answer to the above question is negative in general. For example, one can pose the following question: How many positive functions $a \in C^\infty(\mathbb{S})$ satisfy (1.3) for a given function $0 < b \in C^\infty(\mathbb{S})$? We tend to believe that, for “almost all” b , such a function a is unique up to a conformal equivalence.

The above problem has two other equivalent forms which we will briefly discuss here (see [2] for more detail).

Let $\Omega \subset \mathbb{R}^2$ be a simply-connected domain bounded by a smooth closed curve $\partial\Omega$. The *Steklov spectrum* $\text{Sp}(\Omega)$ of Ω consists of those $\lambda \in \mathbb{R}$ for which the boundary value problem

$$\Delta u = 0 \quad \text{in } \Omega, \quad \partial u / \partial \nu|_{\partial\Omega} = -\lambda u|_{\partial\Omega}$$

has a nontrivial solution. Here ν is the unit outer normal to the boundary. It is known that the spectrum $\text{Sp}(\Omega)$ is discrete and nonnegative. The classical inverse problem is as follows: To what extent is a simply-connected domain $\Omega \subset \mathbb{R}^2$ determined by its Steklov spectrum? Here some natural conjecture can be formulated as follows:

Conjecture 1.3. *A smooth simply-connected bounded domain $\Omega \subset \mathbb{R}^2$ is defined by its Steklov spectrum uniquely up to an isometry of the plane \mathbb{R}^2 endowed with the standard Euclidean metric e .*

Conjectures 1.2 and 1.3 are equivalent if we consider multisheeted domains (see [2] for more detail). The correspondence between two types of the Steklov spectrum is established as follows: If $\Phi : \mathbb{D} \rightarrow \Omega$ is a conformal mapping then $\text{Sp}(\Omega) = \text{Sp}(a\Lambda_e)$, where $1/a = |\Phi'|_{\mathbb{S}}$.

If g is a Riemannian metric on the disk \mathbb{D} then denote by Δ_g the Laplace–Beltrami operator of this metric. The DN-operator of this metric is defined as

$$\Lambda_g : C^\infty(\mathbb{S}) \rightarrow C^\infty(\mathbb{S}), \quad \Lambda_g(f) = -\partial u / \partial \nu|_{\mathbb{S}},$$

where ν is the unit outer normal to \mathbb{S} with respect to g and u is the solution to the Dirichlet problem

$$\Delta_g u = 0 \quad \text{in } \mathbb{D}, \quad u|_{\mathbb{S}} = f.$$

This coincides with (1.1) for the Euclidean metric. The spectrum $\text{Sp}(\Lambda_g)$ is again discrete and nonnegative. Again we pose the inverse problem: To what extent is the metric g on the disk \mathbb{D} determined by the spectrum $\text{Sp}(\Lambda_g)$? Here some natural conjecture is as follows:

Conjecture 1.4. *A Riemannian metric on the unit disk is determined by its Steklov spectrum uniquely up to conformal equivalence. More exactly, if g and g' are two metrics on \mathbb{D} then the equality $\text{Sp}(\Lambda_g) = \text{Sp}(\Lambda_{g'})$ holds if and only if there exist a diffeomorphism $\Psi : \mathbb{D} \rightarrow \mathbb{D}$ and a function $0 < \rho \in C^\infty(D)$ with $\rho|_{\mathbb{S}} = 1$ and $g' = \rho\Psi^*g$.*

Conjectures 1.2 and 1.4 are equivalent, as was proved in [2]. The first version of the inverse problem possibly looks simpler from the analytical viewpoint since it is concerned with finding one function of a real variable. On the other hand, the last two forms of the inverse problem look more natural from the geometric standpoint. Of course, progress in any of these statements would imply the corresponding results in the other two.

2. Zeta-Invariants

Our main construction is in fact a generalization of some of Edward's arguments [1, Theorem 2]. Recall that $\mathbb{S} = \{e^{i\theta}\}$ is the unit circle. Given a function $a \in C^\infty(\mathbb{S})$, denote its Fourier coefficients by \hat{a}_n ; i.e.,

$$a(\theta) = \sum_{n=-\infty}^{\infty} \hat{a}_n e^{in\theta}.$$

Given an integer $k \geq 1$, put

$$Z_k(a) = \sum_{j_1 + \dots + j_{2k} = 0} N_{j_1 \dots j_{2k}} \hat{a}_{j_1} \hat{a}_{j_2} \cdots \hat{a}_{j_{2k}}, \tag{2.1}$$

where, for $j_1 + \dots + j_{2k} = 0$,

$$N_{j_1 \dots j_{2k}} = \sum_{n=-\infty}^{\infty} [|n(n+j_1)(n+j_1+j_2) \cdots (n+j_1+\dots+j_{2k-1})| - n(n+j_1)(n+j_1+j_2) \cdots (n+j_1+\dots+j_{2k-1})]. \tag{2.2}$$

The quantities $Z_k(a)$ ($k = 1, 2, \dots$) will be called the *zeta-invariants* of the function a (or of the operator $a\Lambda_e$). Note that only finitely many summands can be nonzero on the right-hand side of (2.2) since the product

$$f(n) = n(n+j_1)(n+j_1+j_2) \cdots (n+j_1+\dots+j_{2k-1}) \tag{2.3}$$

is a polynomial of degree $2k$ in n taking positive values for sufficiently large $|n|$.

Series (2.1) converges absolutely because the Fourier coefficients \hat{a}_n decrease fast. We will give the corresponding estimates at the end of this section.

We emphasize that definition (2.1) makes sense for every (complex-valued) function $a \in C^\infty(\mathbb{S})$. Thus, the quantities $Z_k(a)$ are explicitly expressed via the Fourier coefficients of a although in a rather sophisticate manner. On the other hand, for a positive function a , the zeta-invariants are uniquely determined by the eigenvalues of the operator $a\Lambda_e$, as is stated by Theorem 2.1 below. Before formulating this theorem, discuss some auxiliary notions.

In the remaining part of the section, we consider a positive function $a \in C^\infty(\mathbb{S})$ normalized by the condition

$$\frac{1}{2\pi} \int_0^{2\pi} \frac{d\theta}{a(\theta)} = 1. \tag{2.4}$$

Let $\{0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \dots\}$ be the spectrum of the operator $a\Lambda_e$. The *zeta-function* of this operator is defined by the equality

$$\zeta_a(s) = \text{Tr}[(a\Lambda_e)^{-s}] = \sum_{n=1}^{\infty} \lambda_n^{-s}. \tag{2.5}$$

Recall [1] that the spectra of the operators $a\Lambda_e$ and Λ_e have the same asymptotics. This implies the convergence of (2.5) in the half-plane $\operatorname{Re} s > 1$ and also the possibility of extending $\zeta_a(s)$ to a function meromorphic on \mathbb{C} with a unique simple pole at $s = 1$. Moreover, the difference $\zeta_a(s) - 2\zeta_R(s)$ is an entire function, where $\zeta_R(s) = \sum_{n=1}^{\infty} n^{-s}$ is the classical Riemann zeta-function.

Theorem 2.1. *For every function $0 < a \in C^\infty(\mathbb{S})$ satisfying (2.4) and for all $k \geq 1$,*

$$Z_k(a) = \zeta_a(-2k).$$

For proving the theorem, we will need

Lemma 2.2. *Introduce the operator $D_\theta = -i \frac{d}{d\theta} : C^\infty(\mathbb{S}) \rightarrow C^\infty(\mathbb{S})$ on the unit circle $\mathbb{S} = \{e^{i\theta}\}$. For an arbitrary function $0 < a \in C^\infty(\mathbb{S})$ satisfying (2.4), the operators aD_θ and D_θ are intertwined, i.e., there exists a diffeomorphism $\varphi : \mathbb{S} \rightarrow \mathbb{S}$ such that $aD_\theta = \varphi^* \circ D_\theta \circ \varphi^{*-1}$, where $\varphi^* u = u \circ \varphi$ for $u \in C^\infty(\mathbb{S})$.*

PROOF. Define the diffeomorphism $\varphi : \mathbb{S} \rightarrow \mathbb{S}$ by the equality

$$\varphi(e^{i\theta}) = \exp \left[i \int_0^\theta \frac{d\tau}{a(\tau)} \right].$$

Then $\frac{d\varphi}{d\theta} = a^{-1}(\theta)$.

For every $u \in C^\infty(\mathbb{S})$,

$$(D_\theta \circ \varphi^*)u = D_\theta(u \circ \varphi) = (D_\theta u) \circ \varphi \cdot \frac{d\varphi}{d\theta} = a^{-1}(D_\theta u) \circ \varphi = a^{-1} \varphi^*(D_\theta u) = a^{-1}(\varphi^* \circ D_\theta)(u).$$

Thus, $a(D_\theta \circ \varphi^*) = \varphi^* \circ D_\theta$. This can be rewritten as $(aD_\theta) \circ \varphi^* = \varphi^* \circ D_\theta$ or $aD_\theta = \varphi^* \circ D_\theta \circ \varphi^{*-1}$. \square

By the lemma, the operators $(aD_\theta)^2$ and $D_\theta^2 = \Lambda_e^2$ are intertwined, and so

$$\operatorname{Tr}[(aD_\theta)^{2s}] = \operatorname{Tr}[\Lambda_e^{2s}] \quad (\operatorname{Re} s < -1).$$

In what follows, we use only this relation.

PROOF OF THEOREM 2.1. Recall that the classical Riemann zeta-function has zeros at even negative integer points: $\zeta_R(-2k) = 0$ ($k = 1, 2, \dots$). Therefore,

$$\zeta_a(-2k) = \zeta_a(-2k) - 2\zeta_R(-2k) = \operatorname{Tr}[(a\Lambda_e)^{2k} - D_\theta^{2k}].$$

From this by Lemma 2.2 we conclude that

$$\zeta_a(-2k) = \operatorname{Tr}[(a\Lambda_e)^{2k} - (aD_\theta)^{2k}]. \quad (2.6)$$

Find the right-hand side of (2.6) by computing the values of $(a\Lambda_e)^{2k}$ and $(aD_\theta)^{2k}$ at the elements of the trigonometric basis $e^{in\theta}$.

