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DEFORMATION AND FRACTURE MECHANICS

Deformation and Long-Term Strength of a Thick-Walled Tube of a Physically Non-Linear Viscoelastic Material under Constant Pressure

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Abstract—An exact solution is constructed for the problem of creep and fracture of a hollow cylinder made of a physically nonlinear rheonomic isotropic incompressible material, which obeys Rabotnov's constitutive viscoelasticity relation with two arbitrary material functions, under the action of internal and external pressures. Closed form equations for long-term strength curves are derived using three versions of a deformation fracture criterion, and the strain intensity, the maximum shear strain, or the maximum tensile strain is chosen as the measure of damage. Their properties are analytically investigated for arbitrary material functions of the constitutive relation.

Keywords: viscoelasticity, physical nonlinearity, Rabotnov's constitutive relation, creep, time to failure, long-term strength curve

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1. INTRODUCTION

The results of testing samples and structural elements show that, under a constant, even a sufficiently low load (causing stresses much lower than the ultimate strength), the strain increases in time (creep is observed) and fracture occurs after certain time t_* after the application of the load (due to creep and related damage accumulation mechanisms). Lifetime t_* depends on the load, the temperature, and other parameters and can be a few years. The dependence of t_* on the load or the related stress in a sample $t_*(\sigma)$ (or the inverse dependence $\sigma(t_*)$) is called the long-term strength curve of a material or structural element. The data of testing various viscoelastoplastic materials [1–9] show that the $t_*(\sigma)$ dependences always decrease, $t_*(\sigma) \to +\infty$ at $\sigma \to \sigma_0$ and $t_*(\sigma) \to 0$ at $\sigma \to \sigma_*$, were $\sigma_0 \geq 0$ is the (conventional) creep threshold and $\sigma_* > 0$ is the instantaneous ultimate tensile strength σ_*.

To predict the lifetime during creep and to simulate the long-term strength of viscoelastic materials, a selected (or constructed) constitutive relation (CR) describing their deformation should be supplemented with a fracture criterion characterizing the fracture time t_* when a critical measure of damage $\omega(t)$ (scalar, vector, or tensor [1–9]) is reached. The simplest type of fracture criteria is represented by the classical deformation fracture criteria postulating that $\omega(t) = C\varepsilon(t)$ and fracture occurs at time $t = t_*$ when a certain measure of strain $\varepsilon(t, \sigma)$ reaches the limiting value ε_*, $\varepsilon(t_*,\sigma) = \varepsilon_*$. They can describe the fracture of both a material and a structural element. It is important that a selected criterion and CR interact well with each other, i.e., make it possible to derive equations for theoretical creep curves $\varepsilon = \varepsilon(t, \sigma, T)$ and a long-term strength curve $t_* = f(\sigma, T)$, or $\sigma = F = (t_*, T)$, where t_* is the time to failure at given stress σ and temperature T; to analytically investigate (in general form) the dependence of the properties of a long-term strength curve on (arbitrary) material functions (MFs), CR parameters, and fracture criterion; and find the restrictions on them that ensure the coincidence the qualitative properties of theoretical curves with the typical properties of experimental curves [9, 10].

This article continues the series of works [11–15] devoted to a systematic analytical study of a physically nonlinear CR for viscoelasticity of the form

$$\varepsilon_{ij}(t) = \frac{3}{2}\Phi(L(t))\sigma(t)^{-1}[\sigma_{ij} - \sigma_0\delta_{ij}] + \frac{1}{3}\Phi_0(L_0(t))\delta_{ij}, \quad (1)$$

$$L(t) := \mathbf{\Pi}\sigma, \quad \Lambda_0(t) := \mathbf{\Pi}_0\sigma_0,$$
$$\mathbf{\Pi}y = \int_0^t \Pi(t-\tau)dy(\tau), \quad (2)$$
$$\mathbf{\Pi}_0 y = \int_0^t \Pi_0(t-\tau)dy(\tau).$$

Here Π and Π_0 are the shear and volumetric creep functions, respectively; Φ and Φ_0 are the nonlinearity functions; $\sigma_0 = \sigma_{ii}(t)/3$ is the average stress (first invariant of tensor $\boldsymbol{\sigma}(t)$); and $\sigma = (1.5s_{ij}s_{ij})^{0.5}$ is the stress intensity (second invariant of the deviator $\mathbf{s} = \boldsymbol{\sigma} - \sigma_0\mathbf{I}$).

Equation (1) describes the isothermal processes of deformation of nonaging isotropic viscoelastic materials by connecting the histories of changes in stress tensors $\boldsymbol{\sigma}(t)$ and small strains $\boldsymbol{\varepsilon}(t)$ at an arbitrary point in a body (stress and time are assumed to be dimensionless). This equation is one of the simplest versions of the generalization of uniaxial Rabotnov's relation with two material functions φ and Π [16–20],

$$\varphi(\varepsilon(t)) = \int_0^t \Pi(t-\tau)d\sigma(\tau),$$
$$\sigma(t) = \int_0^t R(t-\tau)\varphi'(\varepsilon(\tau))d\varepsilon(\tau), \quad t > 0, \tag{3}$$

to the case of a complex state of stress. It was obtained under the assumption of isotropy and tensor linearity of a material and the absence of mutual influence of the spherical and deviator parts of the tensors (independence of volumetric strain $\theta = 3\varepsilon_0 = \varepsilon_{ii}(t)$ from shear stresses and independence of shear strains from average stress σ_0) and by neglecting the influence of their third invariants ($\Phi = \varphi^{-1}$).

The main problems of this work are as follows: (1) to analytically analyze the evolution of the deformed state of a thick-walled tube made of nonlinear hereditary material, which obeys Eq. (1), during creep under constant (internal, external) pressures; (2) to obtain a general expression for the time to failure of a tube through the pressures and MF using three variants of the deformation criterion of fracture (strain intensity, maximum shear strain, and maximum tensile strain were chosen as the measure of damage); and (3) to derive equations for the corresponding long-term strength curves and to analytically investigate their properties at arbitrary material functions of Rabotnov's CRs. These problems have not been solved in general form even for a tube made of an isotropic hereditary material, which obeys a linear viscoelasticity CR with arbitrary functions of shear and volumetric creep, although the problem of calculating a thick-walled tube in terms of the theory of elasticity and elastoplasticity is a classical comprehensively studied problem [3, 21, 22].

