ON FINITE SOLUBLE GROUPS WITH ALMOST FIXED-POINT-FREE AUTOMORPHISMS OF NONCOPRIME ORDER

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Abstract: It is proved that if a finite p-soluble group G admits an automorphism φ of order p^n having at most m fixed points on every φ-invariant elementary abelian p'-section of G, then the p-length of G is bounded above in terms of p^n and m; if in addition G is soluble, then the Fitting height of G is bounded above in terms of p^n and m. It is also proved that if a finite soluble group G admits an automorphism ψ of order $p^a q^b$ for some primes p and q, then the Fitting height of G is bounded above in terms of $|\psi|$ and $|C_G(\psi)|$.

1. Introduction

Studying groups with "almost fixed-point-free" automorphisms means the obtaining of restrictions on the structure of groups which depend on the automorphisms of the groups and certain restrictions on the fixed-point subgroups. In this paper we consider the questions of bounding the p-length and Fitting height of finite p-soluble and soluble groups admitting almost fixed-point-free automorphisms of noncoprime order.

Let $\varphi \in \operatorname{Aut} G$ be an automorphism of a finite group G. Studying the structure of G depending on φ and the fixed-point subgroup $C_G(\varphi)$ is one of the most important and fruitful avenues in finite group theory. The celebrated Brauer–Fowler theorem [1] (bounding the index of the soluble radical in terms of the order of $|C_G(\varphi)|$ when $|\varphi| = 2$) and Thompson's theorem [2] (giving the nilpotency of G when φ is of prime order and acts fixed-point-freely, that is, $C_G(\varphi) = 1$) lie in the foundations of the classification of finite simple groups. The classification was used for obtaining further results on solubility of G or a suitable "large" subgroup. For example, using the classification Hartley [3] generalized the Brauer–Fowler theorem to every order of φ: the group G has a soluble subgroup of index bounded in terms of $|\varphi|$ and $|C_G(\varphi)|$.

Suppose now that G is soluble. Further information on the structure of G is sought first of all in the form of bounds for the Fitting height (nilpotent length). A bound for the Fitting height naturally reduces further studies to the case of nilpotent groups with (almost) fixed-point-free automorphisms, for which, in turn, the problems arise of bounding the derived length, or the nilpotency class of the group or of a suitable "large" subgroup. Such bounds for nilpotent groups so far have been obtained in the cases of φ of prime order or order 4 in [4–9]. In addition, the definitive general results have been obtained in the study of almost fixed-point-free p-automorphisms of finite p-groups [10–15].

On bounding the Fitting height, especially strong results were obtained in the case of soluble groups of automorphisms $A \le \operatorname{Aut} G$ of coprime order. Thompson in [16] proved that if both groups G and A are soluble and have coprime orders, then the Fitting height of G is bounded in terms of the Fitting height of $C_G(A)$ and the number $\alpha(A)$ of prime factors of $|A|$ with multiplicity counted. Later the bounds in Thompson's theorem were improved in numerous papers, with the definitive results obtained

by Turull [17] and Hartley and Isaacs [18] on linear bounds in terms of $\alpha(A)$ for the Fitting height of the group or of a "large" subgroup.

The case of noncoprime orders of G and $A \leq \operatorname{Aut} G$ is more difficult. Bell and Hartley [19] constructed some examples that show that for every nonnilpotent finite group A there are soluble groups G of arbitrarily high Fitting height admitting A as a fixed-point-free group of automorphisms. But if A is nilpotent and $C_G(A) = 1$, then the Fitting height of G is bounded in terms of $\alpha(A)$ by a special case of Dade's theorem [20]. In the almost fixed-point-free situation, even for a cyclic group of automorphisms $\langle \varphi \rangle \leq \operatorname{Aut} G$ it is still an open problem to obtain a bound for the Fitting height of a finite soluble group G in terms of $|\varphi|$ and $|C_G(\varphi)|$ (this question is equivalent to the one recorded by Belyaev in *The Kourovka Notebook* [21] as Hartley's Problem 13.8(a)). Beyond the fixed-point-free case of Dade's theorem, so far the only cases where an affirmative solution is known are the cases of automorphisms of primary order p^n (Hartley and Turau [22]) and of biprimary order $p^a q^b$ (which is discussed in the present paper).

