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CODING THEORY

On Completely Regular Codes

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Abstract—This work is a survey on completely regular codes. Known properties, relations with other combinatorial structures, and construction methods are considered. The existence problem is also discussed, and known results for some particular cases are established. In addition, we present several new results on completely regular codes with covering radius $\rho = 2$ and on extended completely regular codes.

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1. INTRODUCTION

In 1959, Shapiro and Slotnick [1] proved that binary perfect codes are completely regular (without using this term). Later, in 1971, Semakov, Zinoviev, and Zaitsev [2] proved that Preparata codes, extended Preparata codes, and extended perfect codes are completely regular, also without using this term. In 1973, *completely regular codes* in Hamming metric were introduced by Delsarte [3]. Such codes have quite good packing density and possess combinatorial properties similar to perfect codes. The class of completely regular codes includes perfect and extended perfect codes [2], uniformly packed codes [2, 4, 5], codes obtained by their extensions [2], and completely transitive codes [6–9]. Known completely regular codes are, for example, Hamming, Golay, Preparata, binary BCH codes (of length $2^{2m+1} - 1$ with $d = 5$), and two Hadamard codes (of lengths 11 and 12). The combinatorial properties of completely regular codes allow to establish different interrelations with other combinatorial structures such as distance-regular graphs, association schemes, and designs. A comprehensive text about these interrelations is a monograph of Brouwer, Cohen, and Neumaier [10, ch. 11] complemented by a survey of van Dam, Koolen, and Tanaka [11]. A table of possible parameters of completely regular codes of finite lengths and their intersection arrays due to Koolen, Krotov, and Martin can be found in [12]. More recently, Koolen, Lee, Martin, and Tanaka studied and classified the class of arithmetic completely regular codes [13].

It is known that completely regular codes exist for an arbitrary large covering radius (see, e.g., the direct construction of Solé [6]). However, there are no known nontrivial completely regular codes with large error-correcting capability. More precisely, there are no known completely regular codes with minimum distance $d > 8$ having more than two codewords. In 1973 the nonexistence of unknown nontrivial perfect codes over finite fields was independently proved by Tietäväinen [14] and by Leontiev and Zinoviev [15]. The same result was obtained in 1975 for quasi-perfect uniformly packed codes by Goethals and van Tilborg [4,5] (infinite families of such codes were ruled out earlier in [2]). For the particular case of binary linear completely transitive codes, in 2001 Borges, Rifà, and Zinoviev also proved the nonexistence of such codes with $d > 8$ and having more than two codewords [16, 17]. In 1992 Neumaier [18] put forth a natural conjecture that the only completely regular code having more than two codewords with $d \geq 8$ is the extended binary Golay code. However, Borges, Rifà, and Zinoviev found a counterexample to Neumaier's conjecture [19]. More precisely, they proved than the even half of the binary Golay code (composed of its even-weight codewords) is also completely regular. Nevertheless, the existence of unknown nontrivial completely regular codes with $d \geq 9$ remains a major open question in this area. In this survey we consider completely regular codes only for finite lengths and finite cardinalities (for completely regular codes in infinite grids, see, e.g., [20] and references therein). We also consider complete regularity in Johnson schemes $J(n, w)$. We discuss the problem of the existence of perfect codes, stated in 1973 by Delsarte [3] and present known results on completely regular codes, perfect colorings, and completely regular designs in Johnson schemes $J(n, w)$.

The structure of the survey is as follows. In Section 2 we give the main definitions and present preliminary results on completely regular codes in Hamming schemes. We conclude this section with giving some necessary conditions for the existence of completely regular codes and extension of such codes. In Section 3 we study completely transitive codes, a particular case of completely regular codes. Section 4 is devoted to known results on completely regular codes in Johnson schemes. Finally, in Section 5, we collect different constructions of completely regular codes in Hamming schemes.

2. PRELIMINARY RESULTS ON COMPLETELY REGULAR CODES

2.1. Completely Regular and Related Codes

We will consider codes over finite fields $\mathbb{F}_q = GF(q)$ only, q being a prime power. The *Hamming distance* between two vectors $x, y \in \mathbb{F}_q^n$, denoted by $d(x, y)$, is the number of coordinates in which they differ. The whole space $\mathbb{F}_q^n$ with the partition $R = \{R_0, R_1, \ldots, R_n\}$ of the Cartesian square $(\mathbb{F}_q^n)^2$ given by the condition $(x, y) \in R_i$ if and only if $d(x, y) = i$ is called the *Hamming association scheme* (or simply the Hamming scheme) $H(q, n)$. A code is a subset of $\mathbb{F}_q^n$ equipped with the Hamming metric. For codes over rings, the Lee metric is often used. In many cases, such codes can be viewed as binary codes under the well-known Gray map; hence, we can consider them as binary codes with again the Hamming distance. As usual, for a code $C \subset \mathbb{F}_q^n$ we denote by n, d, e, and ρ the length, minimum distance, packing radius (or error-correcting capability), and covering radius of C. If C is a linear code, then k denotes its dimension (or the number of information symbols). We use the standard notation $(n, M, d)_q$ to denote a q-ary code of length n, size M, and minimum distance d. If the code is linear, then its dimension is specified instead of the size, and the notation is $[n, k, d]_q$. A code with packing radius e is often called an e-code. If we also want to specify the covering radius of the code, then we write $(n, M, d; \rho)_q$ for a nonlinear code, or $[n, k, d; \rho]_q$ for a linear code. For the binary case ($q = 2$), we usually omit the subscript q. Unless stated otherwise, we always assume that C is a distance-invariant code [21] containing the zero vector. For a q-ary linear $[n, k, d]_q$ code C, its dual code $C^\perp$ is defined in the following standard way:

$$C^\perp = \{x \in \mathbb{F}_q^n : \langle x \cdot y \rangle = 0 \text{ for every } y \in C\},$$

where $\langle x \cdot y \rangle$ denotes the inner product of vectors x and y in $\mathbb{F}_q^n$.

Recall several simple commonly known operations over codes which permit to obtain new codes from those already known. For a binary code C of length n with minimum distance $d = 2e + 1$, denote by C^* the code obtained from C by adding one more coordinate position, the *overall parity check position*. The code C^* (the *extension* of C) has length $n + 1$, minimum distance $d^* = d + 1$, and the same cardinality $M^* = M$. For a q-ary code C, the extension means the code obtained by adding one more position to every codeword so that for every codeword in the resulting code the sum in $\mathbb{F}_q$ of its coordinates is equal to zero.

For a code C of length n, the *i-punctured code* is obtained by deleting the coordinate i in all codewords. When the coordinate i is not specified, it is assumed that the parameters of the resulting code do not depend on the choice of the ith deleted position. The *a-shortened code* of C is obtained by taking all codewords that have the value a in a fixed coordinate followed by deleting this coordinate. Similarly, we say that a code is obtained from a code C by shortening if its parameters do not depend on the choice of a. In a more general case, for fixed vectors $x_1, \ldots, x_r$ of length $j < n$, the $\{x_1, \ldots, x_r\}$-shortened code of C is obtained by fixing some j coordinate positions, taking all codewords of C that have one of the vectors $x_1, \ldots, x_r$ in these j coordinates, and then deleting the j fixed coordinates. Thus, the resulting code has length $n - j$ and minimum distance

$$d' \geq d - \max_{\substack{i,i' \in \{1,\ldots,r\} \\ i \neq i'}} d(x_i, x_{i'}).$$

For a binary code C, $\mathrm{Aut}(C)$ denotes the automorphism group of C, i.e., the set of coordinate permutations that fixes C setwise. For a q-ary code C, $\mathrm{Aut}(C)$ denotes the set of coordinate permutations (followed by multiplications of the coordinates by nonzero elements of the field $\mathbb{F}_q$). We say that C is a *trivial* code if either $M = |C| \leq 2$ or $C = \mathbb{F}_q^n$.

Let $C \subset \mathbb{F}_q^n$ be a code. Given a vector $x \in \mathbb{F}_q^n$, we denote by $B_{x,i}$ the number of codewords at distance i $(0 \leq i \leq n)$ from x. The *outer distribution matrix* of C is a $q^n \times (n+1)$ matrix B with entries

$$B_{x,i} = |\{v \in C \mid d(x, v) = i\}|.$$

Hence, the row B_x is the weight distribution of the translate $C + x$. Denote by $b + 1$ the number of distinct rows of B.