2. RABOTNOV'S CONSTITUTIVE RELATION AND RESTRICTIONS ON ITS MATERIAL FUNCTIONS

A uniaxial version of CR (3) was proposed by Rabotnov [16–20] to describe nonlinear creep as a generalization of the one-dimensional linear viscoelasticity CR

$$\varepsilon(t) = \int_0^t \Pi(t-\tau)d\sigma(\tau) = \Pi\sigma,$$
$$\sigma(t) = \int_0^t R(t-\tau)d\varepsilon(\tau) = \mathbf{R}\varepsilon, \quad t > 0, \tag{4}$$

by introducing an additional MF $\varphi(u)$. The creep and relaxation functions $\Pi(t)$ and $R(t)$ in Eqs. (4) and (3) are related by the integral equation

$$\int_0^t R(t-\tau)\Pi(\tau)d\tau = t, \tag{5}$$

which expresses the condition of mutual inverseness of operators (4) (and (3)). In English-language works, CR (3) is called the quasi-linear viscoelasticity (QLV) equation and its author is considered to be Fung [23–32]. In papers [16–20, 33–35], CR (3) was applied to describe the one-dimensional behavior of graphite, metals, alloys, and composites; in [23–32], it was used to describe ligaments, tendons, and other biological tissues. Detailed reviews of the literature and the fields of application of CR (3) are given in [12–15].

In the one-dimensional case, inverse CR (3) has the form $\sigma = \mathbf{R}\varphi(\varepsilon)$ (composition of the operator of action of function φ and linear operator $\mathbf{R}$ from Eq. (4). The inversion of three-dimensional CR (1) for any increasing MFs Φ and Φ_0 is written as

$$\sigma_0 = \mathbf{R}_0\varphi_0(\theta), \quad \sigma = \mathbf{R}\varphi(\varepsilon),$$
$$s_{ij}(t) = \frac{2}{3}\sigma(t)\varepsilon(t)^{-1}e_{ij}(t), \tag{6}$$

where $\varphi = \Phi^{-1}$, $\varphi_0 = \Phi_0^{-1}$, and the $R(t)$ and $R_0(t)$ relaxation functions are related by equations of form (5). Among the three material functions φ, Π, and R in CR (3), only two are independent, and CR (1) contains four independent MFs.

We impose the same minimal restrictions on the creep and relaxation functions in CRs (3) and (1) as in the linear theory of viscoelasticity: $\Pi(t)$, $\Pi_0(t)$, $R(t)$, and $R_0(t)$ are assumed to be positive and differentiable in $(0, \infty)$; functions Π and Π_0 are assumed to be increasing and convex up [36]; R and R_0 are decreasing and convex down in $(0, \infty)$; and functions $R(t)$ and $R_0(t)$ can have an integrable singularity or δ singularity at point $t = 0$ (term $\eta\delta(t)$, where $\eta > 0$ and $\delta(t)$ is a delta function). These conditions imply the existence of the limit $\Pi(0) = \inf\Pi(t) \geq 0$ ($y(0){:} = y(0+)$ is the designation of the limit of function $y(t)$ on the right at point $t = 0$).

We impose the following minimum requirements on MFs φ and φ_0 in CRs (3) and (6) and on MFs $\Phi(x)$ and $\Phi_0(x)$ in CR (1) [12–15]: function $\varphi(u)$ is continuously differentiable and strictly increases on $(0; \omega)$, where $\omega > 0$ and $\varphi_0(u)$, on the set $(\omega_-; 0) \cup (0; \omega_+)$, where $\omega_-\omega_+ < 0$; here, $\varphi(0+) = 0$ and $\varphi(0+) = \varphi_0(0-) = 0$ (otherwise, a nonzero response $\boldsymbol{\sigma}(t)$ corresponds to the input process $\varepsilon(t) \equiv 0$). An increase in $\varphi(u)$ and $\varphi_0(u)$ leads to the existence and increase of inverse functions $\Phi(x) = \varphi^{-1}$, $x \in (0; X)$, $X := \sup\varphi(u)$ and $\Phi_0(x) = \varphi_0^{-1}$, $x \in (\underline{x}; \overline{x})$, where $\underline{x} = \varphi_0(\omega_-;+0)$, $\overline{x} = \varphi_0(\omega_+; -0)$, and the reversibility of CR (1). Similarly, the reversibility of CR (1) follows from an increase in Φ and Φ_0. The families of the functions that can be conveniently used to set MFs Φ, Φ_0 or φ, φ_0 are given in [12–15].

3. FORMULATION AND SOLUTION OF THE BOUNDARY PROBLEM

Consider the problem of determining the stresses and strains in a hollow cylinder made of a hereditary incompressible material obeying nonlinear Rabotnov's CR (1) under the action of constant pressures p_1 and p_2 set on the inner and outer cylinder surfaces at $t > 0$. We use a cylindrical coordinate system. Let r_1 and r_2 be the inner and outer radii of the unloaded cylinder (at $t = 0$). Then, the boundary conditions have the form

$$\begin{gathered} \sigma_r|_{r_1} = -p_1(t), \quad \sigma_r|_{r_2} = -p_2(t), \\ \sigma_{r\theta}|_{r_1} = \sigma_{rz}|_{r_1} = 0, \quad \sigma_{r\theta}|_{r_2} = \sigma_{rz}|_{r_2} = 0. \end{gathered} \tag{7}$$

The solution will also be valid in the case where pressures $p_1(t)$ and $p_2(t)$ depend on time and change slowly, so that the influence of the inertial terms in the equations of motion can be neglected.