Using induction on k , validate the formula

$$\begin{aligned} (a\Lambda_e)^{2k} e^{in\theta} &= \sum_{r_1, \dots, r_k} \sum_{j_1+j_2=r_1-n} \sum_{j_3+j_4=r_2-r_1} \cdots \sum_{j_{2k-1}+j_{2k}=r_k-r_{k-1}} |nr_1 \cdots r_{k-1}| \\ &\quad \times |(n+j_1)(r_1+j_3)(r_2+j_5) \cdots (r_{k-1}+j_{2k-1})| \hat{a}_{j_1} \hat{a}_{j_2} \cdots \hat{a}_{j_{2k}} e^{ir_k\theta}. \end{aligned} \quad (2.7)$$

Begin with the obvious equality $(a\Lambda_e)e^{in\theta} = |n|a e^{in\theta}$. Applying $a\Lambda_e$ to this equality, we infer

$$\begin{aligned} (a\Lambda_e)^2 e^{in\theta} &= |n|a\Lambda_e(ae^{in\theta}) = |n|a\Lambda_e \left(\sum_{j_1} \hat{a}_{j_1} e^{i(n+j_1)\theta} \right) \\ &= |n|a \sum_{j_1} \hat{a}_{j_1} |n+j_1| e^{i(n+j_1)\theta} = \sum_{j_2} \hat{a}_{j_2} e^{ij_2\theta} \sum_{j_1} \hat{a}_{j_1} |n(n+j_1)| e^{i(n+j_1)\theta} \\ &= \sum_r \left(\sum_{j_1+j_2=r-n} |n(n+j_1)| \hat{a}_{j_1} \hat{a}_{j_2} \right) e^{ir\theta}. \end{aligned} \quad (2.8)$$

This coincides with (2.7) for $k = 1$.

Make the induction step, for which apply $(a\Lambda_e)^2$ to (2.7):

$$(a\Lambda_e)^{2(k+1)}e^{in\theta} = \sum_{r_1, \dots, r_k} \sum_{j_1+j_2=r_1-n} \sum_{j_3+j_4=r_2-r_1} \cdots \sum_{j_{2k-1}+j_{2k}=r_k-r_{k-1}} |nr_1 \cdots r_{k-1}| \\ \times |(n+j_1)(r_1+j_3) \cdots (r_{k-1}+j_{2k-1})| \hat{a}_{j_1} \cdots \hat{a}_{j_{2k}} (a\Lambda_e)^2 e^{ir_k\theta}.$$

Using (2.8), we get

$$(a\Lambda_e)^{2(k+1)}e^{in\theta} = \sum_{r_1, \dots, r_k} \sum_{j_1+j_2=r_1-n} \sum_{j_3+j_4=r_2-r_1} \cdots \sum_{j_{2k-1}+j_{2k}=r_k-r_{k-1}} |nr_1 \cdots r_{k-1}| \\ \times |(n+j_1)(r_1+j_3) \cdots (r_{k-1}+j_{2k-1})| \hat{a}_{j_1} \cdots \hat{a}_{j_{2k}} \\ \times \sum_{r_{k+1}} \sum_{j_{2k+1}+j_{2k+2}=r_{k+1}-r_k} |r_k(r_k+j_{2k+1})| \hat{a}_{j_{2k+1}} \hat{a}_{j_{2k+2}} e^{ir_{k+1}\theta}.$$

Changing the order of summation here gives (2.7) for $k := k + 1$. Thus, (2.7) is proved.

The formula

$$(aD_\theta)^{2k}e^{in\theta} = \sum_{r_1, \dots, r_k} \sum_{j_1+j_2=r_1-n} \sum_{j_3+j_4=r_2-r_1} \cdots \sum_{j_{2k-1}+j_{2k}=r_k-r_{k-1}} nr_1 \cdots r_{k-1} \\ \times (n+j_1)(r_1+j_3)(r_2+j_5) \cdots (r_{k-1}+j_{2k-1}) \hat{a}_{j_1} \cdots \hat{a}_{j_{2k}} e^{ir_k\theta} \quad (2.9)$$

is proved by repeating the arguments for (2.7). There is no actual need in repetition; it suffices only to compare the equalities

$$(a\Lambda_e)e^{in\theta} = |n|ae^{in\theta}, \quad (aD_\theta)e^{in\theta} = nae^{in\theta},$$

from which we see that all formulas for $a\Lambda_e$ become valid also for aD_θ if we delete the modulus signs therein.

Subtracting (2.9) from (2.7), we obtain

$$[(a\Lambda_e)^{2k} - (aD_\theta)^{2k}]e^{in\theta} = \sum_{r_1, \dots, r_k} \sum_{j_1+j_2=r_1-n} \sum_{j_3+j_4=r_2-r_1} \\ \cdots \sum_{j_{2k-1}+j_{2k}=r_k-r_{k-1}} N(n; r_1, \dots, r_{k-1}; j_1, j_3, \dots, j_{2k-1}) \hat{a}_{j_1} \cdots \hat{a}_{j_{2k}} e^{ir_k\theta}, \quad (2.10)$$

where we used the provisional notation

$$N(n; r_1, \dots, r_{k-1}; j_1, j_3, \dots, j_{2k-1}) = |nr_1 \cdots r_{k-1}(n+j_1)(r_1+j_3) \cdots (r_{k-1}+j_{2k-1})| \\ - nr_1 \cdots r_{k-1}(n+j_1)(r_1+j_3) \cdots (r_{k-1}+j_{2k-1}).$$

For computing the trace of $(a\Lambda_e)^{2k} - (aD_\theta)^{2k}$, we must distinguish the coefficient at $e^{in\theta}$ on the right-hand side of (2.10); i.e., put $r_k = n$ and carry out summation over n :

$$\text{Tr}[(a\Lambda_e)^{2k} - (aD_\theta)^{2k}] = \sum_n \sum_{r_1, \dots, r_{k-1}} \sum_{j_1+j_2=r_1-n} \sum_{j_3+j_4=r_2-r_1} \\ \cdots \sum_{j_{2k-3}+j_{2k-2}=r_{k-1}-r_{k-2}} \sum_{j_{2k-1}+j_{2k}=n-r_{k-1}} N(n; r_1, \dots, r_{k-1}; j_1, j_3, \dots, j_{2k-1}) \hat{a}_{j_1} \cdots \hat{a}_{j_{2k}}. \quad (2.11)$$

Now, change the summation order in (2.11) so that summation over n become most internal (the possibility of changing summation order is easy to justify). For this, fixing the value of n , put

$$r_1 = j_1 + j_2 + n = n + j_1 + j_2, \\ r_2 = j_3 + j_4 + r_1 = n + j_1 + j_2 + j_3 + j_4, \\ \dots \dots \dots \\ r_{k-1} = j_{2k-3} + j_{2k-2} + r_{k-2} = n + j_1 + j_2 + \cdots + j_{2k-2}.$$

Then (2.11) takes the form

$$\mathrm{Tr}[(a\Lambda_e)^{2k} - (aD_\theta)^{2k}] = \sum_{j_1 + \dots + j_{2k} = 0} \sum_n \tilde{N}(n; j_1, \dots, j_{2k-1}) \hat{a}_{j_1} \cdots \hat{a}_{j_{2k}}, \quad (2.12)$$

where

$$\begin{aligned} & \tilde{N}(n; j_1, \dots, j_{2k-1}) \\ &= N(n; n + j_1 + j_2, n + j_1 + j_2 + j_3 + j_4, \dots, n + j_1 + \dots + j_{2k-2}; j_1, j_3, \dots, j_{2k-1}) \\ &= |n(n + j_1)(n + j_1 + j_2) \cdots (n + j_1 + j_2 + \dots + j_{2k-1})| \\ &\quad - n(n + j_1)(n + j_1 + j_2) \cdots (n + j_1 + j_2 + \dots + j_{2k-1}). \end{aligned}$$

The right-hand of (2.12) coincides with that of (2.1). Thus, we have proved that

$$\mathrm{Tr}[(a\Lambda_e)^{2k} - (aD_\theta)^{2k}] = Z_k(a).$$

This together with (2.6) gives the theorem. \square

Let us inspect (2.1) in more detail. The coefficients of the series possess the evenness

$$N_{-j_1, \dots, -j_{2k}} = N_{j_1 \dots j_{2k}} \quad (j_1 + \dots + j_{2k} = 0) \quad (2.13)$$

which is proved by the change $m = -n$ of the summation index in (2.2). These coefficients do not change either under cyclic permutations of the indices:

$$N_{j_1 j_2 \dots j_{2k}} = N_{j_2 j_3 \dots j_{2k} j_1} = \cdots = N_{j_{2k} j_1 \dots j_{2k-1}} \quad (j_1 + \dots + j_{2k} = 0) \quad (2.14)$$

which is proved by the change $m = n + j_1$ of the summation index in (2.2). But the coefficients $N_{j_1 \dots j_{2k}}$ are in general not invariant under arbitrary permutations of indices.

It makes sense to symmetrize the coefficients of the $2k$ -form (2.1), i.e. to rewrite this form as

$$Z_k(a) = \sum_{j_1, \dots, j_{2k} = -\infty}^{\infty} Z_{j_1 \dots j_{2k}} \hat{a}_{j_1} \cdots \hat{a}_{j_{2k}}, \quad (2.15)$$

where

$$\begin{aligned} Z_{j_1 \dots j_{2k}} &= 0 \quad \text{for } j_1 + \dots + j_{2k} \neq 0, \\ Z_{j_1 \dots j_{2k}} &= \frac{1}{(2k)!} \sum_{\pi \in \Pi_{2k}} N_{j_{\pi(1)} \dots j_{\pi(2k)}} \quad \text{for } j_1 + \dots + j_{2k} = 0. \end{aligned} \quad (2.16)$$

Here Π_{2k} is the group of all permutations of the set $\{1, 2, \dots, 2k\}$. The coefficients $Z_{j_1 \dots j_{2k}}$ are symmetric, i.e., invariant under every permutation of the indices (j_1, \dots, j_{2k}) . Of course, symmetrization preserves evenness (2.13); i.e.,

$$Z_{-j_1, \dots, -j_{2k}} = Z_{j_1 \dots j_{2k}}. \quad (2.17)$$

This implies the important assertion: All zeta-invariants are real for a real function a . Indeed, applying complex conjugation to (2.15) and reckoning with the reality of $Z_{j_1 \dots j_{2k}}$, we have

$$\overline{Z_k(a)} = Z_k(\bar{a}) = \sum_{j_1, \dots, j_{2k} = -\infty}^{\infty} Z_{j_1 \dots j_{2k}} \bar{\hat{a}}_{j_1} \cdots \bar{\hat{a}}_{j_{2k}}.$$

The Fourier coefficients of a real function satisfy $\bar{\hat{a}}_j = \hat{a}_{-j}$. Therefore, the last formula takes the form

$$\overline{Z_k(a)} = \sum_{j_1, \dots, j_{2k} = -\infty}^{\infty} Z_{-j_1, \dots, -j_{2k}} \hat{a}_{j_1} \cdots \hat{a}_{j_{2k}}.$$

The right-hand side of this formula coincides with the right-hand side of (2.15) because the coefficients are even.