Define the sets

$$C(i) = \{x \in \mathbb{F}_q^n \mid d(x, C) = i\},$$

where $d(x, C) = \min_{v \in C}\{d(x, v)\}$ denotes the distance of x to the code C, i.e., the distance between x and the closest codeword. The sets $C, C(1), \ldots, C(\rho)$ are referred to as *subconstituents* in [18] and are called *layers* by some other authors. Note that $C(0) = C$ and that $C(t) \neq \varnothing$ if and only if $t \leq \rho$.

We say that a binary code C of length n is *antipodal* if for every codeword $c \in C$ there exists a codeword $\bar{c}$ which is at distance n from c, i.e., $\bar{c} = c + (1, 1, \ldots, 1)$. Obviously, if C is a binary distance-invariant code and contains a codeword of weight n, then C is antipodal. But if $(1, \ldots, 1) \notin C$, then C is nonantipodal.

Definition 1 [4]. A code C with covering radius ρ is *t-regular* $(0 \leq t \leq \rho)$ if for all $i = 0, \ldots, \rho$ the value $B_{x,i}$ depends only on i and on the distance $d(x, C)$ of x to C, for all x such that $d(x, C) \leq t$.

In other words, C is t-regular if B_x depends only on $d(x, C)$ for $d(x, C) \leq t$. By the definition, if C is t-regular, then it is j-regular for all $j = 0, \ldots, t$. Intuitively, C is t-regular if we “see” the same amount of codewords at the same distances from any vector which is at distance no greater than t from the code. For example, a 0-regular code is exactly a distance-invariant code.

There are several equivalent definitions of completely regular codes (CR codes, for short).

Definition 2 [3]. A code C is *completely regular* if it is ρ-regular.

Clearly, the following definitions are equivalent:

(i) A code C is CR if for all $x \in \mathbb{F}_q^n$ the value $B_{x,i}$ depends only on i and $d(x, C)$;
(ii) A code C is CR if for all $x \in \mathbb{F}_q^n$ the weight distribution B_x depends only on $d(x, C)$;
(iii) A code C is CR if $b = \rho$.

However, it is not so straightforward to see the equivalence of the definitions above with the next definition. Two vectors x and y in $\mathbb{F}_q^n$ are said to be *neighbors* if they are at distance $d(x,y)=1$ from each other.

Definition 3 [18]. A code C is *completely regular* if for all $\ell \geq 0$ every vector $x \in C(\ell)$ has the same number c_ℓ of neighbors in $C(\ell-1)$ and the same number b_ℓ of neighbors in $C(\ell+1)$, where we set $c_0 = b_\rho = 0$.

It is clear that the sets $C, C(1), \ldots, C(\rho)$ induce a partition of the whole space $\mathbb{F}_q^n$, called the *distance partition*. If the conditions of Definition 3 are satisfied, then the partition is said to be *equitable*. Hence, C is CR if and only if its distance partition is equitable.

For the equivalence between Definitions 2 and 3, see [18, Theorem 4.1] or [22]. For $\ell \geq 0$, let $a_\ell = n(q-1)-b_\ell-c_\ell$. Thus, a_ℓ is the number of neighbors in $C(\ell)$ of any vector in $C(\ell)$. The parameters a_ℓ, b_ℓ, and c_ℓ are called the *intersection numbers* and the sequence $\mathrm{IA} = \{b_0, \ldots, b_{\rho-1}; c_1, \ldots, c_\rho\}$ is referred to as the *intersection array* of C.

Let us give examples of CR codes. Recall that a code C is *perfect* if $\rho = e$ and *quasi-perfect* if $\rho = e+1$. The following codes are trivial perfect codes:

1. Single codeword codes $C = \{x\}$, $x \in \mathbb{F}_q^n$;
2. The whole space $C = \mathbb{F}_q^n$;
3. Binary repetition codes of odd length which consist of two antipodal vectors of length n with $d = n$.

Nontrivial perfect codes exist only for $e \leq 3$ [14,15]. In $\mathbb{F}_q^n$, the following perfect codes are known:

1. The binary Golay code with $n = 23$, $k = 12$, $d = 7$;
2. The ternary Golay code with $n = 11$, $k = 6$, $d = 5$;
3. Perfect codes of size $M = q^{n-m}$ and length $n = (q^m-1)/(q-1)$ with minimum distance $d = 3$.

In the last case, for every prime power q and every $n = (q^m-1)/(q-1)$ there is a unique linear version, called the Hamming code.

An interesting class of codes closely related to CR codes is the class of *uniformly packed* (UP) codes. There are three mostly used different concepts of UP codes. Note that uniformly packed codes were first introduced in 1971 in [2].

Definition 4 [2]. A binary quasi-perfect code C with minimum distance $d = 2e+1$ is *uniformly packed in the narrow sense* if there exist a natural number μ such that $B_{x,e} + B_{x,e+1} = \mu$ for any vector $x \in C(e) \cup C(e+1)$.

We emphasize that all binary perfect codes fall into this category with the parameter $\mu = (n+1)/(e+1)$, which was called *packing density* in [2]; for all other codes, μ is less than $(n+1)/(e+1)$. Uniformly packed codes in the narrow sense include 1-shortened binary perfect codes, Preparata codes, binary primitive BCH codes with designed distance 5 of length $2^{2m+1}-1$, the Hadamard code of length 11. In [2] it was proved, in terms of Definition 2 (iii), that UP codes in the narrow sense and codes obtained as their extensions (which are not uniformly packed in this sense but are UP in the wide sense; see Definition 6 below) are CR.

UP codes in the narrow sense are a subclass of UP codes introduced in 1975 by Goethals and van Tilborg [4].

Definition 5 [4]. A quasi-perfect e-error-correcting q-ary code C is called *uniformly packed* if there exist natural numbers λ and μ such that for any vector x the quantity $B_{x,e+1}$ assumes two values:

$$B_{x,e+1} = \begin{cases} \lambda & \text{if } d(x,C) = e, \\ \mu & \text{if } d(x,C) = e+1. \end{cases}$$

For the case $\mu = \lambda + 1$ any such binary code is UP in the narrow sense. Note that binary extended perfect codes fall into this category for $\lambda = 0$ and $\mu = (n+1)/(e+1)$ [2]. These codes include also the ternary Golay code, its extension, and the code obtained from it by shortening. Van Tilborg [5] (see also [2, 23]) showed that no other nontrivial codes of this kind exist for $e > 3$.

The following result is a generalization of the result in [2] to the case of UP codes.

Proposition 1 [4]. *A uniformly packed code is completely regular.*

Note that the extension of any UP code which is not perfect, in general, need not be UP. This was one of motivations for introducing the following class of UP codes.

Definition 6 [24]. A code C with covering radius ρ is *uniformly packed in the wide sense* if there exist rational numbers $\beta_0, \ldots, \beta_\rho$ such that for any $x \in \mathbb{F}_q^n$

$$\sum_{i=0}^{\rho} \beta_i B_{x,i} = 1.$$

The numbers $\beta_0, \ldots, \beta_\rho$ are called *packing parameters* (for example, for perfect codes all these numbers are equal: $\beta_i = 1$ for all $i = 0, 1, \ldots, \rho = e$). The above notion is much more general than those in Definitions 4 and 5. In other words, a code C is UP in the wide sense if the all-one column is a linear combination of the first ρ columns of the outer distribution matrix B. Below we will see that any CR code is UP in the wide sense.

Now we give some simple examples of CR codes. But, before, we again refer to the paper by Goethals and van Tilborg [4], where the so-called *uniformly packed codes of order j* were introduced, which form an intermediate class of codes between UP codes and UP codes in the wide sense. Here are examples of CR codes:

1. As we have already mentioned, any perfect code is a CR code [1];
2. The set of all even-weight vectors is a linear CR code with $\rho = 1$ [2];
3. An extended binary perfect code is a quasi-perfect UP code, and thus a CR code [2];
4. For any CR code C, the subconstituent $C(\rho)$ is also CR, with a reversed intersection array [18].

2.2. Designs and Distance-Regular Graphs

Two combinatorial structures related to CR codes are of special interest: designs and distance-regular graphs.

Definition 7. A $T(v,k,t,\lambda)$-*design* T (also called a $t-(v,k,\lambda)$-design or simply a t-design) is an incidence structure $(X, \mathcal{B})$ where X is a v-set of elements (called *points*) and $\mathcal{B}$ a collection of k-subsets of points (called *blocks*) such that every t-subset of points is contained in exactly $\lambda > 0$ blocks, $0 < t \le k \le v$.