The problem is axisymmetric; therefore, at any point (r, θ, z) at any time, all components of displacements, strains, and stresses do not depend on angle Θ,

$$\begin{gathered} \sigma_{r\theta} \equiv 0, \quad \sigma_{r\theta} \equiv 0, \quad u_\theta(t) \equiv 0, \\ \varepsilon_\theta(r,t) = (u_{\theta,\theta} + u_r)/r = u/r, \end{gathered} \tag{8}$$

$$\varepsilon_r(r,t) = u_{r,r} = \partial u/\partial r, \quad \varepsilon_z(r,t) = u_{z,z}, \tag{9}$$

where the notation $u := u_r(r, t)$ is introduced for the radial displacement.

The tube is assumed to be fixed at the ends so that the axial displacement is $u_z = 0$ tangential stresses are absent at the ends, $\sigma_{z\theta} = 0$ and $\sigma_{rz} = 0$. Then, the tube is in the state of plane deformation, u_r and σ_z do not depend on z, and (apart from Eq. (8)), the equalities

$$\begin{gathered} \sigma_{rz} \equiv 0, \quad \varepsilon_{z\theta} \equiv 0, \quad \varepsilon_{r\theta} \equiv 0, \quad \varepsilon_z \equiv 0, \quad u_z \equiv 0, \\ \varepsilon_{rz} = \frac{1}{2}(u_{r,z} + u_{z,r}) \equiv 0, \quad \varepsilon_{\theta z} = \frac{1}{2}(u_{\theta,z} + r^{-1}u_{z,\theta}) \equiv 0, \end{gathered} \tag{10}$$

hold true.

As follows from Eqs. (8) and (10), the strain and stress tensors are diagonal, $\boldsymbol{\varepsilon} = \mathbf{diag}\{\varepsilon_r, \varepsilon_\theta, 0\}$ and $\sigma := \mathbf{diag}\{\sigma_r, \sigma_\theta, \sigma_z\}$, and the coordinate dependences of the nonzero components have the form $u_r(r, t)$, $\varepsilon_r(r, t)$, $\varepsilon_\theta(r, t)$, $\sigma_r(r, t)$, $\sigma_\theta(r, t)$, and $\sigma_z(t)$.

Due to the symmetry of the stress field (Eqs. (8), (10)), the set of equations of equilibrium of the medium is equivalent to the following equation in the projection onto a radius:

$$\sigma_{r,r} + r^{-1}(\sigma_r - \sigma_\theta) = 0. \tag{11}$$

We use the assumption of the incompressibility of the material, $\varepsilon_r + \varepsilon_\theta + \varepsilon_z = 0$. Since $\varepsilon_z \equiv 0$, it takes the form $\varepsilon_r + \varepsilon_\theta = 0$. Using Eq. (9), we obtain the ordinary differential equation $\partial u/\partial r + u/r = 0$ for $u(r, t)$ and have

$$u = C(t)r^{-1}, \quad r_1 \le r \le r_2, \quad t > 0. \tag{12}$$

From Eqs. (12) and (9), all nonzero components of the strain tensor are expressed through one unknown function $C(t)$,

$$\begin{gathered} \varepsilon_\theta(r,t) = u/r = C(t)r^{-2}, \\ \varepsilon_r(r,t) = \partial u/\partial r = -C(t)r^{-2}. \end{gathered} \tag{13}$$

CR (1) is applied. Due to the incompressibility of the material, the strain deviator coincides with this CR and CR (1) is reduced to one-dimensional CR $\varepsilon = \Phi(\mathbf{\Pi}\sigma)$ with two arbitrary MFs (Φ and Π or φ and R), which relates the stress and strain intensities, and to the condition of proportionality of deviators (6),

$$s_{ij}(t) = \frac{2}{3}\sigma(t)\varepsilon(t)^{-1}e_{ij}(t), \quad \sigma = \mathbf{R}\varphi(\varepsilon). \tag{14}$$

The first equation in CR (6) is not used, and the average stress is found by solving the boundary problem, as usually under incompressibility condition.

Since $\varepsilon_{ij}(t) \equiv 0$ at $i \ne j$ and $\varepsilon_z \equiv 0$ at any point, the strain deviator has the form $\mathbf{e} = \mathbf{diag}\{\varepsilon_r, \varepsilon_\theta, 0\}$ and the strain intensity is

$$\begin{aligned} \varepsilon &= \frac{\sqrt{2}}{3}\left[\left((\varepsilon_r - \varepsilon_\theta)^2 + \varepsilon_r^2 + \varepsilon_\theta^2\right)\right]^{0.5} \\ &= \frac{2}{3}\left[\varepsilon_r^2 + \varepsilon_\theta^2 - \varepsilon_r\varepsilon_\theta\right]^{0.5} = \frac{2}{\sqrt{3}}|C(t)|\,r^{-2}. \end{aligned} \tag{15}$$

Depending on the ratio of the histories of pressures $p_1(\tau)$ and $p_2(\tau)$, any sign of $C(t)$ is possible.

The stress deviator is also diagonal at any point due to the tensor linearity of CR,

$$\mathbf{s} := \mathbf{diag}\{\sigma_r - \sigma_\theta, \sigma_\theta - \sigma_0, \sigma_z - \sigma_0\},$$

where $\sigma_0(r, t) = (\sigma_r + \sigma_\Theta + \sigma_z)/3$ is the average stress.