Formula (2.16) can be simplified a slightly by using (2.14). Indeed, denote by Π_{2k-1} the subgroup in Π_{2k} consisting of all permutations fixing the last element; i.e.,

$$\Pi_{2k-1} = \{\pi = (\pi(1), \dots, \pi(2k-1), 2k)\} \subset \Pi_{2k}.$$

Let $\zeta = (2, 3, \dots, 2k, 1)$ be a cyclic permutation. Represent Π_{2k} as the union of cosets

$$\Pi_{2k} = \bigcup_{\ell=0}^{2k-1} \zeta^\ell \Pi_{2k-1}$$

and partition the set of all summands of the sum (2.16) into $2k$ subsets corresponding to this representation. The so-obtained partial sums coincide by (2.14), and (2.16) is simplified to the following:

$$Z_{j_1 \dots j_{2k}} = \frac{1}{(2k-1)!} \sum_{\pi \in \Pi_{2k-1}} N_{j_{\pi(1)} \dots j_{\pi(2k-1)} j_{2k}} \quad \text{for } j_1 + \dots + j_{2k} = 0. \quad (2.18)$$

Prove the absolute convergence of series (2.15). To this end, let us first validate the following estimate for the coefficients of the series:

$$0 \leq Z_{j_1 \dots j_{2k}} \leq 2(2(|j_1| + \dots + |j_{2k}|))^{2k+1}. \quad (2.19)$$

Indeed, fix $j = (j_1, \dots, j_{2k})$ and put $|j| = |j_1| + \dots + |j_{2k}|$. Denote by x_- and x_+ the minimal and maximal roots of the polynomial $f(n)$ defined by (2.3). They satisfy the inequality $|x_\pm| \leq |j|$. A summand in (2.2) is nonzero only for $n \in (x_-, x_+)$, the number of these summands is at most $2|j|$, and each of them is at most $2(|n| + |j|)^{2k} \leq 2(2|j|)^{2k}$. Therefore,

$$N_{j_1 \dots j_{2k}} \leq 2(2|j|)^{2k}(2|j|) = 2(2|j|)^{2k+1}.$$

This proves (2.19).

The Fourier coefficients of a smooth function a decrease rapidly, i.e., they satisfy the estimate $|\hat{a}_n| \leq C_M(|n| + 1)^{-M}$ for every $M > 0$. Together with (2.19), this implies the absolute convergence of (2.15). Indeed,

$$|Z_{j_1 \dots j_{2k}} \hat{a}_{j_1} \dots \hat{a}_{j_{2k}}| \leq 2^{2k+2} C_M^{2k} (|j| + 1)^{-M+2k+1}, \quad \text{where } |j| = |j_1| + \dots + |j_{2k}|.$$

Therefore,

$$\sum_{j_1 + \dots + j_{2k} = 0} |Z_{j_1 \dots j_{2k}} \hat{a}_{j_1} \dots \hat{a}_{j_{2k}}| \leq 2^{2k+2} C_M^{2k} \sum_{\ell=0}^{\infty} (\ell + 1)^{-M+2k+1} K(\ell),$$

where

$$K(\ell) = \#\{(j_1, \dots, j_{2k}) \mid j_1 + \dots + j_{2k} = 0, |j_1| + \dots + |j_{2k}| = \ell\} \leq (2\ell + 1)^{2k}.$$

Finally,

$$\sum_{j_1 + \dots + j_{2k} = 0} |Z_{j_1 \dots j_{2k}} \hat{a}_{j_1} \dots \hat{a}_{j_{2k}}| \leq 2^{4k+2} C_M^{2k} \sum_{\ell=0}^{\infty} (\ell + 1)^{-M+4k+1}.$$

The series on the right-hand side of this inequality converges for M sufficiently large.

The first zeta-invariant was in fact introduced by Edward [1]. We will reproduce his calculations here. By (2.1)–(2.2),

$$Z_1(a) = \sum_{j+\ell=0} Z_{j\ell} \hat{a}_j \hat{a}_\ell = \sum_j Z_{j,-j} \hat{a}_j \hat{a}_{-j}, \quad (2.20)$$

where

$$Z_{j,-j} = N_{j,-j} = \sum_n (|n(n+j)| - n(n+j)).$$

Obviously,

$$|n(n+j)| - n(n+j) = \begin{cases} -2n(n+j) & \text{if } 0 < n < -j, \\ -2n(n+j) & \text{if } -j < n < 0, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, for positive j ,

$$Z_{j,-j} = -2 \sum_{n=-j}^{-1} n(n+j) = -2 \sum_{n=-j}^{-1} n^2 - 2j \sum_{n=-j}^{-1} n = -2 \sum_{n=1}^j n^2 + 2j \sum_{n=1}^j n = \frac{1}{3}(j^3 - j).$$

We have used the familiar equalities

$$\sum_{n=1}^j n = \frac{1}{2}j(j+1), \quad \sum_{n=1}^j n^2 = \frac{1}{6}j(j+1)(2j+1). \quad (2.21)$$

Similarly, $Z_{j,-j} = \frac{1}{3}|j^3 - j|$ for negative j . Thus, for all j ,

$$Z_{j,-j} = \frac{1}{3}|j^3 - j|. \quad (2.22)$$

Inserting these values in (2.20), we infer

$$Z_1(a) = \frac{1}{3} \sum_{j=-\infty}^{\infty} |j^3 - j| \hat{a}_j \hat{a}_{-j} = \frac{2}{3} \sum_{n=2}^{\infty} (n^3 - n) \hat{a}_n \hat{a}_{-n}. \quad (2.23)$$

If, for two positive functions $a, b \in C^\infty(\mathbb{S})$, the operators $a\Lambda_e$ and $b\Lambda_e$ are isospectral; then, by Theorem 2.1,

$$Z_k(a) = Z_k(b) \quad (k = 1, 2, \dots). \quad (2.24)$$

In particular, (2.24) holds for conformally equivalent positive functions a and b .

The *completeness problem of the system of zeta-invariants* is posed as follows: Given a function $0 < b \in C^\infty(\mathbb{S}^1)$, find all $0 < a \in C^\infty(\mathbb{S}^1)$ satisfying the system of equations

$$Z_k(a) = b_k \quad (k = 1, 2, \dots), \quad (2.25)$$

where $b_k = Z_k(b)$. Note that the spectrum of $a\Lambda_e$ is not involved in the statement of this problem and one should not recall it in solving the problem. This is a purely algebraic problem since the left-hand side of (2.25) is a $(2k)$ -form in Fourier coefficients of a . By analogy with spectral geometry, we tend to believe that, for a function $0 < b \in C^\infty(\mathbb{S}^1)$ in “general position,” the set of solutions to system (2.25) coincides with the family of functions conformally equivalent to b . At the same time, we conjecture the existence of “exceptional” examples of the functions b for which the set of solutions to (2.25) is wider (although such examples are not found yet). The further content of the article focuses on solving this problem but we are still far from the final solution.

3. Conformal Equivalence in Terms of Fourier Coefficients

Given $\rho \in (-1, 1)$, denote by Φ_ρ the conformal transformation of the unit disk defined by the equality

$$\Phi_\rho(z) = \frac{z - \rho}{1 - \rho z}. \quad (3.1)$$

Given $a \in C^\infty(\mathbb{S})$, let b be the function conformally equivalent to a by means of Φ_ρ , i.e.,

$$b = a \circ \varphi \left(\frac{d\varphi}{d\theta} \right)^{-1}, \quad \text{where } \varphi = \Phi_\rho|_{\mathbb{S}}. \quad (3.2)$$

This fact will be denoted by the equality $b = a\Phi_\rho$, and the notation will be explained in Section 4. As is seen from (3.2), the Fourier coefficients \hat{b}_n of b depend linearly on the Fourier coefficients \hat{a}_n of a ; i.e.,

$$\hat{b}_n = \sum_k \mu_{nk}(\rho) \hat{a}_k.$$

In this section, we find the (infinite) matrix $M(\rho) = (\mu_{nk}(\rho))_{n,k=-\infty}^\infty$ and establish some of its properties.