In terms of a (binary) incident matrix, a $T(v,k,t,\lambda)$-design is a binary code C of length $n = v$ with codewords of weight $w = k$ such that any binary vector of length n and weight t is covered by exactly λ codewords. A $T(v,k,t,\lambda)$-design is *simple* if it has no repeated blocks, and *nontrivial* if it is simple and does not contain all the k-subsets of points. A design with $\lambda = 1$ is called a *Steiner system* and is denoted also by $S(v,k,t)$. The following properties are well known and can be found, e.g., in monographs [25–27].

Proposition 2. *For a given $T(v,k,t,\lambda)$-design, every i-subset of points, $0 \le i \le t$, is contained in exactly λ_i blocks, where*

$$\lambda_i = \lambda \frac{\binom{v-i}{t-i}}{\binom{k-i}{t-i}}.$$

Corollary 1. *Let T be a $T(v,k,t,\lambda)$-design. Then*

(i) *T is a $T(v,k,i,\lambda_i)$-design for any $i \leq t$;*
(ii) $\lambda = \lambda_t$;
(iii) *The number of blocks of T is $b = \lambda_0$;*
(iv) *Each point of X is contained in the same number of blocks, namely, $r = \lambda_1 = bk/v$ (the number r is called the* replication number*).*

Designs with $t \leq 5$ have been known since 1938 (starting from the famous Witt designs [28], i.e., Steiner systems $S(24,8,5)$ and $S(12,6,5)$). The first nontrivial designs for $t = 6$ were found in 1983 by Magliveras and Leavitt [29]. In 1987, Teirlinck [30] proved that there are nontrivial simple t-designs for any natural number t.

There is a natural generalization of such designs to the q-ary case (see [1, 3, 4, 15, 31, 32]). Let $E = \{0, 1, \ldots, q-1\}$. A collection $\mathcal{B}$ of b vectors $x_1, \ldots, x_b$ of length v and weight k over E is called a $T(v,k,t,\lambda)_q$ design if for every vector y over E of length v and weight t there are exactly λ vectors $x_{i_1}, \ldots, x_{i_\lambda}$ from $\mathcal{B}$ such that $d(y, x_{i_j}) = k - t$ for all $j = 1, \ldots, \lambda$. If $\lambda = 1$, then we obtain a q-ary Steiner system, denoted by $S(v,k,t)_q$.

For a given code C, denote by C_w the set of all codewords of C of weight w. For a vector $x = (x_1, \ldots, x_n)$ in E^n denote by $\mathrm{supp}(x)$ its *support*, i.e., the set of indices of its nonzero positions:

$$\mathrm{supp}(x) = \{i : \; x_i \neq 0\}.$$

Regularity of a code C implies that the sets C_w induce t-designs, as we see in the next statement. For UP codes in the narrow sense this fact was stated in 1971 in [2].

Theorem 1 [4]. *Let C be an e-regular code with minimum distance $d \geq 2e$. Then the supports of its codewords in any nonempty set C_w form an incidence matrix of an e-design (form an e-design).*

Directly from the definition of CR codes we have the following result.

Theorem 2. *Let C be a q-ary CR code of length n with distance d.*

(i) *If $d = 2e+1$, then any nonempty set C_w is a $T(n,w,e,\lambda_w)_q$ design;*
(ii) *If $d = 2e+2$, then any nonempty set C_w is a $T(n,w,e+1,\lambda_w)_q$ design;*
(iii) *If C is a q-ary perfect code, then any nonempty set C_w is a $T(n,w,e+1,\lambda_w)_q$ design and C_d is a Steiner system $S(n, 2e+1, e+1)_q$;*
(iv) *If C is an extended binary perfect code with distance $d = 2e+2$, then any nonempty set C_w is a $T(n,w,e+2,\lambda_w)_q$ design and C_d is a Steiner system $S(n, 2e+2, e+2)_q$.*

Let Γ be a finite connected simple graph (i.e., an undirected graph without loops and multiple edges). Let $d(\gamma,\delta)$ be the distance between two vertices γ and δ (i.e., the number of edges in the minimal path between γ and δ). The *diameter* D of Γ is the largest distance between two of its vertices. Two vertices γ and δ from Γ are said to be *neighbors* if $d(\gamma,\delta) = 1$. Denote

$$\Gamma_i(\gamma) = \{\delta \in \Gamma : \; d(\gamma,\delta) = i\}.$$

An *automorphism* of a graph Γ is a permutation π of the vertex set of Γ such that for all $\gamma, \delta \in \Gamma$ we have $d(\gamma,\delta) = 1$ if and only if $d(\pi(\gamma),\pi(\delta)) = 1$.

Definition 8 [10]. A simple connected graph Γ is said to be *distance-regular* if it is regular of valency k and for any two vertices $\gamma, \delta \in \Gamma$ at distance i from each other there are precisely c_i neighbors of δ in $\Gamma_{i-1}(\gamma)$ and b_i neighbors of δ in $\Gamma_{i+1}(\gamma)$. Furthermore, this graph is called *distance-transitive* if for any pair of vertices γ, δ at distance $d(\gamma,\delta)$ there is an automorphism π from $\mathrm{Aut}(\Gamma)$ which takes this pair (γ,δ) to any other given pair γ', δ' of vertices at the same distance $d(\gamma,\delta) = d(\gamma',\delta')$.

The sequence $\{b_0, b_1, \ldots, b_{D-1}; c_1, c_2, \ldots, c_D\}$, where D is the diameter of Γ, is called (as well as for CR codes) the *intersection array* of Γ. Let $a_i = k - b_i - c_i$. The numbers a_i, b_i, and c_i are called *intersection numbers* of this graph. Clearly, $b_0 = k$, $b_D = c_0 = 0$, and $c_1 = 1$.

Let C be a linear CR code with covering radius ρ and intersection array IA $= \{b_0, \ldots, b_{\rho-1}; c_1, \ldots c_\rho\}$. Let $\{A\}$ be the set of cosets of C. Define the graph Γ_C, which is called the *coset graph of* C, having all different cosets $A = C + x$ as vertices, with two vertices $\gamma = \gamma(A)$ and $\gamma' = \gamma(A')$ adjacent (i.e., neighboring) if and only if the cosets A and A' contain vectors $v \in A$ and $u \in A'$ such that $d(v, u) = 1$.

2.3. Parameters and Properties of CR Codes

For a code C, we denote by $s + 1$ the number of nonzero terms in the the MacWilliams transform [21] of the distance distribution of C. The parameter s was called *external distance* by Delsarte [3], and it is equal to the number of nonzero weights of $C^\perp$ if C is linear. This is a key parameter for CR codes, as we will see in several their properties. Recall that B is the outer distribution matrix of the code C and $b + 1$ denotes the number of different rows of B.

Theorem 3. *The following statements hold*:

(i) $\operatorname{rank}(B) = s + 1$ [3];
(ii) $b \geq s$ [6];
(iii) $\rho \leq s$ [3].

Hence, we have the inequalities $e \leq \rho \leq s \leq b$. Now, we have the following characterizations of codes under consideration.

Theorem 4. *The following statements hold*:

(i) *A code C is perfect if and only if $e = s$* [3];
(ii) *A code C is UP if and only if $s = e + 1$* [2,4];
(iii) *If a code C is CR, then $\rho = s$* [6];
(iv) *A code C is UP in the wide sense if and only if $\rho = s$* [33].

The converse of (iii) is not true in general. Delsarte [3] gives an example of a $[48, 24, 12]$ extended quadratic residue code with $\rho = s = 8$ and $b = 14$. However, this condition is necessary and sufficient for UP codes in the wide sense. Therefore, we obtain

Corollary 2. *If C is CR, then C is UP in the wide sense.*

Many infinite families of UP codes in the wide sense which are not CR were constructed in [34,35]. The following properties of CR codes are due to Delsarte [3].

Theorem 5 [3]. *The following statements hold*:

(i) *If $t \geq d - s \geq 0$, then C is t-regular*;
(ii) *If C is t-regular with $t \geq s - 1$ and if $d \geq 2s - 1$, then C is CR.*

We can strengthen these conditions if all weights of C are even (such a code is said to be *even*).

Corollary 3. *Let C be an even code. Then*

(i) *If $t \geq d - s + 1 \geq 0$, then C is t-regular* [33];
(ii) *If $d \geq 2s - 2$, then C is CR* [10].