According to Eq. (14), the deviators are proportional; therefore, $\sigma_z - \sigma_0 \equiv 0$ (at t when $\varepsilon(t) \ne 0$, i.e., $C(t) \ne 0$) from the condition $e_z \equiv 0$. From whence, we have

$$\sigma_z = \sigma_0 \quad \text{and} \quad \sigma_z = (\sigma_r + \sigma_\theta)/2. \tag{16}$$

Then, $|\sigma_r - \sigma_0| = |\sigma_\theta - \sigma_0|$, $|\sigma_r - \sigma_z| = |\sigma_\theta - \sigma_z| = 0.5|\sigma_r - \sigma_\theta|$ and the stress intensity expression is simplified,

$$\sigma = \frac{\sqrt{2}}{2}\left[(\sigma_r - \sigma_\theta)^2 + (\sigma_\theta - \sigma_r)^2 + (\sigma_z - \sigma_r)^2\right]^{0.5}$$
$$= \frac{\sqrt{2}}{2}\left[(\sigma_r - \sigma_\theta)^2 + 0.5(\sigma_r - \sigma_\theta)^2\right]^{0.5} = \frac{\sqrt{3}}{2}|\sigma_r - \sigma_\theta|.$$

From the condition of proportionality of deviators (14), we have $\sigma_r - \sigma_0 = \frac{2}{3}\varepsilon_r\sigma/\varepsilon$, and $\sigma_\theta - \sigma_0 = \frac{2}{3}\varepsilon_\theta\sigma/\varepsilon$. From Eqs. (13) and (15) we find

$$\begin{aligned} \varepsilon_r/\varepsilon &= -\frac{\sqrt{3}}{2}C(t)/|C(t)| = -\frac{\sqrt{3}}{2}\operatorname{sgn} C(t), \\ \varepsilon_\theta/\varepsilon &= -\frac{\sqrt{3}}{2}C(t)/|C(t)| = -\frac{\sqrt{3}}{2}\operatorname{sgn} C(t), \\ \sigma_r - \sigma_0 &= -\frac{\sqrt{3}}{3}\operatorname{sgn} C(t)\sigma, \\ \sigma_\theta - \sigma_0 &= -\frac{\sqrt{3}}{3}\operatorname{sgn} C(t)\sigma, \end{aligned} \tag{17}$$

where the stress intensity is

$$\sigma = \mathbf{R}\varphi(\varepsilon) = \mathbf{R}\varphi\left(\frac{2}{\sqrt{3}}|C(t)|r^{-2}\right). \tag{18}$$

due to Eqs. (14) and (15).

The equilibrium of equilibrium in the projection onto a radius has form (11). Subtracting the formulas in Eq. (17) from each other, we find $\sigma_r - \sigma_\theta = -\frac{2}{\sqrt{3}}\operatorname{sgn} C(t)\sigma$; substituting this expression into Eq. (11), we obtain $\sigma_{r,r} = \frac{2}{\sqrt{3}}\operatorname{sgn} C(t)\sigma r^{-1}$, i.e.,

$$\sigma_{r,r} = \frac{2}{\sqrt{3}}\operatorname{sgn} C(t)\mathbf{R}\left[r^{-1}\varphi\left(\frac{2}{\sqrt{3}}|C(t)|r^{-2}\right)\right]. \tag{19}$$

We integrate Eq. (19) over the range from r_1 to r, use the permutability of the integration operators in r and τ, change the variable, and have $x = \frac{2}{\sqrt{3}}|C(t)|\rho^{-2}$:

$$\sigma_r(r) - \sigma_1(r_1)$$
$$= \frac{2}{\sqrt{3}}\operatorname{sgn} C(t)\mathbf{R}\left[\int_{r_1}^{r}\rho^{-1}\varphi\left(\frac{2}{\sqrt{3}}|C|\rho^{-2}\right)d\rho\right]$$
$$= -\frac{1}{\sqrt{3}}\operatorname{sgn} C(t)\mathbf{R}\left[\int_{\frac{2}{\sqrt{3}}|C|r_1^{-2}}^{\frac{2}{\sqrt{3}}|C|r^{-2}}\varphi(x)x^{-1}dx\right].$$

Introducing the designations $\bar{r} := r/r_1$, $q := (r_1/r_2)^2 \in (0;1)$, $y(t) := \frac{2}{\sqrt{3}}C(t)r_1^{-2}$ (due to Eq. (15), $|y(t)| = \varepsilon(r_1)$ is the strain intensity at $r = r_1$ and $|y(t)| = \varepsilon(r_2)/q)$, and

$$F(s) := \int_0^s \varphi(x)x^{-1}dx, \quad s > 0, \tag{20}$$

we have

$$C(t) = \frac{\sqrt{3}}{2}y(t)r_1^2,$$

and

$$\sigma_r(\bar{r}) + p_1 = -\frac{1}{\sqrt{3}}\operatorname{sgn} y(t)\mathbf{R}\left[\int_{|y(t)|}^{|y(t)/\bar{r}^2|}\varphi(x)x^{-1}dx\right],$$

i.e.,

$$\begin{gathered} \sigma_r(\bar{r}) = -p_1 + \frac{1}{\sqrt{3}}\operatorname{sgn} y(t) \\ \times \mathbf{R}\left[F(|y(t)|) - F(|y(t)|/\bar{r}^2)\right], \\ \bar{r} \in [1, r_2/r_1], \quad t \geq 0. \end{gathered} \tag{21}$$

Assuming $r = r_2$ in Eq. (21), from second boundary condition (7) we obtain the integral equation

$$p_1 - p_2 = \frac{1}{\sqrt{3}}\operatorname{sgn} y(t)\mathbf{R}\left[F(|y(t)|) - F(q|y(t)|)\right],$$
$$t > 0,$$

to determine $y(t)$.

The increase in $\varphi(x)$ and the condition $\varphi(0) = 0$ (then $\varphi(x) > 0$) lead to an increase in $F(s)$ (Eq. (20)) in the rage $s > 0$. Therefore, the inequality $F(|y(t)|) - F(q|y(t)|) > 0$ is always valid (since $q \in (0; 1)$). Since relaxation function $R(t)$ is positive, the function $f(t) = \mathbf{R}[F(|y(t)|) - F(q|y(t)|)]$ is positive at any time and, hence, the sign sgn$y(t)$ coincides with $z(t) := \operatorname{sgn}(p_1(t) - p_2(t))$. As a result, we have

$$\begin{gathered} p_1 - p_2 = \frac{1}{\sqrt{3}}z(t)\mathbf{R}\left[F(|y(t)|) - F(|y(t)|/\bar{r}^2)\right], \\ q := (r_1/r_2)^2 \in (0;1), \quad t > 0. \end{gathered} \tag{22}$$

Applying linear operator $\mathbf{\Pi}$, which is inverse to $\mathbf{R}$, to Eq. (22), we obtain the following functional equation for $Y := |y(t)|$:

$$\begin{gathered} F(|y(t)|) - F(q|y(t)|) = P(t), \\ P(t) := \sqrt{3}\Pi\left[z(p_1 - p_2)\right] = \sqrt{3}\Pi|p_1 - p_2|, \end{gathered} \tag{23}$$

where $P(t)$ is a known function if a creep function is specified.