Another generalization of fixed-point-free automorphisms in the noncoprime case is Thompson's problem on bounding the p-length of a finite p-soluble group G admitting a p-group of automorphisms P that acts fixed-point-freely on every P-invariant p'-section of G. Rae [23] and Hartley and Rae [24] solved this problem in the affirmative for $p \neq 2$, as well as for cyclic P for any p. A special case of this problem is when a p-soluble group G admits a so-called p^n-splitting automorphism φ, which means that $xx^\varphi x^{\varphi^2} \cdots x^{\varphi^{p^n - 1}} = 1$ for all $x \in G$ (this also implies $\varphi^{p^n} = 1$); then of course φ automatically acts fixed-point-freely on φ-invariant p'-sections. This case was actually considered earlier by Kurzweil [25] who obtained bounds for the Fitting height of a soluble group G, and these bounds were improved to linear ones by Meixner [26]. If it is only known that φ induces a p^n-splitting automorphism on a φ-invariant Sylow p-subgroup of G, then there is already a bound in terms of n for the p-length of G: for $p \neq 2$ such a bound was obtained by Wilson [27], and for all primes p in [28] even under a weaker assumption.

In this paper we consider the natural generalization of Thompson's problem for a p-soluble group G admitting an automorphism φ of order p^n in which the condition that φ acts fixed-point-freely on φ-invariant p'-sections is replaced by the condition that φ acts almost fixed-point-freely on these sections. It is actually sufficient to impose the restriction on the number of fixed points of φ only on elementary abelian φ-invariant p'-sections.

Theorem 1.1. *If a finite p-soluble group G admits an automorphism φ of order p^n such that φ has at most m fixed points on every φ-invariant elementary abelian p'-section of G, then the p-length of G is bounded above in terms of p^n and m.*

It would be interesting to obtain a bound of the p-length in terms of n (or at least in terms of p^n) for some subgroup of index bounded in terms of p^n and m.

REMARK 1.2. There is a certain similarity with the situation for a p^n-splitting automorphism described above. Namely, if, for a p-soluble group G with an automorphism φ of order p^n, instead of a restriction on the number of fixed points on p'-sections, we have a restriction $|C_P(\varphi)| = p^m$ on the number of fixed points of φ in a φ-invariant Sylow p-subgroup P, then we also obtain a bound for the p-length of G. Indeed, then the derived length of P is bounded in terms of p, n, and m by Shalev's theorem [12], and so the bound for the p-length is immediate from the Hall–Higman theorems [29] for $p \neq 2$, and the theorems of Hoare [30], Berger and Gross [31], and Bryukhanova [32]. Moreover, by [13] the group P even has a (normal) subgroup of index bounded in terms of p, n, and m that has p^n-bounded derived length. Therefore, by the Hall–Higman–Hartley Theorem 2.3 (see below) there is a characteristic subgroup H of G such that the p-length of H is p^n-bounded and a Sylow p-subgroup of the quotient G/H has order bounded in terms of p, n, and m.

For soluble groups, Theorem 1.1 can be combined with the available results to give a bound for the Fitting height.

Corollary 1.3. *If a finite soluble group G admits an automorphism φ of order p^n such that φ has at most m fixed points on every φ-invariant elementary abelian p'-section of G, then the Fitting height of G is bounded above in terms of p^n and m.*

The technique of the proof of Theorem 1.1 is also applied in the proof of the soluble case of the following theorem on an almost fixed-point-free automorphism of biprimary order; the reduction to the soluble case is given by Hartley' theorem [3] (based on the classification of finite simple groups).

Theorem 1.4. *If a finite group G admits an automorphism φ of order $p^a q^b$ for some primes p, q and nonnegative integers a and b, then G has a soluble subgroup whose index and Fitting height are bounded above in terms of $p^a q^b$ and $|C_G(\varphi)|$.*

The standard inverse limit arguments yield the following corollary for locally finite groups:

Corollary 1.5. *If a locally finite group G contains an element g of order $p^a q^b$ for some primes p and q and nonnegative integers a and b with finite centralizer $C_G(g)$, then G has a finite index subgroup having a finite normal series with locally nilpotent factors.*

Another corollary is of more technical nature but it may be useful in further studies.

Corollary 1.6. *If a finite group G admits an automorphism φ such that there are at most two primes dividing both $|\varphi|$ and $|G|$, then G has a soluble subgroup whose index and Fitting height are bounded above in terms of $|\varphi|$ and $|C_G(\varphi)|$.*

REMARK 1.7. After this paper had been prepared for publication, the author became aware of an unpublished manuscript of Brian Hartley which contains the result of Theorem 1.4; the author together with A. Borovik and P. Shumyatsky published this manuscript as [33] on the web-site of the University of Manchester.

2. Preliminaries

The induced automorphisms of invariant sections are denoted by the same letters. The following lemma is well known:

Lemma 2.1. *If φ is an automorphism of a finite group G and N is a normal φ-invariant subgroup, then $|C_{G/N}(\varphi)| \leq |C_G(\varphi)|$.*

The next lemma is also a well-known consequence of considering the Jordan normal form of a linear transformation of order p^k in characteristic p.

Lemma 2.2. *If an elementary abelian p-group P admits an automorphism φ of order p^k such that $|C_P(\varphi)| = p^m$, then the rank of P is bounded in terms of p^k and m.*

We will use the following consequence of the Hall–Higman–type theorems in Hartley's paper [34].