Theorem 6. *Let C be a nonantipodal CR code with covering radius ρ.*

(i) *The set $C(\rho)$ is a translate of C by the vector $(1, \ldots, 1)$* [19];
(ii) *The set $C \cup C(\rho)$ is a CR code* [36].

2.4. Necessary Conditions for CR Codes

For a given CR code C with covering radius ρ and intersection numbers a_i, b_i, c_i, a tridiagonal matrix A called the *intersection matrix* is defined as follows:

$$A = \begin{bmatrix} a_0 & b_0 & 0 & \dots & 0 & 0 \\ c_1 & a_1 & b_1 & \dots & 0 & 0 \\ 0 & c_2 & a_2 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & a_{\rho-1} & b_{\rho-1} \\ 0 & 0 & 0 & \dots & c_\rho & a_\rho \end{bmatrix}.$$

The following statement is called Lloyd's theorem for CR codes. Recall that the eigenvalues of the Hamming scheme $H(q,n)$ are the eigenvalues of the intersection matrix of $H(q,n)$, which are equal to $(q-1)n - qj$, $j = 0, 1, \dots, n$.

Theorem 7 [10, 18]. *Let C be a CR code of length n with covering radius ρ and with intersection matrix A. Then A has ρ integer eigenvalues, which are eigenvalues of $H_q(n)$.*

Since any CR code is UP in the wide sense, there exists another variant of this theorem, which in some cases might be more useful and which is a natural generalization of the classical Lloyd theorem for perfect codes (see [21]). Denote by K_i the cardinality of $C(i)$. Define $\varkappa_i$ by letting $K_i = \varkappa_i |C|$. It is easily seen that

$$\varkappa_i = \beta_i (q-1)^i \binom{n}{i},$$

where $\beta_0, \beta_1, \dots, \beta_\rho$ are the packing parameters of a UP code (see Definition 7).

Theorem 8 [24]. *Let C be a UP code in the wide sense of length n with covering radius ρ and with parameters $\beta_0, \beta_1, \dots \beta_\rho$. Then the generalized Lloyd polynomial of degree ρ in ξ*

$$L_\rho(n,\xi) = \sum_{r=0}^{\rho} \beta_r P_r(n,\xi), \tag{1}$$

where $P_r(n,\xi)$ is the Krawtchouk polynomial

$$P_r(n,\xi) = \sum_{j=o}^{r} (-1)^{r-j} (q-1)^j \binom{n-\xi}{j} \binom{\xi}{r-j}$$

and where for any real number a

$$\binom{a}{i} = \frac{1}{i!} a(a-1)\dots(a-i+1),$$

has ρ distinct integer roots in the range between 0 and n.

The next theorem generalizes the classical *sphere packing condition* for perfect codes to UP codes in the wide sense (hence, to any CR codes).

Theorem 9 [24]. *Let C be a UP code in the wide sense of length n with parameters $\beta_0, \beta_1, \dots \beta_\rho$. Then*

$$|C| = \frac{q^n}{\sum\limits_{i=0}^{\rho} \beta_i (q-1)^i \binom{n}{i}}. \tag{2}$$

Some more interesting properties of CR codes (which are also necessary existence conditions) can be found in [10, 18].

We consider two illustrative examples [24]. The Preparata codes P with parameters ($n = 2^{2m} - 1$, $M = 2^{n+1-4m}$, $d = 5$) for $m = 2, 3, \ldots$ have the following packing parameters β_i and roots ξ_i of the polynomial $P_\rho(n, \xi)$:

$$\beta_0 = \beta_1 = 1, \quad \beta_2 = \beta_3 = \frac{3}{n},$$
$$\xi_1 = \frac{1}{2}(n + 1 - \sqrt{n+1}), \quad \xi_2 = \frac{(n+1)}{2}, \quad \xi_3 = \frac{1}{2}(n + 1 + \sqrt{n+1}).$$

An interesting fact is that these codes are not only CR in $H(2, n)$ but are also CR in the Hamming code which contains this code P (see [2]).

The binary primitive BCH codes with parameters $[n = 2^{2m+1} - 1, k = n - 4m - 2, d = 5]$ for $m = 2, 3, \ldots$ have the following packing parameters β_i and the roots ξ_i of the Lloyd polynomial:

$$\beta_0 = \beta_1 = 1, \quad \beta_2 = \beta_3 = \frac{6}{(n-1)},$$
$$\xi_1 = \frac{n+1}{2} - \sqrt{\frac{n+1}{2}}, \quad \xi_2 = \frac{n+1}{2}, \quad \xi_3 = \frac{n+1}{2} + \sqrt{\frac{n+1}{2}}.$$

2.5. Extension of CR Codes

Recall that for a binary code C, the extended code C^* is obtained by adding a parity (or antiparity) check bit to each codeword. One interesting open question for CR codes concerns their extensions and is as follows: *Given a CR (n, N, d) code C with odd distance $d = 2e + 1$, what are the conditions for its extension, i.e., the code C^*, to be again a CR code*? Here we restrict our attention to binary codes only, though it seems that many results can be extended to the nonbinary case as well.

In [22] it is proved that puncturing an even CR code at any position gives also a CR code (answering a question posed in [24]). Therefore, if C^* is CR, so is C. However, the converse is not true in general. Bassalygo and Zinoviev [33] gave an example of a CR code C such that C^* is not CR: the $[21, 12, 5]$ code obtained from the binary Golay code by double puncturing is CR, but its extension is not. Moreover, in this case, the extended code, i.e., a $[22, 12, 6]$ code C^*, is not even UP in the wide sense. Therefore the extension of a UP code in the wide sense could be non-UP in the wide sense in general. For such uniformly packed codes this question was completely solved in [33]. A necessary and sufficient condition for a UP code in the wide sense to remain UP under extension is given by the following theorem.

Theorem 10 [33]. *A binary UP code C in the wide sense of length n with covering radius ρ and with packing parameters $\beta_0, \ldots, \beta_\rho$ remains to be UP in the wide sense under extension if and only if the following system of equations holds*:

$$\beta_{\rho-2i} = \beta_{\rho-2i-1}, \quad \textit{for all } i,\ 0 \le i \le [(\rho-1)/2].$$

Furthermore, the packing parameters $\gamma_0, \ldots, \gamma_\rho, \gamma_{\rho+1}$ of the extended code C^* are defined by the following formulas:

$$\gamma_{\rho-2i} = \beta_{\rho-2i}, \quad \forall\, i = 0, 1, \ldots, \lfloor \rho/2 \rfloor,$$
$$\gamma_{\rho-2i+1} = \frac{1}{n+1}\big((\rho + 1 - 2i)\beta_{\rho-2i} + (n - \rho + 2i)\beta_{\rho-2i+2}\big), \quad \forall\, i = 0, \ldots, \left\lfloor \frac{\rho+1}{2} \right\rfloor.$$

The condition in the above theorem is obviously necessary for a code to be CR. It is an open question whether or not the condition is sufficient for CR codes.

We also give a useful necessary condition for CR codes. As we know (Theorem 2), the set C_d of a binary UP code C of length n and distance $d = 2e+1$ induces a $T(n, d, e, \lambda)$-design.

Proposition 3 [33]. *Let C^* be the extension of a binary UP code C in the wide sense. Let C be of length n and minimum distance $d = 2e + 1$. If C^* is UP, then the set C^*_{d+1} induces a $T(n + 1, d + 1, e + 1, \lambda)$-design.*

In the case where C and C^* are UP and the covering radius of C is $\rho = e + 1$, both codes are also CR:

Proposition 4 [2, 37]. *Let C be a quasi-perfect UP code (hence, a CR code). If C^* is UP, then C^* is CR.*

Another necessary condition is as follows.

Proposition 5. *If C is a CR code of even length and C is antipodal, then C^* is not UP in the wide sense (and hence not CR).*

Proof. Assume that C^* is UP in the wide sense. Then the external distance is $s^* = s + 1$ implying that for each weight w in the dual distribution, the number $n + 1 - w$ is also a weight in the dual distribution. But w must be even (since C contains the all-one vector); hence, $n + 1 - w$ is odd, getting a contradiction.

The following property is a strengthening of a result in [38].