By the definition of Fourier coefficients,

$$\hat{b}_n = \frac{1}{2\pi} \int_0^{2\pi} e^{-in\theta} b(\theta) d\theta = \frac{1}{2\pi} \int_0^{2\pi} e^{-in\theta} a(\varphi(\theta)) \left(\frac{d\varphi}{d\theta} \right)^{-1} d\theta$$

or

$$\hat{b}_n = \frac{1}{2\pi} \int_0^{2\pi} e^{-in\theta(\varphi)} a(\varphi) \left(\frac{d\theta}{d\varphi}(\varphi) \right)^2 d\varphi.$$

Perform the change of variables by the formulas

$$z = e^{i\varphi}, \quad d\varphi = \frac{1}{i} z^{-1} dz, \quad e^{i\theta(\varphi)} = \frac{z + \rho}{1 + \rho z}, \quad \frac{d\theta}{d\varphi} = \frac{1 - \rho^2}{|1 + \rho z|^2}.$$

Then

$$\hat{b}_n = (1 - \rho^2)^2 \frac{1}{2\pi i} \oint_{|z|=1} \left(\frac{1 + \rho z}{z + \rho} \right)^n z^{-1} |1 + \rho z|^{-4} a(z) dz.$$

Inserting $a(z) = \sum_k \hat{a}_k z^k$, we obtain the formula

$$\hat{b}_n = \sum_{k=-\infty}^\infty \mu_{nk} \hat{a}_k, \quad (3.3)$$

where

$$\mu_{nk} = (1 - \rho^2)^2 \frac{1}{2\pi i} \oint_{|z|=1} \left(\frac{1 + \rho z}{z + \rho} \right)^n z^{k-1} |1 + \rho z|^{-4} dz. \quad (3.4)$$

Expand the last factor under the integral in (3.4) in the powers of z taking into account the relation $|z| = 1$:

$$|1 + \rho z|^2 = (1 + \rho z)(1 + \rho z^{-1}) = \frac{1}{z}(1 + \rho z)(z + \rho), \quad |1 + \rho z|^{-4} = z^2(1 + \rho z)^{-2}(z + \rho)^{-2}.$$

Inserting this in (3.4), we obtain the final formula

$$\mu_{nk} = \mu_{nk}(\rho) = (1 - \rho^2)^2 \frac{1}{2\pi i} \oint_{|z|=1} \frac{(1 + \rho z)^{n-2}}{(z + \rho)^{n+2}} z^{k+1} dz. \quad (3.5)$$

Introduce also the constant matrix $D = (d_{nk})_{n,k=-\infty}^\infty$ by setting

$$d_{nk} = (n - 2)\delta_{n-1,k} - (n + 2)\delta_{n+1,k} = \begin{cases} n - 2 & \text{if } k = n - 1, \\ -(n + 2) & \text{if } k = n + 1, \\ 0 & \text{otherwise.} \end{cases} \quad (3.6)$$

Proposition 3.1. *The matrix $M(\rho)$ is expressed via D and $\rho \in (-1, 1)$ by the equality*

$$M(\rho) = e^{tD}, \quad \text{where } \tanh t = \rho. \quad (3.7)$$

The mapping $\rho \mapsto M(\rho)$ satisfies

$$M(\rho)M(\rho') = M(\rho''), \quad \text{where } \rho'' = \frac{\rho + \rho'}{1 + \rho\rho'}. \quad (3.8)$$

In particular, $M(\rho)$ and $M(\rho')$ commute, and $M(\rho)$ commutes with D .

PROOF. It suffices to prove (3.7); the remaining assertions follow from this equality, which is in turn equivalent to the differential equation $\frac{dM}{dt} = DM$. Assuming that ρ and t are so that $\tanh t = \rho$, rewrite the last equation in the form $(1 - \rho^2)\frac{dM}{d\rho} = DM$, whence from (3.6) we obtain

$$(1 - \rho^2)\frac{d\mu_{nk}}{d\rho} - (n - 2)\mu_{n-1,k} + (n + 2)\mu_{n+1,k} = 0. \quad (3.9)$$

Thus, everything amounts to proving (3.9).

Differentiating (3.5), we have

$$\begin{aligned} \frac{d\mu_{nk}}{d\rho} &= \frac{(1 - \rho^2)}{2\pi i} \oint_{|z|=1} \frac{(1 + \rho z)^{n-3}}{(z + \rho)^{n+3}} [-4\rho(1 + \rho z)(z + \rho) \\ &+ (n - 2)(1 - \rho^2)z(z + \rho) - (n + 2)(1 - \rho^2)(1 + \rho z)] z^{k+1} dz. \end{aligned}$$

Inserting the expression for the derivative $d\mu_{nk}/d\rho$ and the expression for $\mu_{n\pm 1,k}$ on the left-hand side of (3.9), we get the equation

$$(1 - \rho^2)\frac{d\mu_{nk}}{d\rho} - (n - 2)\mu_{n-1,k} + (n + 2)\mu_{n+1,k} = \frac{(1 - \rho^2)^2}{2\pi i} \oint_{|z|=1} \frac{(1 + \rho z)^{n-3}}{(z + \rho)^{n+3}} f(n, \rho, z) z^{k+1} dz,$$

where

$$\begin{aligned} f(n, \rho, z) &= -4\rho(1 + \rho z)(z + \rho) + (n - 2)(1 - \rho^2)z(z + \rho) - (n + 2)(1 - \rho^2)(1 + \rho z) \\ &- (n - 2)(z + \rho)^2 + (n + 2)(1 + \rho z)^2. \end{aligned}$$

As is easy to verify, the function $f(n, \rho, z)$ is identically zero. \square

Integral (3.5) can be calculated by the residue theorem. First of all, the integrand in (3.5) is a holomorphic function in the unit disk for $n \leq -2$ and $k \geq -1$. Therefore,

$$\mu_{nk} = 0 \quad \text{for } n \leq -2, k \geq -1. \quad (3.10)$$

Changing the integration variable in (3.5) by the equality $z = 1/\zeta$, we have

$$\begin{aligned} \mu_{nk} &= (1 - \rho^2)^2 \frac{1}{2\pi i} \oint_{|\zeta|=1} \frac{(1 + \rho/\zeta)^{n-2}}{(1/\zeta + \rho)^{n+2}} \zeta^{-k-3} d\zeta = (1 - \rho^2)^2 \frac{1}{2\pi i} \oint_{|\zeta|=1} \frac{(\zeta + \rho)^{n-2}}{(1 + \rho\zeta)^{n+2}} \zeta^{-k+1} d\zeta \\ &= (1 - \rho^2)^2 \frac{1}{2\pi i} \oint_{|\zeta|=1} \frac{(1 + \rho\zeta)^{-n-2}}{(\zeta + \rho)^{-n+2}} \zeta^{-k+1} d\zeta = \mu_{-n, -k}. \end{aligned}$$

Thus,

$$\mu_{nk} = \mu_{-n, -k}. \quad (3.11)$$

Together with (3.10), this gives

$$\mu_{nk} = 0 \quad \text{for } n \geq 2, k \leq 1. \quad (3.12)$$

By (3.11), it suffices to consider only the case of $n \geq 0$.

Assuming that $k \geq -1$, from (3.5) we obtain

$$\mu_{nk} = (1 - \rho^2)^2 \operatorname{Res} \left[\frac{(1 + \rho z)^{n-2}}{(z + \rho)^{n+2}} z^{k+1} \right]_{z=-\rho} \quad (k \geq -1). \quad (3.13)$$

For finding this residue, we must expand the function $\frac{(1+\rho z)^{n-2}}{(z+\rho)^{n+2}} z^{k+1}$ in the powers of $(z + \rho)$.

First of all,

$$z^{k+1} = \sum_{\ell=0}^{k+1} (-1)^{k-\ell+1} \binom{k+1}{\ell} \rho^{k-\ell+1} (z + \rho)^\ell. \quad (3.14)$$

Here and below, $\binom{r}{s} = \frac{r!}{s!(r-s)!}$ are the binomial coefficients, which are assumed to be defined for all integers r and s with account taken of the agreement

$$\binom{r}{s} = 0, \quad \text{if } r < 0 \text{ or } s < 0, \text{ or } s > r. \quad (3.15)$$

Further, using the identity $1 + \rho z = \rho(z + \rho) + (1 - \rho^2)$, we find

$$(1 + \rho z)^{n-2} = \sum_{p=0}^{n-2} \binom{n-2}{p} \rho^p (1 - \rho^2)^{n-p-2} (z + \rho)^p. \quad (3.16)$$

For $n \geq 2$ formulas (3.14) and (3.16) give

$$\frac{(1 + \rho z)^{n-2} z^{k+1}}{(z + \rho)^{n+2}} = \sum_{m=0}^{n+k-1} \left(\sum_{\ell+p=m} (-1)^{k-\ell+1} \binom{n-2}{p} \binom{k+1}{\ell} \rho^{k+p-\ell+1} (1 - \rho^2)^{n-p-2} \right) (z + \rho)^{m-n-2}.$$

In accordance with (3.13), we must find the coefficient at $(z + \rho)^{-1}$ on the right-hand side of the last formula, i.e., put $m = n + 1$. Thus,

$$\mu_{nk} = (1 - \rho^2)^2 \sum_{\ell+p=n+1} (-1)^{k-\ell+1} \binom{n-2}{p} \binom{k+1}{\ell} \rho^{k+p-\ell+1} (1 - \rho^2)^{n-p-2}.$$

Putting here $p = n - \ell + 1$, we get the final formula

$$\mu_{nk} = (-1)^{k+1} \frac{\rho^{n+k+2}}{1 - \rho^2} \sum_{\ell} (-1)^\ell \binom{n-2}{\ell-3} \binom{k+1}{\ell} \rho^{-2\ell} (1 - \rho^2)^\ell \quad (n \geq 2, k \geq -1). \quad (3.17)$$

In this equality the sum is actually taken over

$$3 \leq \ell \leq \min(n + 1, k + 1). \quad (3.18)$$

Formulas (3.12) and (3.17) give explicit expressions for μ_{nk} for $n \geq 2$ and all k . Together with (3.11), this gives μ_{nk} for $|n| \geq 2$ and all k . It remains to consider the cases of $n = 0, \pm 1$. Present the results for these cases without giving their proofs which are analogous to those above:

$$\mu_{-1,k} = \frac{(-\rho)^{k+1}}{1 - \rho^2} \quad \text{for } k \geq -1, \quad (3.19)$$

$$\mu_{0,k} = \mu_{0,-k} = \frac{(-\rho)^k}{1 - \rho^2} ((k + 1) - (k - 1)\rho^2) \quad \text{for } k \geq -1, \quad (3.20)$$

$$\mu_{1,k} = \frac{(-\rho)^{k-1}}{1 - \rho^2} \left(\frac{k(k+1)}{2} - (k^2 - 1)\rho^2 + \frac{k(k-1)}{2}\rho^4 \right) \quad \text{for } k \geq -1. \quad (3.21)$$

In particular,

$$\mu_{n,-1} = \mu_{n,0} = \mu_{n,1} = 0 \quad \text{for } |n| \geq 2, \quad (3.22)$$

$$\begin{pmatrix} \mu_{-1,-1} & \mu_{-1,0} & \mu_{-1,1} \\ \mu_{0,-1} & \mu_{0,0} & \mu_{0,1} \\ \mu_{1,-1} & \mu_{1,0} & \mu_{1,1} \end{pmatrix} = \frac{1}{1-\rho^2} \begin{pmatrix} 1 & -\rho & \rho^2 \\ -2\rho & 1+\rho^2 & -2\rho \\ \rho^2 & -\rho & 1 \end{pmatrix}. \quad (3.23)$$

As is easy to see, $(1/2, 1, 1/2)^t$ is an eigenvector of matrix (3.23) with eigenvalue $\frac{1-\rho}{1+\rho}$. This means that, for any $\rho \in (0, 1)$, the function $1 + \cos \theta$ is an eigenfunction of the operator $a \mapsto a\Phi_\rho$ with eigenvalue $\frac{1-\rho}{1+\rho}$.