Theorem 2.3 (Hall–Higman–Hartley). *Let P be a Sylow p-subgroup of a p-soluble group G. If R is a normal subgroup of P and the derived length of R is d, then $R \leq O_{p',p,p',\ldots,p',p}(G)$, where p occurs on the right-hand side d times if $p > 3$, $2d$ times if $p = 3$, and $3d$ times if $p = 2$.*

PROOF. As a refinement of some of the Hall–Higman theorems [29], Hartley [34] proved that if A is an abelian normal subgroup of a Sylow p-subgroup of G, then

$$A \leq O_{p',p}(G) \qquad \text{if} \quad p > 3,$$

$$A \leq O_{3',3,3',3}(G) \qquad \text{if} \quad p = 3,$$

and

$$A \leq O_{2',2,2',2,2'2}(G) \qquad \text{if} \quad p = 2.$$

The result follows from these inclusions for $A = R^{(d-1)}$ by a straightforward induction on the derived length d.

We now recall some definitions and notation from representation theory. If V is a kG-module for a field k and a group G, we use the right operator notation vg for $v \in V$ and $g \in G$. We use the centralizer notation for fixed points, like $C_V(g) = \{v \in V \mid vg = v\}$. We also use the commutator

notation $[v, g] = -v + vg$ for $v \in V$ and $g \in G$. The commutator subspaces are defined accordingly: if $B \leq G$, then $[V, B]$ is the span of all commutators $[v, b]$, where $v \in V$ and $b \in B$. The subspace $[V, B]$ coincides with the commutator subgroup $[V, B]$ in the natural semidirect product VG when V is regarded as the additive group acted upon by G. In particular, $[V, B]$ is B-invariant, and thus can be regarded as a kB-submodule.

For a group G and a field k, a free kG-module of rank n is a direct sum of n copies of the group algebra kG each of which is regarded as a vector space over k of dimension $|G|$ with a basis $\{b_g \mid g \in G\}$ labeled by elements of G on which G acts in a regular permutation representation: $b_g h = b_{gh}$. In other words, a free kG-module $V = \bigoplus_{g \in G} V_g$ is a direct sum of subspaces that are regularly permuted by G so that $V_g h = V_{gh}$.

The following lemma is known in the literature (see, for example, [22, Lemma 4.5]), but we give a proof for completeness.

Lemma 2.4. *Suppose that an abelian p-group M is acted upon by a cyclic group $\langle \varphi \rangle$ of order p^n and V is a $kM\langle \varphi \rangle$-module for a field k of characteristic different from p. If the subgroup $[M, \varphi^{p^{n-1}}]$ acts nontrivially on V, then the subspace $[V, [M, \varphi^{p^{n-1}}]]$ is a free $k\langle \varphi \rangle$-module.*

Here, of course, $\varphi^{p^{n-1}} = \varphi$ if $n = 1$.

PROOF. The subspace $[V, [M, \varphi^{p^{n-1}}]]$ is clearly $M\langle \varphi \rangle$-invariant, so is an $kM\langle \varphi \rangle$-module. We extend the ground field to its algebraic closure $\overline{k}$ and denote by $W = V \otimes_k \overline{k}$ the resulting $\overline{k}M\langle \varphi \rangle$-module. Then $[W, [M, \varphi^{p^{n-1}}]]$ is a $\overline{k}M\langle \varphi \rangle$-module obtained from $[V, [M, \varphi^{p^{n-1}}]]$ by the field extension.

Since the characteristic of the ground field is coprime to $|M\langle \varphi \rangle|$, by Maschke's theorem

$$W = C_W([M, \varphi^{p^{n-1}}]) \oplus [W, [M, \varphi^{p^{n-1}}]]$$

is a completely reducible $\overline{k}M\langle \varphi \rangle$-module. Let U be an irreducible $\overline{k}M\langle \varphi \rangle$-submodule of $[W, [M, \varphi^{p^{n-1}}]]$ on which $[M, \varphi^{p^{n-1}}]$ acts nontrivially.

By Clifford's theorem, $U = U_1 \oplus \cdots \oplus U_m$ decomposes into homogeneous $\overline{k}M$-submodules U_i (Wedderburn components). The group $\langle \varphi \rangle$ transitively permutes U_i. If the kernel of this permutational action was nontrivial, then $\varphi^{p^{n-1}}$ would stabilize all U_i. But the abelian group M acts by scalar transformations on each homogeneous component U_i. Hence $[M, \varphi^{p^{n-1}}]$ would act trivially on each U_i and therefore on U, contrary to our assumption. Thus, U is a free $\overline{k}\langle \varphi \rangle$-module.