After determining $y(t)$ and $C = \frac{\sqrt{3}}{2}y(t)r_1^2$ (approximately in the general case) from Eq. (23), we find the displacement, strain, and stress fields using Eqs. (12), (13), and (21),

$$u_r(r,t) = C(t)r^{-1} = \frac{\sqrt{3}}{2}y(t)\,r_1/\bar{r}, \quad u_\theta \equiv 0,\ u_z \equiv 0, \tag{24}$$

$$\varepsilon_r(r,t) = -C(t)r^{-2} = -\frac{\sqrt{3}}{2}y(t)\bar{r}^{-2}, \tag{25}$$

$$\varepsilon_\theta(r,t) = -\varepsilon(r,t) = -\frac{\sqrt{3}}{2}y(t)\bar{r}^{-2}, \tag{26}$$

$$\sigma_r(r,t) = -p_1 + \frac{1}{\sqrt{3}}z(t) \times \mathbf{R}\left[F(|y(t)|) - F(|y(t)|/\bar{r}^2)\right], \tag{27}$$

where $\bar{r}: = r/r_1 \in [1, r_2/r_1]$, and $z = \operatorname{sgn}(p_1 - p_2)$.

Stresses σ_θ, $\sigma_z = (\sigma_r + \sigma_\theta)/2$, and $\sigma_0 = \sigma_z$ can be expressed from Eqs. (17) and (18),

$$\sigma_\theta = \sigma_r + \frac{2}{\sqrt{3}}\operatorname{sgn} C(t)\sigma = \sigma_r + \frac{2}{\sqrt{3}}z(t)\mathbf{R}\varphi\left(|y(t)|/\bar{r}^2\right);$$

i.e., we have

$$\sigma_\theta(r,t) = -p_1 + \frac{1}{\sqrt{3}}z(t) \times \mathbf{R}\left[F(|y(t)|) - F(|y(t)|/\bar{r}^2) + 2\varphi\left(|y(t)|/\bar{r}^2\right)\right], \tag{28}$$

$$\sigma_z(r,t) = -p_1 + \frac{1}{\sqrt{3}}z(t) \times \mathbf{R}\left[F(|y(t)|) - F(|y(t)|/\bar{r}^2) + \varphi\left(|y(t)|/\bar{r}^2\right)\right]. \tag{29}$$

4. TIME MONOTONICITY OF STRAINS AT ANY POINT IN A TUBE

Let $p_1(t) - p_2(t) = \text{const}$ at $t > 0$; that is, the problem of tube creep is considered. Then, due to Eq. (23), we have $P(t) = \sqrt{3}|p|\Pi(t)$ and $p := p_1 - p_2$ and Eq. (23) takes the form

$$F(Y(t)) - F(qY(t)) = \sqrt{3}|p|\Pi(t), \quad q: = (r_1/r_2)^2. \tag{30}$$

We differentiate it with respect to time and prove that $Y := |y(t)|$ increases,

$$F'(Y(t))\dot{Y} - qF'(qY(t))\dot{Y} = \sqrt{3}|p|\dot{\Pi}(t),$$

$$\dot{Y}[\varphi(Y)Y^{-1} - \varphi(qY(t))Y^{-1}] = \sqrt{3}|p|\dot{\Pi}(t),$$

$$\dot{Y} = \sqrt{3}|p|\dot{\Pi}(t)Y/[\varphi(Y) - \varphi(qY)].$$

Since $\varphi(x)$ increases, $\varphi(Y(t)) - \varphi(qY(t)) > 0$, $\dot{Y} > 0$ from the condition $\dot{\Pi}(t) > 0$, and $Y(t)$ increases. Therefore, with allowance for Eqs. (24)–(26) and (15), the absolute values of displacement $u_r(r, t)$ and strains $\varepsilon_r(r, t)$, $\varepsilon_\theta(r, t)$, and $\varepsilon(r, t)$ increase with t. Since $y(t) = Y(t)\operatorname{sgn}(p_1 - p_2)$, function $y(t)$ is positive and increases at $p_1 - p_2 > 0$ and is negative and decreases at $p_1 - p_2 < 0$.

5. LONG-TERM STRENGTH CURVE FOR THE MEASURE OF DAMAGE EQUAL TO THE MAXIMUM TENSILE STRAIN

We derive an expression for the time to failure of a tube during creep at $p := p_1 - p_2 > 0$. Then, we have $y(t) = Y(t) > 0$ and $\varepsilon_\theta(r, t) > 0$, and circumferential tension strain $\varepsilon_\theta(r, t)$ increases with t (strain intensity also increases). A fracture criterion is taken to be

$$\max\{\varepsilon_\theta(r,t) | r_1 \le r \le r_2\} = \varepsilon_*;$$

that is, the fracture condition is reaching limiting value ε_* by the maximum tensile strain at a certain point. From Eq. (26), we have

$$\varepsilon_\theta(r,t) = c^{-1}y(t)\bar{r}^{-2}, \quad c = 2/\sqrt{3}, \quad \bar{r}: = r/r_1 \in [1, r_2/r_1].$$

At any time, $\varepsilon_\theta(r, t)$ reaches its maximum at $r = r_1$; therefore, the chosen fracture condition leads to the equation $y(t_*) = c\varepsilon_*$ for fracture time t_*, where $y(t)$ is the solution to functional equation (30). Substituting t_* and $y(t_*) = c\varepsilon_*$ into it, we obtain the following equation for t_*:

$$\Pi(t_*) = C(q,\varepsilon_*)/p, \quad C(q,\varepsilon_*): = [F(c\varepsilon_*) - F(qc\varepsilon_*)]/\sqrt{3}, \tag{31}$$

where $C(q, \varepsilon_*)$ is a known constant, which depends on parameters q and ε_* and MF φ and does not depend on pressure and creep function $\Pi(t)$.