Proposition 6 [39]. *Let C be a binary linear CR code of length $n = 2^m - 1$ with minimum distance $d = 3$, covering radius $\rho = 3$, and intersection array $\{n, b_1, 1; 1, c_2, n\}$. Let the dual code $C^\perp$ have nonzero weights w_1, w_2, and w_3. Then the extended code C^* is CR with covering radius $\rho^* = 4$ if and only if $w_1 + w_3 = 2w_2 = n + 1$. In this case the intersection array of C^* is $\{n + 1, n, b_1, 1; 1, c_2, n, n + 1\}$.*

Hence, weights of the dual code (when C is linear) play an important role. Another important criterion is related to puncturing the extended code C^* at different coordinates: the resulting i-punctured code C should be almost the same independently of a chosen position i.

Proposition 7 [6]. *Let C be a binary CR code. Assume that the i-punctured code obtained from C^* gives the same code C independently of the choice of the position i. Let s^* be the external distance of C^*. If $s^* \leq s + 1$, then C^* is CR.*

Corollary 4 [6]. *If C is a binary linear CR code, the weights of $C^\perp$ are even and symmetric with respect to $(n + 1)/2$, and $\mathrm{Aut}(C^*)$ is transitive, then C^* is CR.*

Let C be a double-error-correcting BCH code with parameters $[2^m - 1,\, 2^m - 1 - 2m,\, 5]$, $m \geq 3$ odd. The weights of the dual code satisfy the conditions of Corollary 4, and C^* is left invariant under the affine group. Hence C^* is CR with $\rho^* = s^* = 4$ and packing radius $e^* = 2$. This result can be deduced from [24], where it was shown that these BCH codes are UP in the narrow sense with

$$\beta_0 = \beta_1 = 1, \qquad \beta_2 = \beta_3 = \frac{6}{(n-1)}.$$

According to Theorem 10, extended BCH codes are UP in the wide sense, and by Proposition 4 these codes are CR.

Now consider the codes studied by Calderbank and Goethals in [40, 41], namely, three-weight cyclic subcodes of the shortened second-order Reed–Muller code. Denote by C the dual of one of these codes. The code C is CR, and C^* is left invariant under the affine group. Again, the three nonzero weights of $C^{\perp}$ satisfy the condition of Corollary 4. Hence C^* is CR with $\rho^* = s^* = 4$ and $e^* = 1$.

Corollary 4 can be generalized to the nonlinear case:

Corollary 5 [6]. *Let C be a binary code whose dual distances are even and symmetric with respect to $(n+1)/2$. If C is CR and* $\mathrm{Aut}(C^*)$ *is transitive, then C^* is CR.*

From the presentation in [42] it is known that the automorphism group of an extended Preparata code is transitive. The punctured code, i.e., the Preparata code of length $n = 2^{2m} - 1$, $m = 2, 3, \dots$, is CR [2], and its dual distances are those of the Kerdock code [43], satisfying the condition of Corollary 5. Hence we conclude that the extended Preparata code is CR [2] with $\rho^* = s^* = 4$ and $e^* = 2$.

Let C^* be a Hadamard $(12, 24, 6)$ code. It is known that C is UP with dual distances $4, 6, 8$ [4,44]. The group $\mathrm{Aut}(C^*)$ is isomorphic to the Mathieu group M_{12}. Hence, C^* is CR with $\rho^* = s^* = 4$ and $e^* = 2$. From [37] we have the following result.

Proposition 8 [37]. *Let C be a binary CR code with parameters (n, d).*

(i) *If $(n, d) = (12, 6)$, then C is equivalent to the Hadamard code;*
(ii) *If $(n, d) = (11, 5)$, then C is equivalent to the punctured Hadamard code.*

The results of Solé [6] were slightly improved by Brouwer [45].

Proposition 9 [45]. *Let C be a binary CR code. If the outer distribution matrices of all codes obtained from C^* by deleting one coordinate position have the same set of rows, in particular, if C^* admits a group which is transitive on the set of coordinate positions, then C^* is CR.*

Corollary 6 [45]. *C^* is CR if and only if all codes obtained from it by deleting one coordinate position are CR with the same outer distribution.*

It is interesting, of course, to find necessary and sufficient conditions on C for C^* to be CR. From the last corollary we can derive some necessary conditions on the punctured codes of C^*, in particular, on C.

For any binary vector $v = (v_1, \dots, v_n)$ and for each $i = 1, \dots, n$, we define a vector

$$\tau_i(v) = (v_1, \dots, v_{i-1}, p(v), v_{i+1}, \dots, v_n),$$

where $p(v)$ denotes the parity of v, i.e.,

$$p(v) \equiv \sum_{i=1}^{n} v_i \pmod 2.$$

For every position $i = 1, \dots, n$, define the code $C_{[i]} = \{\tau_i(x) \mid x \in C\}$.

Lemma 1. *C^* is a CR code if and only if C and $C_{[i]}$ are CR for all $i \in \{1, \dots, n\}$.*

Proof. For any code D, denote by $D^{(i)}$ the punctured code obtained by deleting the ith coordinate in D. Also, denote by $\sigma_{i,j}$ the transposition of the coordinates i and j. Assume that the parity check coordinate is at position $n+1$. Hence, it is clear that

$$C_{[i]} = (\sigma_{i,n+1}(C^*))^{(i)} \qquad \text{and} \qquad C = (C^*)^{(n+1)}.$$

The result then follows by Corollary 6. △

Proposition 10. *If C^* is a CR code, then for all $i = 1, \dots, n$*

(i) *The weight distributions of C and $C_{[i]}$ coincide;*
(ii) *The minimum distances of C and $C_{[i]}$ coincide and are odd;*

(iii) *The external distances of* C *and* $C_{[i]}$ *coincide*;
(iv) *The covering radii of* C *and* $C_{[i]}$ *coincide.*

Proof. (i) follows since for each codeword x, B_x must be the same for all codes C and $C_{[i]}$. (ii) and (iii) directly follow from (i).

If the covering radius of a code C is ρ, then the covering radius of C^* is $\rho^* = \rho + 1$. Hence follows (iv). △

Now we have the following necessary condition on C (or $C_{[i]}$).

Corollary 7. *If* C^* *is a CR code of length* $n+1$ *with minimum distance* $d^* = 2e+2 \geq 4$, *then, for all odd* w,

$$(n-w)A_w = (w+1)A_{w+1}, \tag{3}$$

where A_w *is the number of codewords of weight* $w \geq 2e+1$ *in* C (*or* $C_{[i]}$).

Proof. Denote by A^*_{w+1} the number of codewords in C^* of weight $w+1$, w odd. This set C^*_{w+1} of codewords of weight $w+1$ form a $T(n+1, w+1, 2, \lambda^*_2)$-design, by Theorem 1. The number of codewords in C^*_{w+1} with a nonzero value at position $n+1$ is r^*, the replication number, and clearly $r^* = A_w$. Therefore,

$$A^*_{w+1}(w+1) = (n+1)r^*. \tag{4}$$

Combining (4) with the obvious equality $A^*_{w+1} = A_{w+1} + A_w$, we obtain the result. △

In particular, any perfect code must satisfy (3). For the case of binary perfect codes with $d = 3$, this recursion is well known (see, e.g., [21]):

$$(n-i+1)A_{i-1} + A_i + (i+1)A_{i+1} = \binom{n}{i}. \tag{5}$$

By combining (3) and (5), we obtain the following.

Corollary 8. *For any binary perfect code with* $d = 3$ *containing the zero codeword, the number of codewords of weight* i *is*

$$A_i = \begin{cases} \binom{n}{i}\dfrac{1}{n-i+1} - A_{i-1}, & i \text{ odd},\\[2mm] \dfrac{n-i+1}{i}A_{i-1}, & i \text{ even}.\end{cases}$$

By [19], we know that the even half of a (binary) $[23, 12, 7]$ Golay code, say C^*, is CR. Puncturing at any coordinate gives a code C with the following weight distribution:

A_0	A_7	A_8	A_{11}	A_{12}	A_{15}	A_{16}
1	176	330	672	616	176	77

Of course, C is CR [19], and hence it verifies (3), as can readily be seen.

3. COMPLETELY TRANSITIVE CODES

Completely transitive (CT) codes were first introduced in 1990 by Solé [6] as a subclass of binary linear CR codes. If C is a binary linear code, then consider the natural action of $\mathrm{Aut}(C)$ on the quotient space $\mathbb{F}_2^n/C$: for any coset $C + x$ and any $\sigma \in \mathrm{Aut}(C)$, set

$$\sigma(C+x) = \sigma(C) + \sigma(x) = C + \sigma(x).$$

Definition 9 [6]. A binary linear code C with covering radius ρ is *completely transitive* if the group $\mathrm{Aut}(C)$ gives $\rho+1$ orbits under the action on (the set of cosets) $\mathbb{F}_2^n/C$.