Since $[W, [M, \varphi^{p^{n-1}}]]$ is the direct sum of such U, we see that $[W, [M, \varphi^{p^{n-1}}]]$ is also a free $\overline{k}\langle \varphi \rangle$-module. Then $[V, [M, \varphi^{p^{n-1}}]]$ is a free $k\langle \varphi \rangle$-module. Indeed, by the Deuring–Noether theorem [35, Theorem 29.7] two representations over a smaller field are equivalent if they are equivalent over a larger field. Being a free $\overline{k}\langle \varphi \rangle$-module, or a free $k\langle \varphi \rangle$-module, means having a basis, as of a vector space over the corresponding field, elements of which are permuted by φ so that all orbits are regular. In such a basis $\langle \varphi \rangle$ is represented by the corresponding permutational matrices, all of which are defined over k. $\square$

3. An Automorphism of Order p^n

First we state the following proposition which will also be used in the next section in a different situation.

Proposition 3.1. *Suppose that a cyclic group $\langle \varphi \rangle$ of order p^n acts by automorphisms on a finite p-group P, and V is a faithful $\mathbb{F}_q P\langle \varphi \rangle$-module, where $\mathbb{F}_q$ is a prime field of order $q \neq p$. Then the derived length of $[P, \varphi^{p^{n-1}}]$ is bounded in terms of $|C_V(\varphi)|$ and p^n.*

PROOF. Let M be a maximal abelian normal subgroup of the semidirect product $P\langle \varphi \rangle$. If $[M, \varphi^{p^{n-1}}] \neq 1$, then by Lemma 2.4 $[V, [M, \varphi^{p^{n-1}}]]$ is a free $\mathbb{F}_q\langle \varphi \rangle$-module. Obviously, in a free $\mathbb{F}_q\langle \varphi \rangle$-module the fixed points of φ are exactly the "diagonal" elements. Hence the order of $[V, [M, \varphi^{p^{n-1}}]]$ is equal to

$$|C_{[V,[M,\varphi^{p^{n-1}}]]}(\varphi)|^{|\varphi|} = |C_{[V,[M,\varphi^{p^{n-1}}]]}(\varphi)|^{p^n}$$

and so is bounded in terms of $|C_V(\varphi)|$ and p^n. The group $[M, \varphi^{p^{n-1}}]$ acts faithfully on V; therefore by Maschke's theorem it also acts faithfully on $[V, [M, \varphi^{p^{n-1}}]]$. Hence the order of $[M, \varphi^{p^{n-1}}]$ is bounded in terms of $|C_V(\varphi)|$ and p^n. The same of course holds if $[M, \varphi^{p^{n-1}}] = 1$.

It follows that the index $|M : C_M(\varphi^{p^{n-1}})|$ is bounded in terms of $|C_V(\varphi)|$ and p^n, since this index is equal to the number of different commutators $[m, \varphi^{p^{n-1}}]$ for $m \in M$. Consider a central series of $P\langle\varphi\rangle$ connecting 1 and M. Since $|M : C_M(\varphi^{p^{n-1}})|$ is bounded in terms of $|C_V(\varphi)|$ and p^n, the number of factors of this series that are not covered by $C_M(\varphi^{p^{n-1}})$ is bounded in terms of $|C_V(\varphi)|$ and p^n. Therefore, there is a normal series of bounded length connecting 1 and M each factor of which is either central in $P\langle\varphi\rangle$ or is covered by $C_M(\varphi^{p^{n-1}})$. Obviously, then $\varphi^{p^{n-1}}$ acts trivially on each factor of this series, and therefore so does $[P, \varphi^{p^{n-1}}]$. By Kaluzhnin's theorem, the automorphism group induced by the action of $[P, \varphi^{p^{a-1}}]$ on M is nilpotent of bounded class. Since M contains its centralizer in $P\langle\varphi\rangle$, it follows that $[P, \varphi^{p^{n-1}}]$ is soluble of bounded derived length, since by the above $\gamma_s([P, \varphi^{p^{n-1}}]) \le M \cap [P, \varphi^{p^{n-1}}]$ for some number s bounded in terms of $|C_V(\varphi)|$ and p^n. $\quad\square$