Since $\Pi(t)$ increases monotonically, Eq. (31) has at most one solution. Since the condition $\Pi(0) < \Pi(t) < \Pi(\infty)$, where $\Pi(0) \ge 0$ and $\Pi(\infty) \le \infty$, should be met at $t > 0$, the following three cases are possible:

(1) if $\Pi(0) < C(q, \varepsilon_*)/p < \Pi(\infty)$, Eq. (31) has one solution and the time to failure is determined by the formula

$$t_* = \Psi(C(q,\varepsilon_*)/p), \quad C(q,\varepsilon_*)E_\infty < p < C(q,\varepsilon_*)E, \tag{32}$$

where $\Psi := \Pi^{-1}$ is the inverse function of $\Pi(t)$ (determined in the range $[\Pi(0), \Pi(\infty)]$) and $E := 1/\Pi(0)$ and $E_\infty = 1/\Pi(\infty)$ are the instantaneous and long-term moduli of the deformation diagrams of linear viscoelasticity CR (4), respectively [36];

(2) if $\Pi(\infty) < \infty$ ($\Pi(t)$ is limited) and $C(q, \varepsilon_*)/p \ge \Pi(\infty)$, i.e., $E_\infty > 0$ and $p \le C(q, \varepsilon_*)E_\infty$, Eq. (31) has no solutions and fracture does not occur in an arbitrary long time;

(3) if $C(q, \varepsilon_*)/p \le \Pi(0)$ ($\Pi(0) > 0$ should be met in this case), i.e., $p \ge p_*$ (where $p_* = C(q, \varepsilon_*)E$), fracture

occurs at time $t = 0$ when a load is applied ($t_* = 0$) and p_* is the limiting pressure of the tube.

Equation (32) demonstrates that the shape and the main qualitative properties of the long-term strength curve $t_*(p)$ and the character of its dependence on the pressure difference p are mainly determined by the creep function and weakly depend on the MF φ, which sets nonlinearity in CR (1), and on the ratio of the radii. This behavior is due to the fact that φ and q affect only positive coefficient $C(q, \varepsilon_*)$, which causes tension of curve (32) along axis p; therefore, they do not affect the presence of extremum or inflection points, the character of monotonicity or convexity, and the horizontal asymptotes. This result is all the more interesting because MF φ significantly affects (as proved in [12, 14]) the qualitative variety of the creep curves generated by CR (1). In particular, it makes it possible to simulate the third stage of creep curves (linear viscoelasticity CR does not allow this, since it generates only up convex creep curves).

Due to the monotonic increase in $\Pi(t)$, function $\Psi := \Pi^{-1}$ also increases; therefore, dependence $t_*(p)$ (32) decreases and the convexity of Ψ and $t_*(p)$ down for any MF CR (1) obeying the minimum restrictions (see Section 2) follows from the convexity up and the increase in $\Pi(t)$. The inverse function

$$p = p(t_*) = C(q, \varepsilon_*)/\Pi(t_*), \quad t_* > 0,$$

also increases and is convex down. For $t_* \to \infty$, the curve $p(t_*)$ has a horizontal asymptote $p = C(q, \varepsilon_*)E_\infty$ and the curve $t_*(p)$ has a vertical asymptote $p = C(q, \varepsilon_*)E_\infty$, to which it tends from the right. If $\Pi(\infty) < \infty$, $E_\infty > 0$; if $\Pi(\infty) = \infty$, $E_\infty = 0$ and the asymptote is $p = 0$. If $\Pi(0) = 0$, $E = \infty$, $\Psi(0) = 0$, function $t_*(p)$ (32) has the horizontal asymptote $t_* = 0$ at $p \to \infty$, and function $p(t_*)$ has the vertical asymptote $t_* = 0$.

For example, for models with an arbitrary φ MF and the creep function

$$\Pi = Bt^w + b, \quad w \in (0;1], \quad B > 0, \quad b > 0 \tag{33}$$

(as in the fractal Maxwell model), we have

$$E = b^{-1}, \quad E_\infty = 0, \quad \Psi = ((x - b)/B)^{1/w}$$

and

$$t_*(p) = B^{-1/w}(C(q, \varepsilon_*)p^{-1} - b)^{1/w},$$
$$p \in (0; p_*), \quad p_* = C(q, \varepsilon_*)/b.$$

In particular, for models with a power function of creep $\Pi = Bt^w$ (where $w \in (0{:}\ 1]$, we have $E = \infty$, $E_\infty = 0$, and $p_* = \infty$ and the long-term strength curve is described by the relation $t_* = (C/B)^{1/w} p^{-1/w}$, at $p > 0$ or $p = C(q, \varepsilon_*)B^{-1}t_*^{-w}$. In this case, the long-term strength curve in the $\log t_*$–$\log p$ coordinates is a straight line with a slope $w < 0$ (this shape of the curve is observed in testing many viscoelastoplastic materials).

In the case $p_1 - p_2 < 0$, we have $\varepsilon_\theta(r, t) < 0$ but the time to failure is expressed by the same equation (Eq. (32)), since $\varepsilon_r = -\varepsilon_\theta$ due to the incompressibility condition.

6. TIME TO FAILURE FOR THE MEASURE OF DAMAGE EQUAL TO THE STRAIN INTENSITY

If the measure of damage is taken to be the maximum shear strain $\gamma_{\max} = (\varepsilon_1 - \varepsilon_3)/2$ rather than the maximum tensile strain $\varepsilon_1 = \varepsilon_\theta$, we obtain the same time to failure as in Eq. (32) at the same limiting strain ε_*, since $\gamma_{\max} = (\varepsilon_1 - \varepsilon_3)/2 = (\varepsilon_\theta - \varepsilon_r)/2 = \varepsilon_\theta$ ($\varepsilon_z \equiv 0$ and $\varepsilon_r = -\varepsilon_\theta$ due to the incompressibility condition).