Since two cosets in the same orbit have identical weight distributions, we obtain the following.

Proposition 11 [6]. *If C is a binary linear CT code, then C is CR.*

The following fact is a strengthening of Theorem 6.

Proposition 12 [36]. *If C is a nonantipodal binary linear CT code, then $C \cup C(\rho)$ is CT too.*

There is a strong relation between CR codes and distance-regular graphs. In particular, the following results hold.

Proposition 13. *Let C be a binary linear CR code with covering radius ρ and intersection array* $\mathrm{IA} = \{b_0, \dots, b_{\rho-1}; c_1, \dots, c_\rho\}$, *and let Γ_C be the coset graph of C. Then*

(i) Γ_C *is distance-regular of diameter $D = \rho$ with the same* IA [10];
(ii) *If C is CT, then Γ_C is distance-transitive* [46].

Therefore, by Proposition 13 any CR or CT code induces a distance-regular or distance-transitive graph, respectively. In this paper we will not deal with the associated distance-regular graphs or distance-transitive graphs, whose existence follows from the corresponding CR or CT codes, respectively. Concerning these graphs, having the same intersection array as the corresponding codes, see [10–12], where they are considered in detail.

For a given permutation group G of degree n (acting on an n-set), we say that G is t-transitive (respectively t-homogeneous) if it sends any t-tuple (respectively t-set) to any t-tuple (respectively t-set). G is transitive if it is 1-transitive. A result of Livingston and Wagner [47] states that if G is i-homogeneous with $i \le n/2$, then G is also j-homogeneous, for $j \le i$. This fact implies the following.

Proposition 14 [6]. *Let C be a binary linear code of length n and covering radius $\rho \le n/2$. If* $\mathrm{Aut}(C)$ *is ρ-homogeneous, then C is CT.*

Using this property, we obtain that the following codes are CT:

(i) All perfect binary linear codes (repetition codes, Hamming codes, and the binary Golay code);
(ii) All extended binary linear perfect codes.

Proposition 14 gives us a sufficient (but not necessary) condition for complete transitivity. This can be seen from the binary $[9,5,3]$ code C which is dual to the code obtained by the Kronecker product of two $[3,2,2]$ parity check codes. The code C is UP in the narrow sense [2,6] with $\rho = s = 2$ and $e = 1$. Moreover, C is CT; however, $\mathrm{Aut}(C)$ is transitive but not 2-homogeneous.

A necessary condition is as follows.

Proposition 15 [6]. *If C is a binary linear CT e-code, then* $\mathrm{Aut}(C)$ *is e-homogeneous.*

As an example of CR code which is not CT, consider a binary primitive BCH $[2^m-1, 2^m-2m-1, 5; 3]$ code, i.e., a cyclic code C with $m > 4$ and m odd. Such code is CR [2,24,33]. The intersection array is $\{n, n-1, (n+3)/2; 1, 2, (n-1)/2\}$, where $n = 2^m - 1$ (these are the codes for $\ell = 3$ in the family of Section 5.13; see also Section 2.4). As can be seen in [48], $\mathrm{Aut}(C)$ is the semilinear group $\Gamma\mathrm{L}(1, 2^m)$ of $\mathbb{F}_{2^m}$ over $\mathbb{F}_{2^m}$ (recall that the semilinear group, denoted by $\Gamma\mathrm{L}(t,q)$, consists of all invertible semilinear transformations of $\mathbb{F}_q^t$ over $\mathbb{F}_q$). If $q = p^r$, then it is well known that $|\Gamma\mathrm{L}(t,q)| = r|\mathrm{GL}(t,q)|$, where $\mathrm{GL}(t,q)$ denotes the general linear group (see, e.g., [25, p. 163]); in our case, $t = 1$. Hence, the order of $\mathrm{Aut}(C)$ is $|\Gamma\mathrm{L}(1,2^m)| = m|\mathrm{GL}(1,2^m)| = m(2^m-1)$. Since C has packing radius $e = 2$, we know that C has exactly $(2^m-1)(2^{m-1}-1)$ cosets of minimum weight 2. Therefore, since the number of such cosets is greater than $|\mathrm{Aut}(C)|$, they cannot belong to the same orbit under $\mathrm{Aut}(C)$. We conclude that C is not CT.

Similarly to perfect and quasi-perfect UP codes, the nonexistence of CT codes for $e > 3$ was also established. In 2000, Borges and Rifà [16], proved the following.

Theorem 11 [16]. *If C is a nontrivial binary linear e-error-correcting CT code, then $e \leq 4$.*

The proof was based on the nonexistence of highly transitive groups and some bounds on the size of a code. Using the Griesmer bound and the nonexistence of certain designs, the result was improved in 2001 by Borges, Rifà, and Zinoviev [17], giving the following result.

Theorem 12 [17]. *If C is a nontrivial binary linear e-error-correcting CT code, then $e \leq 3$.*

Clearly, Definition 9 can be extended to nonbinary linear codes. Giudici and Praeger [7, 8] studied this more general case. They called these codes, including the binary CT codes introduced by Solé, *coset-completely transitive* [6]. In an earlier preprint, Godsil and Praeger generalized the concept of complete transitivity to the nonlinear case. A newer version of that preprint is [49].

Definition 10 [49]. A code $C \subset \mathbb{F}_q^n$ is *G-completely transitive*, or simply *completely transitive*, if there exists a subgroup G of $\mathrm{Aut}(\mathbb{F}_q^n)$ such that each subconstituent $C(i)$ of the distance partition is a G-orbit.

Proposition 16 [8]. *If $C \subset \mathbb{F}_q^n$ is CT in the sense of Definition* 10, *then it is CR.*

Note that, in general, a coset-completely transitive code may be not $\mathrm{Aut}(C)$-completely transitive in the sense of Definition 10. This is because $\mathrm{Aut}(C)$ is often not even transitive on C, for example, when C has codewords of different weights.

For a linear code $C \subset \mathbb{F}_q^n$, define N_C as the set of all translations of $\mathbb{F}_q^n$ by vectors in C, i.e., $N_C = \{\tau_x \mid x \in C\}$, where $\tau_x(v) = v + x$, for every $v \in \mathbb{F}_q^n$. Clearly, N_C is a subgroup, since C is linear. Define now the semidirect product $G = N_C \rtimes \mathrm{Aut}(C)$. G fixes C set-wise, and the N_C-orbits are the cosets of C; in particular, C is a G-orbit [8]. From all these observations, the following result is obtained.

Theorem 13 [8]. *Let $C \subset \mathbb{F}_q^n$ be a linear code. Then C is coset-completely transitive if and only if C is $(N_C \rtimes \mathrm{Aut}(C))$-completely transitive.*

However, for $q \leq 3$, the notions are equivalent:

Theorem 14 [8]. *Let $C \subset \mathbb{F}_q^n$ be a linear code with $q \leq 3$. Then C is coset-completely transitive if and only if C is completely transitive.*

Let $q \geq 7$ be a prime power, $q \neq 8$, and let C be the repetition code in $\mathbb{F}_q^3$. Taking $G = S_q \rtimes S_3$, one can directly verify that C is G-completely transitive; however, C is not coset-completely transitive [8].

Complete transitivity is a quite special property, which, in general, has no relation to the optimality of codes. For example, Preparata codes (which have the maximal possible packing number after perfect codes [2]) are not completely transitive except for the case where they coincide with Kerdock codes, i.e., except for the Nordstrom–Robinson code.

Theorem 15 [9]. *Let C and C^* be the Preparata code of length n and its extension, respectively. These codes are CT if and only if $n = 15$.*

4. COMPLETELY REGULAR CODES IN JOHNSON SCHEMES

The *Johnson scheme* $J(n, w)$ is the set of all binary vectors of length n with weight w (i.e., with w nonzero positions). A subset C of $J(n, w)$ is a constant weight code which has four parameters: length n, constant weight w, minimum Hamming distance $d = 2\delta$, and cardinality $N = |C|$. If we define the distance between two words x and y of weight w as half their Hamming distance (indeed, the Hamming distance between two words of weight w is always even), we obtain a new distance d_J

(which has also the properties of a metric), called the *Johnson distance*. Two vectors x and y from $J(n,w)$ are called *neighbors* if they are at the distance $d_J = 1$ from each other.