In Section 4, we will need the estimate

$$|\mu_{n,k}(\rho)| \leq C_k |n|^{|k|} |\rho|^{|n|/2} \quad \text{for } |n| \geq 2 |k| \geq 2. \quad (3.24)$$

It easily follows from (3.17). Indeed, suppose first that $n \geq 2$ and $k \geq 0$. From (3.17) we conclude that

$$|\mu_{nk}| \leq \sum_{\ell} \binom{n-2}{\ell-3} \binom{k+1}{\ell} |\rho|^{n+2k-2\ell+2} (1-\rho^2)^{\ell-1}.$$

As was observed above, the sum here is in fact taken over ℓ satisfying (3.18). Therefore, the last factor on the right-hand side is bounded above by unity and the inequality is simplified to the following:

$$|\mu_{nk}| \leq \sum_{\ell} \binom{n-2}{\ell-3} \binom{k+1}{\ell} |\rho|^{n+2k-2\ell+2}.$$

Assuming that $n \geq 2k \geq 0$, from (3.18) we deduce the inequality $n+2k-2\ell+2 \geq n/2$, which enables us to write our estimate as

$$|\mu_{nk}| \leq |\rho|^{n/2} \sum_{\ell} \binom{n-2}{\ell-3} \binom{k+1}{\ell}.$$

Finally,

$$\binom{n-2}{\ell-3} = \frac{(n-2)(n-3)\dots(n-\ell+2)}{(\ell-3)!} \leq \frac{n^{\ell-2}}{(\ell-3)!} \leq \frac{n^{k-1}}{(\ell-3)!}$$

and

$$|\mu_{nk}| \leq n^{k-1} |\rho|^{n/2} \sum_{\ell=3}^{k+1} \frac{1}{(\ell-3)!} \binom{k+1}{\ell} = C_k n^{k-1} |\rho|^{n/2}.$$

This proves (3.24) for $n \geq 2$ and $k \geq 0$. If $n \geq 2$ and $k \leq 0$ then estimate (3.24) holds trivially in view of (3.12). Finally, for validating (3.24) for negative n , it suffices to recall evenness (3.11).

4. The Zeta-Invariants and the Conformal Group

Denote by G the group of all conformal and anticonformal transformations of the unit disk \mathbb{D} (it is a Lie group with two connected components, and the component of the identity transformation is isomorphic to $PSL(2, \mathbb{R})$). Restricting every $\Phi \in G$ to $\mathbb{S} = \partial\mathbb{D}$, regard G as the three-dimensional Lie group of diffeomorphisms of the unit circle \mathbb{S} . Consequently, G acts from the right on the vector space $C^\infty(\mathbb{S})$ by the rule

$$a\Phi = a \circ \varphi |d\varphi/d\theta|^{-1} \quad \text{for } \Phi \in G, a \in C^\infty(\mathbb{S}), \text{ where } \varphi = \Phi|_{\mathbb{S}}. \quad (4.1)$$

In these notations, (2.24) means that

$$Z_k(a\Phi) = Z_k(a) \quad (\Phi \in G, k = 1, 2, \dots) \quad (4.2)$$

for every positive function $a \in C^\infty(\mathbb{S})$.

Proposition 4.1. *Equality (4.2) holds for all $a \in C^\infty(\mathbb{S})$.*

PROOF. Fixing k and $\Phi \in G$, put $Q(a) = Z_k(a\Phi) - Z_k(a)$. By (2.1) and (2.2), Q is a $2k$ -form on $C^\infty(\mathbb{S})$. Let us prove that this form is identically zero. Consider $C^\infty(\mathbb{S})$ as a topological vector space with the C^∞ -topology. The form Z_k is continuous, as is shown by our estimates at the end of Section 2. The form Q is also continuous. We know that $Q(a) = 0$ for a positive function a . Positive functions form an open convex cone in the space $C^\infty_{\mathbb{R}}(\mathbb{S})$ of real functions. If a continuous form vanishes on an open set then it is identically zero. Thus, $Q(a) = 0$ for every real function a .

Obviously, any $2k$ -form on $C^\infty(\mathbb{S})$ is uniquely determined by its restriction to $C^\infty_{\mathbb{R}}(\mathbb{S})$. Therefore, Q is identically zero. \square

Show that the conformal invariance (4.2) is equivalent to some linear relations between the coefficients $Z_{j_1 \dots j_{2k}}$ of (2.15).

The group G is generated by the three subgroups:

(1) the group of rotations $R_\alpha : z \mapsto e^{i\alpha} z$;

(2) the two-element group $\{I, J\}$, where I is the identity transformation and $J : z \mapsto \bar{z}$ is complex conjugation;

(3) the group $T = \{\Phi_\rho \mid -1 < \rho < 1\}$, where Φ_ρ is defined by (3.1).

From the standpoint of hyperbolic geometry, Φ_ρ is the shift of the hyperbolic plane ($\text{Int } D$, $ds^2 = \frac{|dz|^2}{(1-|z|^2)^2}$) along the real straight line $(-1, 1)$ by the distance t such that $\rho = \tanh t$. This means that $\Phi_\rho(x) \in (-1, 1)$ for $x \in (-1, 1)$ and $\text{dist}(x, \Phi_\rho(x)) = t$, where dist is the hyperbolic distance. The shift Φ_ρ has two fixed points ± 1 on the circle \mathbb{S} at infinity.

For the first two subgroups, the situation is obvious, the invariants $Z_k(a)$ do not change if the function a is transformed by a rotation or complex conjugation. Indeed, in these cases, the factor $|d\varphi/d\theta|$ on the right-hand side of (4.1) is an identical unity and (4.2) coincides with one of the equalities:

$$Z_k(a \circ R_\alpha) = Z_k(a), \quad Z_k(a \circ J) = Z_k(a). \quad (4.3)$$

The Fourier coefficients of the function $a \circ R_\alpha$ are expressed via the Fourier coefficients of a by the equalities

$$(\widehat{a \circ R_\alpha})_j = e^{i\alpha j} \hat{a}_j.$$

Therefore,

$$(\widehat{a \circ R_\alpha})_{j_1} \dots (\widehat{a \circ R_\alpha})_{j_{2k}} = e^{i\alpha(j_1 + \dots + j_{2k})} \hat{a}_{j_1} \dots \hat{a}_{j_{2k}} = \hat{a}_{j_1} \dots \hat{a}_{j_{2k}}$$

if $j_1 + \dots + j_{2k} = 0$; hence, all summands in (2.1) are unchanged when a is replaced by $a \circ R_\alpha$. Similarly, $(\widehat{a \circ J})_j = \hat{a}_{-j}$, and the second equality in (4.3) is equivalent to the evenness property (2.17).

It remains to consider the shift Φ_ρ defined by (3.1). For $a \in C^\infty(\mathbb{S})$, let $b = a\Phi_\rho$. By (3.3), the Fourier coefficients of a and b are connected by the relation

$$\hat{b}_n = \sum_{k=-\infty}^{\infty} \mu_{nk}(\rho) \hat{a}_k.$$

Inserting this expression in the formula

$$Z_k(b) = \sum_{j_1, \dots, j_{2k} = -\infty}^{\infty} Z_{j_1 \dots j_{2k}} \hat{b}_{j_1} \dots \hat{b}_{j_{2k}},$$

we obtain

$$Z_k(b) = \sum_{\ell_1, \dots, \ell_{2k} = -\infty}^{\infty} \left(\sum_{j_1, \dots, j_{2k} = -\infty}^{\infty} Z_{j_1 \dots j_{2k}} \mu_{j_1 \ell_1}(\rho) \dots \mu_{j_{2k} \ell_{2k}}(\rho) \right) \hat{a}_{\ell_1} \dots \hat{a}_{\ell_{2k}}.$$

Since a is arbitrary, $Z_k(a)=Z_k(b)$ amounts to

$$\sum_{j_1, \dots, j_{2k} = -\infty}^{\infty} Z_{j_1 \dots j_{2k}} \mu_{j_1 \ell_1}(\rho) \cdots \mu_{j_{2k} \ell_{2k}}(\rho) = Z_{\ell_1 \dots \ell_{2k}}. \quad (4.4)$$

This equality must hold for each $k = 1, 2, \dots$, all finite indices $(\ell_1, \dots, \ell_{2k})$, and every $\rho \in (-1, 1)$.

Verify the absolute convergence of the series on the left-hand side of (4.4). Fix the indices $(\ell_1, \dots, \ell_{2k})$, put $\ell = |\ell_1| + \dots + |\ell_{2k}|$, and estimate from above the modulus of the left-hand side of (4.4) by the expression

$$\sum_{j_1, \dots, j_{2k} = \ell-1}^{\ell+1} Z_{j_1 \dots j_{2k}} |\mu_{j_1 \ell_1}(\rho) \cdots \mu_{j_{2k} \ell_{2k}}(\rho)| + \sum_{j=\ell+2}^{\infty} \sum_{|j_1| + \dots + |j_{2k}| = j} Z_{j_1 \dots j_{2k}} |\mu_{j_1 \ell_1}(\rho) \cdots \mu_{j_{2k} \ell_{2k}}(\rho)|.$$

The first sum is finite. Check the convergence of the second series. For this, assuming that $|j_1| + \dots + |j_{2k}| = j$, $|j_\alpha| \geq \ell + 2$ ($1 \leq \alpha \leq 2k$), use estimates (2.19) and (3.24) to obtain the inequality

$$|Z_{j_1 \dots j_{2k}} \mu_{j_1 \ell_1}(\rho) \cdots \mu_{j_{2k} \ell_{2k}}(\rho)| \leq C_k j^{2k+1} C_{\ell_1} |j_1|^{|\ell_1|} |\rho|^{|j_1|/2} \cdots C_{\ell_{2k}} |j_{2k}|^{|\ell_{2k}|} |\rho|^{|j_{2k}|/2} \leq C_{k, \ell_1 \dots \ell_{2k}} |\rho|^{j/2+2k+1}.$$

Hence,

$$\sum_{j=\ell+2}^{\infty} \sum_{|j_1| + \dots + |j_{2k}| = j} Z_{j_1 \dots j_{2k}} |\mu_{j_1 \ell_1}(\rho) \cdots \mu_{j_{2k} \ell_{2k}}(\rho)| \leq C_{k, \ell_1 \dots \ell_{2k}} \sum_{j=\ell+2}^{\infty} (j+1)^{4k+1} |\rho|^{j/2+2k+1}.$$

The series on the right-hand side converges for $|\rho| < 1$.