Using another fracture criterion of a deformation type, namely, reaching the limiting strain intensity $\max\{\varepsilon(r, t) | r_1 \le r \le r_2\} = \varepsilon_*$, we obtain a qualitatively similar result (another time to failure and another long-term strength curve), since strain intensity (15) $\varepsilon(r, t) = y(t)/\bar{r}^2$ differs from $\varepsilon_\theta(r, t)$ only in constant factor $c = 2/\sqrt{3}$ and also reaches the maximum value on the inner tube surface, $\varepsilon(r_1, t) = y(t)$. This criterion gives the equation $y(t_{**}) = \varepsilon_*$ ($c = 1$ instead of $c = 2/\sqrt{3} > 1$) for the time to failure. Since $y(t)$ increases, $t_{**} < t_*$ for any MF and p and, instead of Eq. (31), we obtain

$$\Pi(t_{**}) = C_2(q, \varepsilon_*)/p,$$
$$C_2(q, \varepsilon_*) := [F(\varepsilon_*) - F(q\varepsilon_*)]/\sqrt{3},$$

where $C_2(q, \varepsilon_*)$ is a constant, which is dependent on parameters q, ε_*, and MF φ and is independent of p and crep function $\Pi(t)$.

Similarly to Eq. (32), we find

$$t_{**} = \Psi(C_2(q, \varepsilon_*)/p),$$
$$E_\infty C_2(q, \varepsilon_*) < p < EC_2(q, \varepsilon_*). \tag{34}$$

Thus, the time to failure as a function of q and ε_* is different, and dependence (34) is derived from Eq. (32) by tension along axis p with the coefficient $k = C_2(q, \varepsilon_*)/C(q, \varepsilon_*) < 1$.

7. LONG-TERM STRENGTH CURVE FOR A MODEL WITH A POWER MATERIAL FUNCTION

We now consider a model with MF $\varphi(x) = Ax^\alpha \operatorname{sgn} x$, $\alpha > 0$, and an arbitrary creep function. Then, according to Eq. (20), we have $F(s) = A\alpha^{-1}s^\alpha \operatorname{sgn} s$ and, hence,

$$C_2(q, \varepsilon_*) = A\alpha^{-1}\left[\varepsilon_*^\alpha - (q\varepsilon_*)^\alpha\right]/\sqrt{3},$$
$$C(q, \varepsilon_*) = c^\alpha C_2(q, \varepsilon_*), \tag{35}$$

that is, the long-term strength curve of type (34) is obtained from Eq. (32) by compression along the pressure axis with coefficient $k = C_2/C = c^{-\alpha}$ independent of q and ε_* (which is true of any homogeneous func-

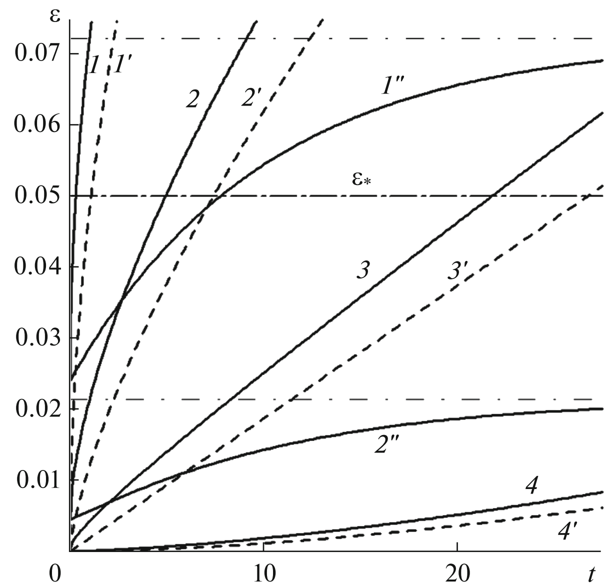

Fig. 1. Creep curves of a tube with $r_1/r_2 = 0.8$ at $p = 1$ (strain intensity $\varepsilon(r_1, t)$ curves) generated by various models (1) with MF $\varphi(x) = Ax^{\alpha}$, where $\alpha = 1$, 0.7, 0.5, and 0.3 and $A = 1$, and a creep function of type (33) or $\Pi = \beta - \gamma e^{-\lambda t}$.

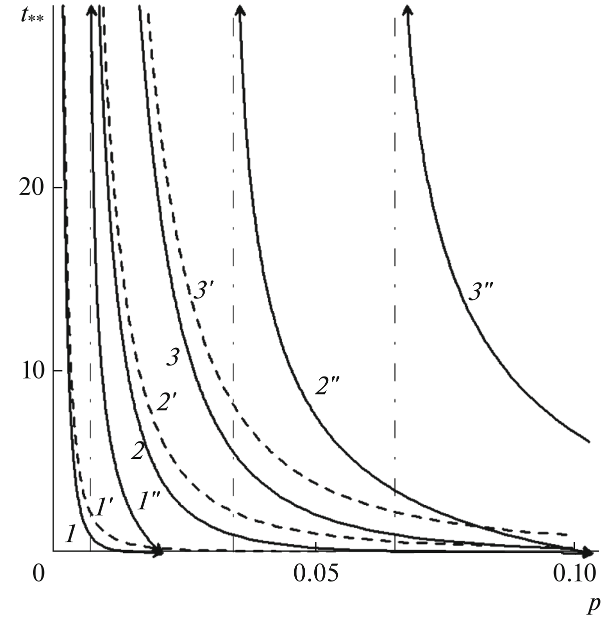

Fig. 2. Long-term strength of a tube with $r_1/r_2 = 0.8$ generated by nine models (1) with MF $\varphi(x) = Ax^{\alpha}$, where $\alpha = 1$, 0.5, and 0.3 and $A = 1$, and a creep function of type (33) or $\Pi = \beta - \gamma e^{-\lambda t}$.

tion φ). For power function φ and an arbitrary creep function, the solution $y(t)$ of Eq. (30) is analytically found,

$$y(t) = z(t)\left[\sqrt{3}A^{-1}\alpha(1 - q^{\alpha})^{-1}|p|\Pi(t)\right]^{1/\alpha}, \quad t \geq 0. \quad (36)$$

Its substitution into Eqs. (24)–(26) gives expressions for the displacement and strain fields during creep under a constant pressure. Due to Eq. (36), the creep curves $y(t) = \varepsilon(r_1, t)$, $\varepsilon_\theta(r, t)$, and $\varepsilon_r(r, t)$ are limited in the $t \geq 0$ semiaxis only when $\Pi(t)$ is limited.