With this Johnson scheme we associate in a natural way the *Johnson graph* (sometimes also denoted by $J(n,w)$) as the graph whose vertices are all binary vectors of length n with weight w with two vertices connected if they are at Hamming distance 2 from each other (i.e., if they are neighbors). Two vertices x and y are at Johnson distance $d_J(x,y)$ if it equals the number of edges in the shortest path between x and y. Clearly, the graphs $J(n,w)$ and $J(n,n-w)$ are isomorphic, so we only consider the case $w \le n/2$.

Definition 11. A perfect m-coloring of the Johnson scheme $J(n,w)$ is a coloring of vertices of $J(n,w)$ in m colors $\{1,\dots,m\}$ such that for a fixed vertex x of color i the number of vertices y of color j connected with x by an edge (i.e., $d_J(x,y) = 1$) equals $a_{i,j}$, which does not depend on x.

Clearly, such m-colorings are nothing else but distance-regular graphs. The matrix $A = [a_{i,j}]$ is called the *matrix of parameters of the perfect coloring*, or shortly the *matrix of parameters*. A *perfect code* in a Johnson scheme is a classical example of a CR code.

Definition 12. A constant w-weight code C of length n and minimum distance $d_J = 2e+1$ is a perfect e-code in $J(n,w)$ if for any vector $x \in J(n,w)$ there exists exactly one codeword $c \in C$ such that $d_J(x,c) \le e$. In other words, all balls of (Johnson) radius e around every codeword of C do not intersect each other and fill all the set of vectors of $J(n,w)$.

Denote by $\Phi_e(n,w)$ the cardinality of a ball of radius e in $J(n,w)$. Clearly,

$$\Phi_e(n,w) = \sum_{i=0}^{e} \binom{w}{i}\binom{n-w}{i}.$$

Then, by the sphere packing condition, for a perfect e-code C in $J(n,w)$, we have the following expression for the cardinality of C:

$$|C| = \frac{\binom{n}{w}}{\Phi_e(n,w)},$$

which means, of course, that the obtained number on the right-hand side is integer. The following trivial examples of perfect e-codes in $J(n,w)$ are known:

1. The Johnson scheme $J(n,w)$ with $e = 0$;
2. Any vector x from $J(n,w)$ with $e = w$;
3. For the case $n = 4e+2$ and $w = 2e+1$, any pair of vectors x and y from $J(n,w)$ at the (Johnson) distance $d_J = w$ is a perfect e-code.

Note that in the special case $n = 2w$, the Johnson scheme $J(2w,w)$ is a CT (and, hence, CR) code in the Hamming scheme $H(2,2w)$ (see Section 5.8).

The existence problem of nontrivial codes in Johnson schemes was formulated in 1973 by Delsarte [3] (who stated a conjecture that there are no nontrivial perfect codes), and till now this problem is still open. Bannai [50] proved the nonexistence of perfect e-codes in the Johnson scheme $J(2w+1,w)$ for $e \ge 2$. Hammond [51] improved this result, showing that $J(n,w)$ cannot contain a nontrivial perfect code for $n \in \{2w-2,\ 2w-1,\ 2w+1,\ 2w+2\}$. Note that the generalization of Lloyd's theorem [3,50] (which was the main tool for proving the nonexistence of perfect codes in Hamming schemes; see [14,15] and references therein) did not lead to any significant results. Many nonexistence results were obtained by Etzion and Schwartz [52–54]. Using divisibility conditions, which rule out perfect codes and which are derived from Steiner systems (which must accompany perfect codes in $J(n,w)$), Etzion [52] proved the following results.

Theorem 16 [52]. *Nontrivial perfect e-codes do not exist*

(i) *in Johnson schemes* $J(2w+e+1,w)$;
(ii) *in Johnson schemes* $J(2w+p,w)$, p *prime*;
(iii) *in Johnson schemes* $J(2w+2p,w)$, p *prime*, $p \neq 3$;
(iv) *in Johnson schemes* $J(2w+3p,w)$, p *prime*, $p \neq 2$, 3, 5.

Shimabukuro [55] added to that list the schemes $J(2w+p^2,w)$, which do not admit the existence of nontrivial perfect e-codes. Another approach to nonexistence results was used by Etzion and Schwartz [53]. They used the so-called *k-regularity* of perfect codes in $J(n,w)$. We will return to this concept when considering perfect 2-colorings. For the case of small $e \leq 2$, Etzion and Schwartz [53] obtained the following result.

Theorem 17 [53]. *There are no perfect* 1*-codes in* $J(n,w)$ *for* $n \leq 50000$ *and no perfect* 2*-codes in* $J(n,w)$ *for* $n \leq 40000$.

Gordon [56] derived necessary conditions for the existence of perfect 1-codes by considering possible prime divisors of $\Phi_e(w,a)$. For that, he used two number theoretical results, which we recall here. The first is the following Kummer's lemma.

Lemma 2 [57]. *Let* p *be a prime. The number of times that* p *appears in the factorization of* $\binom{a}{b}$ *equals the number of carries when adding* b *to* $a-b$ *modulo* p.

The second result is the following lemma due to Loxton [58] (in the proof of which, an omission was corrected by Bernstein [59]).

Lemma 3 [58, 59]. *The number of perfect powers* (*i.e., powers of the type* a^m *with* $a, m > 1$) *in the interval* $[w, w+\sqrt{w}\,]$ *is at most*

$$\exp(40\sqrt{\log\log w \log\log\log w}).$$

Using these two lemmas, Gordon [56] derived the following result.

Theorem 18 [56]. *There are no perfect* 1*-codes in* $J(n,w)$ *for* $n \leq 2^{250}$.

At present, there are many known necessary conditions for the existence of perfect codes in Johnson schemes. We consider some of them. Roos [60] (see also [53] for a simpler proof) upper bounded the length n, and Movsisyan [61] lower bounded it. This gives the following range of possible lengths n for the existence of such perfect e-codes.

Theorem 19 [60, 61]. *If* C *is a perfect e-code in the Johnson scheme* $J(n,w)$, *then its length* n *is bounded as follows*:

$$\frac{2e+1}{e+1}(w+1) \leq n \leq \frac{2e+1}{e}(w-1). \tag{6}$$

Looking at the lower bound on the length of a perfect e-code, we have the following result.

Corollary 9 [61]. *For any fixed* w, *the number of perfect e-codes in* $J(n,w)$ *is finite.*

Let $C \subset J(n,w)$ be a perfect e-code. For any vector y from $J(n,w+1)$, denote by $R(y)$ the set of vectors x from C such that $d_J(x,y) = 2e+1$.

Corollary 10 [61]. *If* C *is a perfect e-code in* $J(n,w)$, *then for every vector* $y \in J(n,w+1)$ *the set* $R(y)$ *has cardinality* $(w+1)/(e+1)$, *which means that* $e+1$ *divides* $w+1$.

From Theorem 19 we immediately deduce that the weight w of vectors in a perfect e-code is lower bounded: $w \geq 2e+1$. Movsisyan [61] improved this bound, showing that w should grow with e as a quadratic function.

Proposition 17 [61]. *If C is a nontrivial perfect e-code in $J(n,w)$, then*

$$w \geq e^2 + 3e + 1.$$

It is clear that two remarkable combinatorial objects in $J(n,w)$—Steiner systems $S(n,w,t)$ and perfect e-codes—must be closely related. Several results concerning this relationship were obtained in [52, 62].

Proposition 18 [62]. *A perfect e-code in $J(n,w)$ coexists with a Steiner system $S(n,w,t)$ if and only if the following equalities are valid*:

$$n = 3w - 3, \qquad t = w - 2, \qquad \text{and} \qquad e = 1. \tag{7}$$

Proposition 19 [52]. *If a perfect e-code exists in $J(n,w)$, then Steiner systems $S(w, 2e+1, e+1)$ and $S(n-w, 2e+1, e+1)$ do exist.*

Proposition 20 [52]. *Except for the* (*trivial*) *Steiner systems $S(2w,w,1)$ and $S(n,w,w)$, there are no more Steiner systems which are also perfect codes in the Johnson scheme $J(n,w)$.*

The existence of perfect constant weight codes is deeply connected with the *D-representation problem* for codes [63, 64], which we discuss now. Let A be any subset of the binary space $\mathbb{F}_2^n$, and let φ be an arbitrary function defined on the set of natural numbers. For an x from A, we define the *Dirichlet region* $D_x(\varphi)$ in the following way:

$$D_x(\varphi) = \{y \in \mathbb{F}_2^n : \varphi(d_J(x,y)) \leq \varphi(d_J(z,y)),\ \forall z \in A\},$$

or simply D_x when φ is the identity function.