Equation (4.4) holds trivially for $\rho = 0$ because $M(0) = I$. Differentiate (4.4) with respect to ρ . The possibility of termwise differentiability is easily justified by the estimates of the previous paragraph. In result, we obtain the equation equivalent to (4.4):

$$\sum_{j_1, \dots, j_{2k} = -\infty}^{\infty} Z_{j_1 \dots j_{2k}} \sum_{\alpha=1}^{2k} \mu_{j_1 \ell_1} \cdots \mu_{j_{\alpha-1} \ell_{\alpha-1}} \frac{d\mu_{j_\alpha \ell_\alpha}}{d\rho} \mu_{j_{\alpha+1} \ell_{\alpha+1}} \cdots \mu_{j_{2k} \ell_{2k}} = 0.$$

By Proposition 3.1,

$$\frac{d\mu_{j_\alpha \ell_\alpha}}{d\rho} = \sum_{p=-\infty}^{\infty} d_{j_\alpha p} \mu_{p \ell_\alpha},$$

where the matrix $D = (d_{nk})$ is defined by (3.6). Insert the so-obtained expression into the previous equation:

$$\sum_{j_1, \dots, j_{2k} = -\infty}^{\infty} Z_{j_1 \dots j_{2k}} \sum_{\alpha=1}^{2k} \sum_{p=-\infty}^{\infty} \mu_{j_1 \ell_1} \cdots \mu_{j_{\alpha-1} \ell_{\alpha-1}} d_{j_\alpha p} \mu_{p \ell_\alpha} \mu_{j_{\alpha+1} \ell_{\alpha+1}} \cdots \mu_{j_{2k} \ell_{2k}} = 0.$$

Permuting the summation indices j_α and p enables us to write this down as (it is easy to justify the change of the summation order)

$$\sum_{j_1, \dots, j_{2k} = -\infty}^{\infty} \left(\sum_{\alpha=1}^{2k} \sum_{p=-\infty}^{\infty} d_{pj_\alpha} Z_{j_1 \dots j_{\alpha-1} p j_{\alpha+1} \dots j_{2k}} \right) \mu_{j_1 \ell_1} \cdots \mu_{j_{2k} \ell_{2k}} = 0.$$

Since the subscripts $(\ell_1, \dots, \ell_{2k})$ are arbitrary and the matrix $M = (\mu_{j\ell})$ is nondegenerate, this is equivalent to the equation

$$\sum_{\alpha=1}^{2k} \sum_p d_{pj_\alpha} Z_{j_1 \dots j_{\alpha-1} p j_{\alpha+1} \dots j_{2k}} = 0.$$

Inserting here the value (3.6) for d_{pj_α} , we arrive at the final equation

$$\sum_{\alpha=1}^{2k} ((j_\alpha - 1)Z_{j_1 \dots j_{\alpha-1}, j_{\alpha+1}, j_{\alpha+1} \dots j_{2k}} - (j_\alpha + 1)Z_{j_1 \dots j_{\alpha-1}, j_{\alpha-1}, j_{\alpha+1} \dots j_{2k}}) = 0, \quad (4.5)$$

which must hold for arbitrary (j_1, \dots, j_{2k}) . Conversely, equation (4.5) with (2.17) implies (4.2) for every function a .

Equation (4.5) can be simplified; the simplification is connected with the Lie algebra of G .

Recall that we regard G as the diffeomorphism group of the unit circle $\mathbb{S} = \{e^{i\theta}\}$. Consequently, the Lie algebra \mathfrak{g} of G is the three-dimensional space of vector fields on \mathbb{S} . It is easy to see that the three vector fields

$$X_0 = \frac{\partial}{\partial \theta}, \quad X_1 = \cos \theta \frac{\partial}{\partial \theta}, \quad X_2 = \sin \theta \frac{\partial}{\partial \theta}$$

constitute a basis for \mathfrak{g} . In this basis the Lie product is expressed by the formulas

$$[X_0, X_1] = -X_2, \quad [X_0, X_2] = X_1, \quad [X_1, X_2] = X_0. \quad (4.6)$$

The group G acts on $C^\infty(\mathbb{S})$ by transforming a function a into a conformally equivalent function as was shown at the beginning of the section. Consequently, the Lie algebra \mathfrak{g} also acts on $C^\infty(\mathbb{S})$: a vector $A \in \mathfrak{g}$ can be regarded as the linear operator $A : C^\infty(\mathbb{S}) \rightarrow C^\infty(\mathbb{S})$. Express this action in terms of the Fourier coefficients.

Start from the rotation group $R \subset G$. A rotation acts by the formula $(aR_\alpha)(\theta) = a(e^{i\alpha\theta})$, which gives $(\widehat{aR_\alpha})_n = e^{in\alpha}\hat{a}_n$. Differentiating this equality with respect to α at $\alpha = 0$, we get $\frac{d}{d\alpha}\big|_{\alpha=0}(\widehat{aR_\alpha})_n = in\hat{a}_n$. We have thus found a first element in \mathfrak{g} :

$$(\widehat{Ca})_n = in\hat{a}_n. \quad (4.7)$$

We have already found the element in \mathfrak{g} corresponding to the one-dimensional subgroup $T \subset G$. This is the operator $D : C^\infty(\mathbb{S}) \rightarrow C^\infty(\mathbb{S})$ of Proposition 3.1. By (3.6), in terms of the Fourier coefficients, this operator acts as follows:

$$(\widehat{Da})_n = (n - 2)\hat{a}_{n-1} - (n + 2)\hat{a}_{n+1}. \quad (4.8)$$

For complementing (C, D) to a basis for \mathfrak{g} , we simply calculate the commutator of (4.7) and (4.8):

$$E = [C, D], \quad (\widehat{Ea})_n = -i[(n - 2)\hat{a}_{n-1} + (n + 2)\hat{a}_{n+1}]. \quad (4.9)$$

In the basis (C, D, E) , the Lie bracket is given by the formulas

$$[C, D] = E, \quad [C, E] = -D, \quad [D, E] = -4C. \quad (4.10)$$

Formulas (4.6) and (4.10) are equivalent, as is seen from the rule for changing the basis:

$$C = X_0, \quad D = 2X_2, \quad E = 2X_1.$$

We stress that \mathfrak{g} is a real Lie algebra. In particular, the operators C, D , and E transform real functions again into real functions. Let $\mathfrak{g}_\mathbb{C}$ be the complexification of \mathfrak{g} . The operators

$$D_0 = -iC, \quad D_- = \frac{1}{2}(D + iE), \quad D_+ = \frac{1}{2}(-D + iE)$$

constitute a basis for the algebra $\mathfrak{g}_\mathbb{C}$. In terms of the Fourier coefficients, these operators are defined by the formulas

$$(\widehat{D_0a})_n = n\hat{a}_n, \quad (\widehat{D_-a})_n = (n - 2)\hat{a}_{n-1}, \quad (\widehat{D_+a})_n = (n + 2)\hat{a}_{n+1}. \quad (4.11)$$

In this basis, the Lie bracket is expressed by the equalities

$$[D_0, D_-] = -D_-, \quad [D_0, D_+] = D_+, \quad [D_-, D_+] = 2D_0. \quad (4.12)$$

Equation (4.5) was in fact obtained by the differentiation of the equality

$$Z_k(a\Phi_\rho) = Z_k(a e^{tD}) = Z_k(a) \quad (\tanh t = \rho)$$

with respect to t . Proceeding likewise for the equation

$$Z_k(a e^{tE}) = Z_k(a),$$

we infer

$$\sum_{\alpha=1}^{2k} ((j_\alpha - 1)Z_{j_1 \dots j_{\alpha-1}, j_\alpha+1, j_{\alpha+1} \dots j_{2k}} + (j_\alpha + 1)Z_{j_1 \dots j_{\alpha-1}, j_\alpha-1, j_{\alpha+1} \dots j_{2k}}) = 0. \quad (4.13)$$

Taking the sum and the difference of (4.5) and (4.13), we arrive at the pair of simpler equations:

$$\sum_{\alpha=1}^{2k} (j_\alpha - 1)Z_{j_1 \dots j_{\alpha-1}, j_\alpha+1, j_{\alpha+1} \dots j_{2k}} = 0, \quad (4.14)$$

$$\sum_{\alpha=1}^{2k} (j_\alpha + 1)Z_{j_1 \dots j_{\alpha-1}, j_\alpha-1, j_{\alpha+1} \dots j_{2k}} = 0. \quad (4.15)$$

Of course, (4.14) and (4.15) correspond to the operators $D_+, D_- \in \mathfrak{g}_\mathbb{C}$ in the same way as (4.5) and (4.13) correspond to $D, E \in \mathfrak{g}$.