PROOF OF THEOREM 1.1. Recall that G is a finite p-soluble group admitting an automorphism φ of order p^n such that φ has at most m fixed points in every φ-invariant elementary abelian p'-section of G. We need to bound the p-length of G in terms of p^n and m. Henceforth in this section, saying for brevity that a certain parameter is simply "bounded" we mean that the parameter is bounded above in terms of p^n and m. We use induction on n. It is convenient to consider the case of $n = 0$ as the base of induction, when $|\varphi| = p^0 = 1$, that is, φ acts trivially on G. Then the hypothesis means that every elementary abelian p'-section of G has bounded order. We claim that the nilpotency class of a Sylow p-subgroup P of $\widehat{G} = G/O_p(G)$ is bounded. Indeed, since the order of P is coprime to $|O_{p'}(\widehat{G})|$, for every prime q dividing $|O_{p'}(\widehat{G})|$ there is a P-invariant Sylow q-subgroup Q of $O_{p'}(\widehat{G})$. The quotient $P/C_P(Q)$ acts faithfully on the Frattini quotient $Q/\Phi(Q)$, which has order at most m by assumption. Hence $P/C_P(Q)$ has bounded order and so bounded nilpotency class. Since P acts faithfully on $O_{p'}(\widehat{G})$, we have $\bigcap C_P(Q_i) = 1$, where Q_i runs over all P-invariant Sylow subgroups of $O_{p'}(\widehat{G})$. Hence P is a subdirect product of groups of bounded nilpotency class and therefore has bounded nilpotency class itself. We now find that the p-length of $\widehat{G} = G/O_p(G)$ is bounded by the Hall–Higman theorem [29]. In result, the p-length of G is bounded.

From now on we assume that $n \ge 1$.

Let $\widehat{G} = G/O_p(G)$. Consider a Sylow p-subgroup of the semidirect product $\widehat{G}\langle\varphi\rangle$ containing $\langle\varphi\rangle$ and let P be its intersection with $\widehat{G}$, so that P is a φ-invariant Sylow p-subgroup of $\widehat{G}$. Since the order of the p-group $P\langle\varphi\rangle$ is coprime to $|O_{p'}(\widehat{G})|$, for every prime q dividing $|O_{p'}(\widehat{G})|$ there is a $P\langle\varphi\rangle$-invariant Sylow q-subgroup Q of $O_{p'}(\widehat{G})$.

The quotient $\overline{P} = P/C_P(Q)$ acts faithfully on the Frattini quotient $V = Q/\Phi(Q)$, which we regard as an $\mathbb{F}_q P\langle\varphi\rangle$-module. By hypothesis, $|C_V(\varphi)| \le m$, and so by Proposition 3.1 the derived length of $[\overline{P}, \varphi^{p^{n-1}}]$ is bounded. In other words, $[P, \varphi^{p^{n-1}}]^{(s)} \le C_P(Q)$ for some bounded number s. Since P acts faithfully on $O_{p'}(\widehat{G})$, we have $\bigcap C_P(Q_i) = 1$, where Q_i runs over all $P\langle\varphi\rangle$-invariant Sylow subgroups of $O_{p'}(\widehat{G})$. Hence, $[P, \varphi^{p^{n-1}}]^{(s)} = 1$.

By the Hall–Higman–Hartley Theorem 2.3 we now find that the normal subgroup $[P, \varphi^{p^{n-1}}]$ of the Sylow p-subgroup P is contained in $H = O_{p',p,p',\ldots,p',p}(\widehat{G})$, where p occurs boundedly many times.

Consider the action of φ on the quotient $\widetilde{G} = \widehat{G}/H$. Since $[P, \varphi^{p^{n-1}}] \le H$, it follows that $\varphi^{p^{n-1}}$ acts trivially on the image of P, which is a Sylow p-subgroup of $\widetilde{G}$. In particular, $\varphi^{p^{n-1}}$ acts trivially on $O_{p',p}(\widetilde{G})/O_{p'}(\widetilde{G})$, and therefore so does $[\widetilde{G}, \varphi^{p^{n-1}}]$. Since $O_{p',p}(\widetilde{G})/O_{p'}(\widetilde{G})$ contains its centralizer in $\widetilde{G}/O_{p'}(\widetilde{G})$, we have $[\widetilde{G}, \varphi^{p^{n-1}}] \le O_{p',p}(\widetilde{G})$. In other words, $\varphi^{p^{n-1}}$ acts trivially on the quotient $\widetilde{G}/O_{p',p}(\widetilde{G})$. Therefore, the order of the automorphism induced by φ on $\widetilde{G}/O_{p',p}(\widetilde{G})$ is at most p^{n-1}. By the induction hypothesis the p-length of this quotient is bounded. Then the p-length of $G/O_{p,p'}(G)$ is bounded, and so the p-length of G is bounded, as required. $\quad\square$

PROOF OF COROLLARY 1.3. Here, G is a finite soluble group admitting an automorphism φ of order p^n such that φ has at most m fixed points in every φ-invariant elementary abelian p'-section of G. By Theorem 1.1 the p-length of G is bounded. It remains to obtain a bound for the Fitting height of every p'-factor T of the upper p-series consisting of the subgroups $O_{p',p,p',p,\ldots}$. It is known that the rank of a finite group is bounded in terms of the ranks of its elementary abelian sections. Here, by definition, the rank of a group is the minimum number r such that every subgroup can be generated by r elements. Of course every elementary abelian section of $C_T(\varphi)$ is a φ-invariant p'-section of G and therefore has bounded order by hypothesis. It is also known that the Fitting height of a soluble finite group is bounded in terms of its rank. Thus $C_T(\varphi)$ has bounded Fitting height and therefore so does G by Thompson's theorem [2].