Figure 1 shows the creep curves of a tube with $r_1/r_2 = 0.8$ at $p = 1$, which are generated by CR (1) with MF $\varphi(x) = Ax^{\alpha}$, where $\alpha = 1$, 0.7, 0.5, and 0.3 and $A = 1$, and creep function of type (33) or $\Pi = \beta - \gamma e^{-\lambda t}$ (as in Kelvin's model). These are the strain intensity curves on the inner tube surface $\varepsilon(r_1, t) = y(t)$, which were calculated using Eq. (36) for eight models of type (33) with parameters $w = 0.5$, $B = 1$, and $b = 0.5$ (curves *1–4* at $\alpha = 1$, 0.7, 0.5, 0.3) or $b = 0$ (curves *1'–4'*) and two models with $\Pi = \beta - \gamma e^{-\lambda t}$, $\lambda = 0.1$, $\beta = 1.5$, and $\gamma = 1$ at $\alpha = 1$ and 0.7 (curves *1''* and *2''* and dot-and-dash lines are their horizontal asymptote at $t \to \infty$). The straight line at $\varepsilon = 0.05$ is the given level of strain at which fracture occurs, i.e., $\varepsilon = \varepsilon_*$. The strain curves $\varepsilon_\theta(r_1, t)$ differ from the $\varepsilon(r_1, t)$ curves only in factor $\sqrt{3/2}$.

Figure 2 shows long-term strength curves (34) of a tube with $r_1/r_2 = 0.8$ for the limiting strain intensity $\varepsilon_* = 0.05$ and models (1) at $\alpha = 1$, 0.5, and 0.3, which were used in Fig. 1 (numbering of curves is retained). Curves *1–3* and *1'–3'* have a common vertical asymptote $p = 0$, and curves *1''–3''* of the models with $\Pi = \beta - \gamma e^{-\lambda t}$ have asymptotes $p = p_{\min}(q, \varepsilon_*, \alpha)$ and $p_{\min} = E_\infty C_2 = \beta^{-1}A\alpha^{-1}[\varepsilon_*^{\alpha} - (q\varepsilon_*)^{\alpha}]/\sqrt{3} > 0$ (dashed vertical lines); that is, they simulate materials with a nonzero creep threshold $p_{\min}$.

Figure 3 shows coefficient C_2 from Eq. (35), which determines the long-term strength of a tube, as a func-

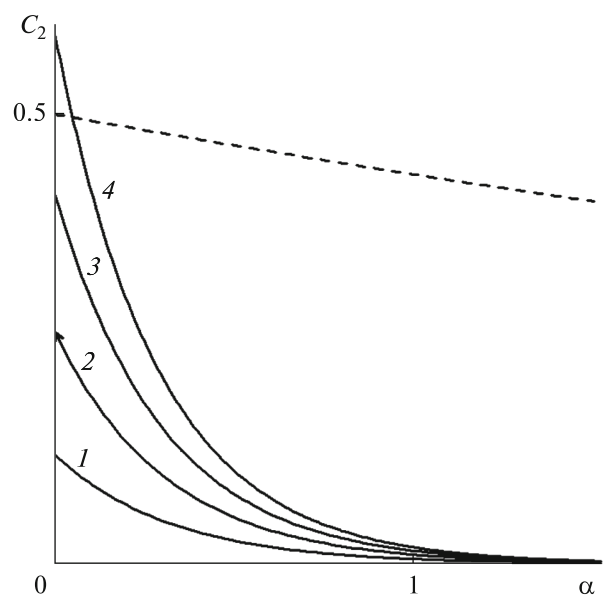

Fig. 3. Coefficients C_2 and $k = C_2/C$, which determine the long-term strength of the tube, vs. parameters α and r_1/r_2.

tion of parameter α at various tube thicknesses $r_1/r_2 =$ 0.9, 0.8, 0.7, and 0.6 (curves *1–4*). The dashed line illustrates the ratio $k(\alpha)/2$, $k = C_2/C = c^{-\alpha}$, i.e., coefficient of compression of a curve of type (34) along axis p, to obtain the curve of type (32) generated by the fracture criterion according to the maximum tensile strain $\varepsilon_1 = \varepsilon_\theta$.

8. CONCLUSIONS

An exact solution is obtained for the problem of creep and fracture of a hollow cylinder made of a physically nonlinear rheonomic material obeying Rabotnov's CR with two arbitrary material functions under constant internal and external pressures. The displacement, strain, stress fields at any time are expressed in terms of one function of time on the assumption of plane deformation and an incompressible material. This function is found by solving a constructed nonlinear functional equation containing material functions of CR and a given load. A general expression is obtained for the time to failure of a tube $t_*(p)$ in terms of the pressure difference, the material functions of CR, and the ratio of the tube radii. As the measure of damage, we chose the maximum tensile strain, the strain intensity, or the maximum shear strain without constructing a solution to a functional equation, which is important.

Equations of the corresponding long-term strength curves are derived, and their properties are analytically investigated for arbitrary material functions of Rabotnov's CR. The shape and the main qualitative properties of the curve $t_*(p)$ were proved to be mainly determined by a creep function and to weakly depend on the function that specifies nonlinearity in CR and on the ratio of the tube radii. For all three fracture criteria and any material functions, the dependence $t_*(p)$ was proved to decrease and to be convex down, the times to failure determined according to the criteria of the maximum tensile strain or the maximum shear strain coincide, and the time to failure found according to the strain intensity criterion is always shorter than them (with the same limiting strain).

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