Note that equations (4.14) and (4.15) are equivalent to each other if we involve the evenness condition (2.17). Indeed, changing the signs of the subscripts (j_1, \dots, j_{2k}) in (4.15) and using (2.17), we obtain (4.14). Therefore, we can exclude (4.15) from consideration without loss of information. Finally, (4.14) holds trivially for $j_1 + \dots + j_{2k} \neq -1$ since, by the definition of Section 2, $Z_{j_1 \dots j_{2k}} = 0$ for $j_1 + \dots + j_{2k} \neq 0$. Thus, relations (4.14) and (4.15) are reduced to the equation

$$\sum_{\alpha=1}^{2k} (j_\alpha - 1)Z_{j_1 \dots j_{\alpha-1}, j_\alpha+1, j_{\alpha+1}, \dots, j_{2k}} = 0 \quad (j_1 + \dots + j_{2k} = -1). \quad (4.16)$$

REMARK. We proved that equation (4.16) together with the evenness condition (2.17) is equivalent to the conformal invariance (4.2) of the zeta-invariants $Z_k(a)$. We emphasize that our proof is based on the use of Theorem 2.1. The question arises: Is it possible to prove (4.16) without involving the Steklov spectrum, i.e. on the basis of the definition (2.2) and (2.16) of the coefficients $Z_{j_1 \dots j_{2k}}$? We have not been able to find such a proof for an arbitrary k . The only exclusions are the cases of $k = 1, 2$. For $k = 1$, equation (4.16) follows easily from Edward's formula (2.22), and for $k = 2$, equation (4.16) can be deduced from the explicit formula for the coefficients $Z_{ijk\ell}$ which are contained in Theorem 5.1 below.

5. An Explicit Formula for the Coefficients of the Second Zeta-Invariant

Edward's formula (2.22) shows that the coefficients of the quadratic form $Z_1(a) = \sum_i Z_{i,-i} \hat{a}_i \hat{a}_{-i}$ are expressed by a piecewise polynomial function of degree 3 of i :

$$Z_{i,-i} = \begin{cases} \frac{1}{3}(i^3 - i) & \text{for } i \geq 0, \\ \frac{1}{3}(-i^3 + i) & \text{for } i \leq 0. \end{cases}$$

We also focus the reader's attention on the interesting circumstance: Both polynomials involved in this formula are odd with respect to i while the coefficient $Z_{i,-i}$ itself is even. An analogous assertion on the second zeta-invariant looks as follows:

Theorem 5.1. *The coefficients of the 4-form*

$$Z_2(a) = \sum_{i,l,k,\ell} Z_{ijkl} \hat{a}_i \hat{a}_j \hat{a}_k \hat{a}_\ell$$

are uniquely determined by the conditions:

- (1) $Z_{ijkl} = 0$ for $i + j + k + \ell \neq 0$;
- (2) Z_{ijkl} is symmetric with respect to (i, j, k, ℓ) and even: $Z_{-i,-j,-k,-\ell} = Z_{i,j,k,\ell}$;
- (3) $Z_{ijk,-i-j-k}$ is expressed via (i, j, k) by the formula

$$Z_{ijk,-i-j-k} = \begin{cases} P_1(i, j, k) & \text{for } i \geq 0, j \geq 0, k \geq 0; \\ P_2(i, j, k) & \text{for } i \leq 0, j \geq 0, k \geq 0, i + j \leq 0, \\ & i + k \leq 0, i + j + k \geq 0, \end{cases} \quad (5.1)$$

in which P_1 and P_2 are the polynomials defined by the equalities

$$P_1(i, j, k) = \frac{1}{15} \sigma_{(ijk)} (3i^5 + 15i^4j + 10i^3j^2 + 10i^3jk - 5i^3 - 25i^2j - 10ijk + 2i), \quad (5.2)$$

$$P_2(i, j, k) = \frac{1}{45} \sigma_{(j_k)} (5i^5 + 25i^4j + 10i^3j^2 + 20i^3jk - 10i^2j^3 - 15ij^4 - 20ij^3k - 4j^5 - 5j^4k + 10j^3k^2 - 5i^3 - 15i^2j + 5ij^2 - 5j^2k + 4j). \quad (5.3)$$

Here $\sigma_{(ijk)}$ ($\sigma_{(j_k)}$) stands for symmetrization over (i, j, k) (over (j, k)).

We stress that P_1 and P_2 are polynomials of degree 5 and these polynomials are odd, i.e.,

$$P_r(-i, -j, -k) = -P_r(i, j, k) \quad (r = 1, 2).$$

Moreover, P_1 and P_2 possess the interesting properties of positivity and divisibility since $3Z_{ijkl}$ is a non-negative even number for every (i, j, k, ℓ) , as is seen from (2.2) and (2.16). It seems that the same holds for the higher zeta-invariants: the coefficients $Z_{j_1 \dots j_{2k-1}, -j_1 \dots -j_{2k-1}}$ of $2k$ -form (2.15) are expressed by a piecewise polynomial function of (j_1, \dots, j_{2k-1}) represented by odd polynomials of degree $2k + 1$. Unfortunately, for $k > 2$, these polynomials are too cumbersome to be useful.

For proving Theorem 5.1, we need

Lemma 5.2. *If items (1) and (2) of Theorem 5.1 hold then the coefficients Z_{ijkl} are uniquely determined by*

the values $Z_{ijk,-i-j-k}$ for $i \geq 0, j \geq 0, k \geq 0$ (case 1);

the values $Z_{ijk,-i-j-k}$ for $i \leq 0, j \geq 0, k \geq 0, i + j \leq 0, i + k \leq 0, i + j + k \geq 0$ (case 2).

PROOF. Consider the set of all ordered quadruples (i, j, k, ℓ) of integers satisfying $i + j + k + \ell = 0$. This set is the union of the following two subsets:

(a) the set of quadruples (i, j, k, ℓ) whose three elements have the same sign (provided that 0 has both signs);

(b) the set of quadruples (i, j, k, ℓ) whose two elements are nonnegative and the other two are non-positive.

By items (1) and (2) of Theorem 5.1, we can permute elements in a quadruple and change the signs of all elements simultaneously. Use this freedom in case (a) to achieve the fulfillment of the inequalities $i \geq 0, j \geq 0, k \geq 0$. This is exactly case 1 in Lemma 5.2.

In case (b), use the above freedom to achieve the fulfillment of the conditions

$$i \leq 0, \quad |i| = \max\{|i|, |j|, |k|, |\ell|\}. \quad (5.4)$$

Now, two elements in the triple (j, k, ℓ) are nonnegative and one is nonpositive. Permute the elements of the triple so that

$$j \geq 0, \quad k \geq 0, \quad \ell \leq 0. \quad (5.5)$$

An easy arithmetic analysis shows that the combination of conditions (5.4) and (5.5) is equivalent to the system

$$i \leq 0, \quad j \geq 0, \quad k \geq 0, \quad i + j \leq 0, \quad i + k \leq 0, \quad i + j + k \geq 0, \quad \ell = -(i + j + k). \quad (5.6)$$

This is exactly case 2 of the lemma. \square

PROOF OF THEOREM 5.1. A detailed proof includes plenty of routine but very bulky calculations with polynomials (multiplication of two polynomials and collection of similar terms). We implemented these calculations on a computer with the use of the MAPLE symbolic computation system. These computations are omitted in the proof below.

Introduce the notation

$$\{x\} = |x| - x = \begin{cases} 0 & \text{for } x \geq 0, \\ -2x & \text{for } x < 0. \end{cases} \quad (5.7)$$

Fix (i, j, k, ℓ) with $i + j + k + \ell = 0$ and define the polynomial

$$f(n) = n(n + i)(n + i + j)(n + i + j + k). \quad (5.8)$$

Formula (2.2) can be written down in the form

$$N_{ijkl} = \sum_n \{f(n)\}. \quad (5.9)$$

The roots of f are elements of the set $\{0, -i, -i - j, -i - j - k\}$. Let (r_1, r_2, r_3, r_4) be a sequence of these roots in increasing order; i.e.,

$$\{r_1, r_2, r_3, r_4\} = \{0, -i, -i - j, -i - j - k\}, \quad r_1 \leq r_2 \leq r_3 \leq r_4.$$

Formula (5.9) can be written down as

$$N_{ijkl} = -2 \sum_{n=r_1}^{r_2} f(n) - 2 \sum_{n=r_3}^{r_4} f(n). \quad (5.10)$$

Transform (5.8) to the form

$$f(n) = n^4 + \alpha_1 n^3 + \alpha_2 n^2 + \alpha_3 n, \quad (5.11)$$

where

$$\alpha_1 = 3i + 2j + k, \quad \alpha_2 = 3i^2 + 4ij + 2ik + j^2 + jk, \quad \alpha_3 = i^3 + 2i^2j + i^2k + ij^2 + ijk. \quad (5.12)$$

Find the first sum on the right-hand side of (5.10). Suppose first that $0 \leq r_1 \leq r_2$. In this case

$$\sum_{n=r_1}^{r_2} f(n) = \sum_{n=0}^{r_2} f(n) - \sum_{n=0}^{r_1} f(n).$$

We have made use of the equality $f(r_1) = 0$. Insert the value (5.11) in the last formula:

$$\begin{aligned} \sum_{n=r_1}^{r_2} f(n) &= \sum_{n=0}^{r_2} n^4 - \sum_{n=0}^{r_1} n^4 + \alpha_1 \left(\sum_{n=0}^{r_2} n^3 - \sum_{n=0}^{r_1} n^3 \right) \\ &+ \alpha_2 \left(\sum_{n=0}^{r_2} n^2 - \sum_{n=0}^{r_1} n^2 \right) + \alpha_3 \left(\sum_{n=0}^{r_2} n - \sum_{n=0}^{r_1} n \right). \end{aligned} \quad (5.13)$$

Using (2.21) and similar equalities [3, § 4.1.1]:

$$\sum_{n=0}^r n^3 = \frac{1}{4}r^2(r+1)^2, \quad \sum_{n=0}^r n^4 = \frac{1}{30}r(r+1)(2r+1)(3r^2+3r-1),$$

from (5.13) we obtain

$$\sum_{n=r_1}^{r_2} f(n) = \varphi(r_2) - \varphi(r_1), \quad (5.14)$$

where

$$\varphi(r) = r(r+1) \left[\frac{1}{30}(2r+1)(3r^2+3r-1) + \frac{\alpha_1}{4}r(r+1) + \frac{\alpha_2}{6}(2r+1) + \frac{\alpha_3}{2} \right] \quad (5.15)$$

is a discrete antiderivative of the function $f(n)$. It is easy to validate (5.14) in the other two cases, when $r_1 \leq 0 \leq r_2$ or $r_1 \leq r_2 \leq 0$. Thus, formula (5.14) is valid for all values of the roots $r_1 \leq r_2 \leq r_3 \leq r_4$. Of course, an analogous formula also holds for the second sum on the right-hand side of (5.10).