If we could obtain in Theorem 1.1 a "strong" bound for the p-length, in terms only of $\alpha(\langle\varphi\rangle)$, for a subgroup of bounded index, then a similar strong bound could be obtained in Corollary 1.3 for the Fitting height of a subgroup of bounded index. This would follow from a rank analog of the Hartley–Isaacs theorem proved in [36], which states that if a finite soluble group K admits a soluble group of automorphisms L of coprime order, then K has a normal subgroup N of Fitting height at most $5(4^{\alpha(L)}-1)/3$ such that the order of K/N is bounded in terms of $|L|$ and the rank of $C_K(L)$. $\square$

4. An Automorphism of Order $p^a q^b$

PROOF OF THEOREM 1.4. Recall that G is a finite group admitting an automorphism φ of order $p^a q^b$. By Hartley's theorem [3] (based on the classification of finite simple groups), G has a soluble subgroup of index bounded in terms of $p^a q^b$ and $|C_G(\varphi)|$. Therefore, we can assume from the outset that G is soluble, so that we need to bound the Fitting height of G in terms of $p^a q^b$ and $|C_G(\varphi)|$. Throughout this section we say for brevity that a certain parameter is "bounded" meaning that this parameter is bounded above in terms of $p^a q^b$ and $|C_G(\varphi)|$. We use without special references the fact that the number of fixed points of φ in every φ-invariant section of G is at most $|C_G(\varphi)|$ by Lemma 2.1.

We use induction on $a+b$. As a base of induction we consider the case when either $a=0$ or $b=0$. Then $|\varphi|$ is a prime power, and by the Hartley–Turau theorem [22] the group G has a subgroup of bounded index that has Fitting height at most $\alpha(\varphi)$. (Actually, for our "weak" bound a simpler argument would suffice: if, say, $|\varphi|=p^a$, then the rank of the Frattini quotient of $O_{p',p}(G)/O_{p'}(G)$ is bounded by Lemma 2.2, which implies a bound for the Fitting height of $G/O_{p'}(G)$, and the Fitting height of $O_{p'}(G)$ is bounded in terms of a by Thompson's theorem [2].) Moreover, the following proposition holds, which apparently was noted by Hartley but may have remained unpublished. We state this proposition in a more general form, without assuming that the automorphism has biprimary order.

Proposition 4.1. *If a finite soluble group G admits an automorphism ψ such that there is at most one prime dividing both $|\psi|$ and $|G|$, then the Fitting height of G is bounded above in terms of $|\psi|$ and $|C_G(\psi)|$.*

PROOF. If $(|\psi|,|G|)=1$, then the result follows from the stronger theorem of Thompson [2]. Now let $\langle\psi\rangle = \langle\psi_r\rangle \times \langle\psi_{r'}\rangle$, where $\langle\psi_r\rangle$ is the Sylow r-subgroup of $\langle\psi\rangle$ and r is the only common prime divisor of $|G|$ and $|\psi|$. The centralizer $C_G(\psi_{r'})$ admits the automorphism ψ_r of prime-power order whose centralizer $C_{C_G(\psi_{r'})}(\psi_r)$ is equal to $C_G(\psi)$. By the Hartley–Turau theorem, the Fitting height of $C_G(\psi_{r'})$ is bounded. We now apply Thompson's theorem to the automorphism $\psi_{r'}$ of G of coprime order to infer that the Fitting height of G is bounded as required.

We return to the proof of Theorem 1.4. Let $a \geq 1$ and $b \geq 1$. Let $\varphi_p = \varphi^{q^b}$ and $\varphi_q = \varphi^{p^a}$, so that $|\varphi_p| = p^a$ and $|\varphi_q| = q^b$, while $\langle\varphi\rangle = \langle\varphi_p\rangle \times \langle\varphi_q\rangle$. The subgroup $O_{q'}(G)$ admits the automorphism φ whose order has at most one prime divisor p in common with $|O_{q'}(G)|$. By Proposition 4.1 the Fitting height of $O_{q'}(G)$ is bounded.

Therefore, we can assume that $O_{q'}(G) = 1$. Then the quotient $\overline{G} = G/O_q(G)$ acts faithfully on the Frattini quotient $V = O_q(G)/\Phi(O_q(G))$, which we regard as an $\mathbb{F}_q\overline{G}\langle\varphi\rangle$-module. The fixed-point subspace $C_V(\varphi_p)$ has bounded order. This follows from Lemma 2.2 applied to the action of the linear

transformation φ_q of order q^b on $C_V(\varphi_p)$, since the fixed points of φ_q in $C_V(\varphi_p)$ belong to the fixed-point subspace $C_V(\varphi)$ of bounded order.