Insert (5.14) and the analogous expression for the second sum in (5.10)

$$N_{ijkl} = 2(\varphi(r_1) - \varphi(r_2) + \varphi(r_3) - \varphi(r_4)). \quad (5.16)$$

Now, symmetrize (5.16) over (i, j, k) to obtain the corresponding formula for Z_{ijkl} ($i + j + k + \ell = 0$). To this end, we use (2.18), which we will reproduce here as follows:

$$3Z_{ijkl} = \frac{1}{2}(N_{ijkl} + N_{ikjl} + N_{jikl} + N_{jkil} + N_{kijl} + N_{kjil}) \quad (i + j + k + \ell = 0). \quad (5.17)$$

The main difficulty is due to the following circumstance: The roots (r_1, r_2, r_3, r_4) must be expressed via (i, j, k) , and this expression is different in different cases. By Lemma 5.2, it suffices to consider the two cases mentioned in the statement of the lemma.

CASE 1. Suppose that $i \geq 0, j \geq 0, k \geq 0$. Then

$$r_1 = -i - j - k, \quad r_2 = -i - j, \quad r_3 = -i, \quad r_4 = 0. \quad (5.18)$$

Inserting these values in (5.15), find the expression $\varphi(r_m)$ ($1 \leq m \leq 4$) by (i, j, k) . Then insert the results for $\varphi(r_m)$ in (5.16) to obtain a formula expressing N_{ijkl} as a polynomial of degree 5 of (i, j, k) . Finally, symmetrize this polynomial, i.e., insert it in (5.17) permuting accordingly the arguments for each of the six summands on the right-hand side of this formula. Here it is important to note that, in case 1, we need not care about the form of the functions $r_1(i, j, k)$, $r_2(i, j, k)$, $r_3(i, j, k)$, and $r_4(i, j, k)$ for different summands on the right-hand side of (5.17); these functions transform by the same permutation. For example, the second summand N_{ikjl} on the right-hand side of (5.17) is obtained from the first summand by the transposition of the subscripts (j, k) . For this summand, $r_1 = -i - j - k$, $r_2 = -i - k$, $r_3 = -i$, and $r_4 = 0$. These formulas are obtained from (5.18) by means of the same transposition. In result, $Z_{ijk, -i-j-k} = P_1(i, j, k)$, where P_1 is the polynomial defined by (5.2).

CASE 2. Suppose that $i \leq 0, j \geq 0, k \geq 0, i + j \leq 0, i + k \leq 0, i + j + k \geq 0$. In this case the roots (r_1, r_2, r_3, r_4) have different expressions for different summands on the right-hand side of (5.17); namely,

$$\begin{aligned} r_1 &= -i - j - k, & r_2 &= 0, & r_3 &= -i - j, & r_4 &= -i & \text{for } N_{ijkl}; \\ r_1 &= -j, & r_2 &= -i - j - k, & r_3 &= 0, & r_4 &= -i - j & \text{for } N_{jikl}; \\ r_1 &= -j - k, & r_2 &= -j, & r_3 &= -i - j - k, & r_4 &= 0 & \text{for } N_{jkil}. \end{aligned}$$

The corresponding formulas for the remaining three summands are obtained by applying the transposition of the subscripts (j, k) to these equalities. Using these expressions, repeat our calculations and come to the equality $Z_{ijk, -i-j-k} = P_2(i, j, k)$, where the polynomial P_2 is defined by (5.3). \square

6. Some Open Problems

Recall the completeness problem of the system of zeta-invariants formulated at the end of Section 2: Given a positive function $b \in C^\infty(\mathbb{S})$, it is required to find all positive functions $a \in C^\infty(\mathbb{S})$ satisfying (2.25), where $b_k = Z_k(b)$. An a priori condition for the solvability of the completeness problem is the positive answer to the following question on the *independence of zeta-invariants*: Are the forms $Z_k(a)$ ($k = 1, 2, \dots$) independent from each other, i.e., does system (2.25) contain infinitely many independent conditions on the Fourier coefficients of the function a ? We think that the answer to this question is positive but we are unable to prove it yet. The first and second zeta-invariants are independent. Indeed, $Z_1(a)$ does not depend on $(\hat{a}_0, \hat{a}_{\pm 1})$, as it is seen from (2.23). On the other hand, the 4-form $Z_2(a)$ contains summands of the form $\hat{a}_0^2 \hat{a}_k a_{-k}$ with nonzero coefficients, as it is easy to see by Theorem 5.1.

Clearly, $Z_k(a) = 0$ ($k = 1, 2, \dots$) for every function a in the three-dimensional subspace

$$L = \{a \in C^\infty(\mathbb{S}) \mid a(\theta) = \hat{a}_0 + \hat{a}_1 e^{i\theta} + \hat{a}_{-1} e^{-i\theta}\}$$

of $C^\infty(\mathbb{S})$. Indeed, by (2.2), $N_{j_1 \dots j_{2k}} = 0$ if each of the subscripts (j_1, \dots, j_{2k}) is zero or ± 1 . The converse holds for $k = 1$ for real functions: If $Z_1(a) = 0$ for a real function $a \in C^\infty(\mathbb{S})$ then $a \in L$. This ensues from Edward's formula (2.23), which takes the following form for a real function a :

$$Z_1(a) = \frac{2}{3} \sum_{n=2}^{\infty} (n^3 - n) |\hat{a}_n|^2. \tag{6.1}$$

What is the structure of the set of all (real) functions $a \in C^\infty(\mathbb{S})$ satisfying $Z_k(a) = 0$ for $k = 2, 3, \dots$; can it be substantially different from L ?

As is seen from (6.1), the estimate

$$Z_1(a) = \frac{2}{3} \sum_{n \geq 2} (n^3 - n) |\hat{a}_n|^2 \geq c_1 \sum_{n \geq 2} n^3 |\hat{a}_n|^2$$

with some absolute constant $c_1 > 0$ holds for all real functions $a \in C^\infty(\mathbb{S})$.

Problem 6.1. *Is it true that the estimate*

$$Z_k(a) \geq c_k \sum_{n \geq 2} n^{2k+1} |\hat{a}_n|^{2k} \tag{6.2}$$

holds for every real function $a \in C^\infty(\mathbb{S})$ and each $k = 2, 3, \dots$, where the coefficient $c_k > 0$ depends only on k ? If the answer is negative then the same question can be posed for positive functions a .

By now even the inequality $Z_k(a) \geq 0$ ($k = 2, 3, \dots$) remains unproved for real a . Alongside (6.1), a “naive” argument in favor of the last inequality is given by the following observation: by (2.5), $Z_k(a) = \zeta_a(-2k) = \text{Tr}(B^2)$ for some selfadjoint operator B . We have numerically checked the inequality $Z_2(a) \geq 0$ for many functions chosen by a more or less random choice of Fourier coefficients satisfying $\bar{\hat{a}}_n = \hat{a}_{-n}$ and $\hat{a}_n = 0$ for $|n| > n_0$ for some n_0 . Thus, the inequality holds in all considered cases.

Compactness theorems of the following form are popular in spectral geometry (see [4] and the references therein): a family of Riemannian manifolds (satisfying some additional constraints) with isospectral Laplacians is (pre)compact in a suitably chosen topology. Discuss one of possible compactness theorems for the Steklov spectrum. Of course, conformal equivalence must be taken into account since the conformal group is noncompact.

Recall that the Hilbert space $H^s(\mathbb{S})$ is the completion of $C^\infty(\mathbb{S})$ in the norm

$$\|a\|_{H^s(\mathbb{S})}^2 = \sum_n (1 + |n|^{2s}) |\hat{a}_n|^2.$$

In our opinion, $\|a\|_{H^{3/2}(\mathbb{S})}$ is the most suitable norm for studying compactness for the Steklov spectrum. Indeed, as we see from (6.1),

$$\|a\|_{H^{3/2}(\mathbb{S})}^2 \sim |\hat{a}_0|^2 + |\hat{a}_1|^2 + Z_1(a)$$

for a real function a .

Consider a sequence of positive functions $a^\nu \in C^\infty(\mathbb{S})$ ($\nu = 1, 2, \dots$) for which the Steklov spectrum $\text{Sp}(a^\nu \Lambda_\epsilon)$ does not depend on ν . Formula (6.1) implies the estimate

$$|\hat{a}_n^\nu| \leq C|n|^{-3/2} \quad \text{for } |n| \geq 2 \quad (6.3)$$

with some constant C independent of ν . Therefore, the sequence $|\hat{a}_n^\nu|$ ($\nu = 1, 2, \dots$) is bounded for each $|n| \geq 2$. The positivity of a^ν implies that $|\hat{a}_1^\nu| \leq \hat{a}_0^\nu$. Thus, the only obstacle to the boundedness of the sequence of norms $\|a^\nu\|_{H^{3/2}(\mathbb{S})}$ ($\nu = 1, 2, \dots$) is the possible unboundedness of the sequence \hat{a}_0^ν ($\nu = 1, 2, \dots$), which indeed can be unbounded as is shown by easy examples. We try to overcome this obstacle by replacing each function a^ν with a conformally equivalent function. Thus, we get the following statement:

Conjecture 6.2. *Let $a^\nu \in C^\infty(\mathbb{S})$ ($\nu = 1, 2, \dots$) be a sequence of functions uniformly bounded from below by some positive constant: $a^\nu(\theta) \geq c > 0$. Suppose that the Steklov spectrum $\text{Sp}(a^\nu \Lambda_\epsilon)$ does not depend on ν . Then there is a subsequence a^{ν_k} such that every function a^{ν_k} is conformally equivalent to some function $b^k \in C^\infty(\mathbb{S})$ and the sequence of the norms $\|b^k\|_{H^{3/2}(\mathbb{S})}$ is bounded. Hence, for every $s < 3/2$, the sequence b^k contains a subsequence converging in $H^s(\mathbb{S})$.*

The main difficulty of our approach is connected with estimate (6.3). We can prove the conjecture if the following stronger estimate holds instead of (6.3):

$$|\hat{a}_n^\nu| \leq C|n|^{-3/2-\epsilon} \quad \text{for } |n| \geq 2 \quad (C \text{ does not depend on } \nu), \quad (6.4)$$

where $\epsilon > 0$ can be arbitrary. The possibility of obtaining estimates of the form (6.4) is closely connected with Problem 6.1.

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