Choose a Sylow p-subgroup of $\overline{G}\langle\varphi_p\rangle$ containing $\langle\varphi_p\rangle$, and let P be its intersection with $\overline{G}$, so that P is a φ_p-invariant Sylow p-subgroup of $\overline{G}$. By Proposition 4.1, the subgroup $[P,\varphi_p^{p^{a-1}}]$ has bounded derived length.

Hence by the Hall–Higman–Hartley Theorem 2.3 the normal subgroup $[P,\varphi_p^{p^{a-1}}]$ of the Sylow p-subgroup P is contained in $H = O_{p',p,p',\ldots,p',p}(\overline{G})$, where p occurs boundedly many times.

Consider the action of φ on the quotient $\widetilde{G} = \overline{G}/H$. Since $[P,\varphi_p^{p^{a-1}}] \le H$, it follows that $\varphi_p^{p^{a-1}}$ acts trivially on the image of P, which is a Sylow p-subgroup of $\widetilde{G}$. In particular, $\varphi_p^{p^{a-1}}$ acts trivially on $O_{p',p}(\widetilde{G})/O_{p'}(\widetilde{G})$, and therefore so does $[\widetilde{G},\varphi_p^{p^{a-1}}]$. Since $O_{p',p}(\widetilde{G})/O_{p'}(\widetilde{G})$ has its centralizer in $\widetilde{G}/O_{p'}(\widetilde{G})$, we have $[\widetilde{G},\varphi_p^{p^{a-1}}] \le O_{p',p}(\widetilde{G})$. In other words, $\varphi_p^{p^{a-1}}$ acts trivially on the quotient $\widetilde{G}/O_{p',p}(\widetilde{G})$. Therefore, the order of the automorphism induced by φ on $\widetilde{G}/O_{p',p}(\widetilde{G})$ divides $p^{a-1}q^b$. By induction, the Fitting height of this quotient is bounded.

It remains to obtain a bound for the Fitting height of each of the boundedly many φ-invariant normal p'-sections that appear in the upper p-series of the groups H and $O_{p',p}(\widetilde{G})$. Such a bound follows from Proposition 4.1.

PROOF OF COROLLARY 1.5. This corollary for locally finite groups follows from Theorem 1.4 by the standard inverse limit argument. $\square$

PROOF OF COROLLARY 1.6. Here, a finite group G admits an automorphism φ such that there are at most two primes dividing both $|\varphi|$ and $|G|$. Again, by Hartley's theorem [3] we can assume from the outset that G is soluble. If $(|\varphi|,|G|)$ is 1 or a prime power, then the result follows from Proposition 4.1. Now let $\langle\varphi\rangle = \langle\varphi_{pq}\rangle \times \langle\psi\rangle$, where $\langle\varphi_{pq}\rangle$ is the Hall $\{p,q\}$-subgroup of $\langle\varphi\rangle$ and p and q are the only common prime divisors of $|G|$ and $|\varphi|$. The centralizer $C_G(\psi)$ admits the automorphism φ_{pq} of biprimary order whose centralizer $C_{C_G(\psi)}(\varphi_{pq})$ is equal to $C_G(\varphi)$. By Theorem 1.4, the Fitting height of $C_G(\psi)$ is bounded in terms of $|\varphi_{pq}|$ and $|C_G(\varphi)|$. We now apply Thompson's theorem [16] to the automorphism ψ of G of coprime order to find that the Fitting height of G is bounded in terms of $|\varphi|$ and $|C_G(\varphi)|$. $\square$

References

1. *Brauer R. and Fowler K. A.*, "On groups of even order," Ann. Math. (2), **62**, 565–583 (1955).
2. *Thompson J. G.*, "Finite groups with fixed-point-free automorphisms of prime order," Proc. Nat. Acad. Sci., **45**, 578–581 (1959).
3. *Hartley B.*, "A general Brauer–Fowler theorem and centralizers in locally finite groups," Pacific J. Math., **152**, 101–117 (1992).
4. *Higman G.*, "Groups and rings which have automorphisms without non-trivial fixed elements," J. London Math. Soc., **32**, No. 2, 321–334 (1957).
5. *Kreknin V. A.*, "Solvability of Lie algebras with regular automorphisms of finite period," Soviet Math. Dokl., **4**, 683–685 (1963).
6. *Kreknin V. A. and Kostrikin A. I.*, "Lie algebras with regular automorphisms," Dokl. Akad. Nauk SSSR, **149**, 249–251 (1963).
7. *Khukhro E. I.*, "Lie rings and groups admitting an almost regular automorphism of prime order," Math. USSR-Sb., **71**, No. 1, 51–63 (1992).
8. *Kovács L. G.*, "Groups with regular automorphisms of order four," Math. Z., Bd 75, 277–294 (1960/1961).
9. *Khukhro E. I. and Makarenko N. Yu.*, "Finite groups with an almost regular automorphism of order four," Algebra and Logic, **45**, No. 5, 326–343 (2006).
10. *Alperin J. L.*, "Automorphisms of solvable groups," Proc. Amer. Math. Soc., **13**, 175–180 (1962).
11. *Khukhro E. I.*, "Finite p-groups admitting an automorphism of order p with a small number of fixed points," Math. Notes, **38**, No. 5, 867–870 (1985).
12. *Shalev A.*, "On almost fixed point free automorphisms," J. Algebra, **157**, 271–282 (1993).
13. *Khukhro E. I.*, "Finite p-groups admitting p-automorphisms with few fixed points," Russian Acad. Sci., Sb. Math., **80**, 435–444 (1995).
14. *Medvedev Yu.*, "p-Divided Lie rings and p-groups," J. London Math. Soc. (2), **59**, 787–798 (1999).
15. *Jaikin-Zapirain A.*, "On almost regular automorphisms of finite p-groups," Adv. Math., **153**, 391–402 (2000).

16. *Thompson J.*, "Automorphisms of solvable groups," J. Algebra, **1**, 259–267 (1964).
17. *Turull A.*, "Fitting height of groups and of fixed points," J. Algebra, **86**, 555–566 (1984).
18. *Hartley B. and Isaacs I. M.*, "On characters and fixed points of coprime operator groups," J. Algebra, **131**, 342–358 (1990).
19. *Bell S. D. and Hartley B.*, "A note on fixed-point-free actions of finite groups," Quart. J. Math. Oxford Ser. (2), **41**, No. 162, 127–130 (1990).
20. *Dade E. C.*, "Carter subgroups and Fitting heights of finite solvable groups," Illinois J. Math., **13**, 449–514 (1969).
21. *Mazurov V. D. and Khukhro E. I.* (Eds.), The Kourovka Notebook: Unsolved Problems in Group Theory, 18th ed., Sobolev Inst. Math., Novosibirsk (2014). http://math.nsc.ru/ alglog/18kt.pdf
22. *Hartley B. and Turau V.*, "Finite soluble groups admitting an automorphism of prime power order with few fixed points," Math. Proc. Cambridge Philos. Soc., **102**, 431–441 (1987).
23. *Rae A.*, "Sylow p-subgroups of finite p-soluble groups," J. London Math. Soc. (2), **71**, 117–123 (1973).
24. *Hartley B. and Rae A.*, "Finite p-groups acting on p-soluble groups," Bull. London Math. Soc., **5**, 197–198 (1973).
25. *Kurzweil H.*, "Eine Verallgemeinerung von fixpunktfreien Automorphismen endlicher Gruppen," Arch. Math. (Basel), **22**, 136–145 (1971).
26. *Meixner T.*, "The Fitting length of solvable H_{p^n}-groups," Israel J. Math., **51**, No. 1–2, 68–78 (1985).
27. *Wilson J. S.*, "On the structure of compact torsion groups," Monatsh. Math., **96**, No. 1, 57–66 (1983).
28. *Khukhro E. I. and Shumyatsky P.*, "Words and pronilpotent subgroups in profinite groups," Austral. Math. Soc. J., **97**, No. 3, 343–364 (2014).
29. *Hall P. and Higman G.*, "The p-length of a p-soluble group and reduction theorems for Burnside's problem," Proc. London Math. Soc. (3), **6**, 1–42 (1956).
30. *Hoare A. H. M.*, "A note on 2-soluble groups," J. London Math. Soc., **35**, 193–199 (1960).
31. *Berger T. R. and Gross F.*, "2-Length and the derived length of a Sylow 2-subgroup," Proc. London Math. Soc. (3), **34**, 520–534 (1977).
32. *Bryukhanova E. G.*, "Connection between the 2-length and the derived length of a Sylow 2-subgroup of a finite solvable group," Math. Notes, **29**, No. 2, 85–90 (1981).
33. *Hartley B.*, "Automorphisms of finite soluble groups. Preliminary version," MIMS EPrint: 2014.52. http://eprints.ma.man.ac.uk/2188/01/covered/MIMS_ep2014_52.pdf.
34. *Hartley B.*, "Some theorems of Hall–Higman type for small primes," Proc. London Math. Soc. (3), **41**, 340–362 (1980).
35. *Curtis C. W. and Reiner I.*, Representation Theory of Finite Groups and Associative Algebras, Interscience, New York and London (1962).
36. *Khukhro E. and Mazurov V.*, "Automorphisms with centralizers of small rank," in: Groups St. Andrews 2005. Vol. II. Selected Papers of the Conference, St. Andrews, UK, July 30–August 6, 2005, London Math. Soc. Lecture Note Ser., vol. 340, Cambridge Univ. Press, Cambridge, 2007, pp. 564–585.

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