Definition 13. A set A of vectors from $\mathbb{F}_2^n$ is called a *$D(\varphi)$-representable* (respectively, D-representable) *code* if and only if the Dirichlet regions $D_x(\varphi)$ (respectively, D_x) of all points of A are pairwise disjoint in $\mathbb{F}_2^n$.

Proposition 21 [63, 64]. *A set C is a perfect e-code in $J(n,w)$ if and only if C is $D(\varphi)$-representable for any strictly monotone function φ.*

Now, consider some perfect colorings in Johnson schemes. We start with a simple observation that a perfect 1-code in $J(n,w)$ induces a perfect 2-coloring of $J(n,w)$ with the following matrix of parameters:

$$A = \begin{bmatrix} 0 & w(n-w) \\ 1 & w(n-w)-1 \end{bmatrix}.$$

For that, we have to color all codewords in one color (usually, white) and all other points of $J(n,w)$ in another color (usually, black) [65].

We start with the notion of *k-regularity*, introduced in [65] for perfect 2-colorings, which extends the corresponding notion for perfect e-codes given in [53]. For two binary vectors x and y of length n, we write $x \preceq y$ if x is covered by y, i.e., if $x_i \leq y_i$ for all $i \in \{1,\ldots,n\}$. For a perfect 2-coloring P of $J(n,w)$, denote by W and B the sets of white and black vertices in $J(n,w)$, respectively. For two binary vectors x and y such that $y \preceq x$ (i.e., x covers y), denote by Γ_x^y the face induced by x and y, i.e., the following set of binary vectors from $\mathbb{F}_2^n$:

$$\Gamma_x^y = \{z + y + x : z \in \mathbb{F}_2^n,\ x \preceq z\}$$

(i.e., Γ_x^y consists of all $2^{n-\mathrm{wt}(x)}$ vectors from $\mathbb{F}_2^n$ that cover y and coincide with y on the support of x).

Definition 14 [65]. A perfect 2-coloring P of a Johnson scheme $J(n,w)$ is said to be k-regular if there exist k^2 numbers γ_{ij} such that for any two binary vectors x, y of length n with $y \preceq x$ and $\mathrm{wt}(x)=i$, $\mathrm{wt}(y)=j$, where $i \le k \le w$, we have

$$|\Gamma_x^y \cap W| = \gamma_{ij}.$$

The following result gives a sufficient condition for k-regularity of a perfect 2-coloring.

Proposition 22 [65]. *Let P be a perfect 2-coloring of the Johnson scheme $J(n,w)$ with matrix $A=[a_{ij}]$, $i,j \in \{1,2\}$. Then*

$$k_1 = \frac{1}{2}\left(n+1-\sqrt{(n-2w+1)^2+4(w+a_{11}-a_{21})}\right)$$

is a natural number, and the coloring P is (k_1-1)-regular.

Corollary 11 [65]. *Let P be a perfect 2-coloring of the Johnson scheme $J(n,w)$ with matrix $A=[a_{ij}]$, $i,j=1,2$. Then the number*

$$\frac{a_{21}}{a_{12}+a_{21}}\binom{n-i}{w-i+j}$$

is an integer for any i and j such that $0 \le j \le i \le k_1-1$, where k_1 is defined in Proposition 22.

Using these results, Mogil'nykh [66] proved the nonexistence of some perfect 2-colorings for $n \le 13$. In particular, this was done for the values

$$(n,w) \in \{(9,3),\ (10,5),\ (11,3),\ (12,4),\ (12,5),\ (12,6),\ (13,4)\}$$

with the corresponding matrices of parameters for every (n,w) from this set.

According to [67], the smallest open case for the existence of a perfect 2-coloring in a Johnson scheme $J(n,w)$ is the case of $J(9,3)$ with the following matrix of parameters:

$$A = \begin{bmatrix} 10 & 8 \\ 8 & 10 \end{bmatrix}.$$

Clearly, a Steiner system $S(n,w,t)$, being a perfect combinatorial object, is a good candidate to induce a perfect coloring. Indeed, this is the case for $w=t+1$, $w=t+2$ [68,69], and some other cases such as, for example, *completely regular designs*, introduced by Martin [68,69]. Any simple $T(n,w,t,\lambda)$-design T (i.e., a set of distinct vectors from the Johnson scheme $J(n,w)$) naturally determines (as well as any code in the Hamming scheme) a partition of all vectors of $J(n,w)$ according to their distance from T. For an integer i, define

$$T(i) := \Big\{x \in J(n,w) : \min_{b\in T} d_J(x,b) = i\Big\}.$$

We denote by ρ the maximum value of i such that the set $T(i)$ is nonempty and call it the *covering radius of the design T*. The partition

$$P = \{T = T(0), T(1), \ldots, T(\rho)\}$$

consisting of nonempty sets $T(i)$ is called the *distance partition*.

Definition 15 [68,70]. A simple $T(n,w,t,\lambda)$-design is a completely regular (CR) design in $J(n,w)$ if, for each $i \in \{0,1,\ldots,\rho\}$, every vector in $T(i)$ has γ_i neighbors in $T(i-1)$, α_i neighbors in $T(i)$, and β_i neighbors in $T(i+1)$.

According to Meyerowitz [70], if T is a CR design in $J(n,w)$ of strength zero, i.e., a $T(n,w,0,\lambda)$-design, then there is a subset V of $J(n,w)$ such that either

$$T = \{B : |B| = w,\ B \subseteq V\} \qquad \text{or} \qquad T = \{B : |B| = w,\ V \subseteq B\}.$$

Martin [68] classified CR designs having strength one, i.e., $T(n,w,1,\lambda)$ designs. To formulate this result, we define the so-called *group-wise complete designs*. Consider the Johnson scheme $J(n,w)$ where $n = h\ell$ and $w = u\ell$ with $\ell \geq 2$ and $h \geq 2u$. Let $\{X_1, X_2, \ldots, X_h\}$ be a partition of the coordinate set $\{1, 2, \ldots, n\}$ into h groups, each of size ℓ. The blocks B of the design T will be all the $\binom{h}{u}$ subsets of the form

$$B = \bigcup_{i \in I} X_i,$$

where I is any u-subset of $\{1, 2, \ldots, h\}$. Since $\ell \geq 2$ and $h > u$, such a design has strength one (i.e., $t = 1$). It is also clear that T has minimum (Johnson) distance ℓ. Such a design was called by Martin [68] a group-wise complete design.

Proposition 23 [68]. *Let T be a group-wise complete design in $J(n,w)$. Then T is a CR design if and only if one of the following conditions hold*:

(i) $\ell = w$, $n = 2w$, *and T is an antipodal pair* (*i.e., two vectors which have no common nonzero positions*);
(ii) $\ell = 2$;
(iii) $\ell = 3$ *and* $u = 1$.

There are also other CR designs of strength one (see [68] and references therein). The next result explains the role of group-wise complete designs.

Theorem 20 [68]. *Let T be a simple CR design in $J(n,w)$ of strength one. Then T is a group-wise complete design.*

Let T be a design $T(n,w,t,\lambda)$. Define the *degree* of T as the number $s = s(T)$ of distinct block intersection values (or the number of distinct distances between two different blocks of T):

$$s(T) := |\{d_J(b,b') : b, b' \in T,\ b \neq b'\}.$$

Similarly to the Hamming scheme, for a design T in the Johnson scheme $J(n,w)$ one can define the *dual degree* (see [3]), denoted by $s^* = s^*(T)$. Let $d_J(T)$ denote the minimum (Johnson) distance between distinct vectors of T.

Proposition 24 [3]. *Any design with dual degree s^* and minimum distance $d_J(T)$ satisfying $d_J(T) \geq 2s^* - 1$ is CR.*

The large Witt design, i.e., the Steiner system $S(24,8,5)$, is CR in $J(24,8)$. It has dual distance $s^* = 2$ and minimum (Johnson) distance 4. Hence, by Proposition 24, this Steiner system $S(24,8,5)$ is CR (which was already observed by Delsarte [3]). Since computing s^* is rather difficult, the following weaker result is often used [69].

Proposition 25 [3, 68]. *Any simple t-design T with minimum distance $d_J(T) \geq 2(w-t) - 1$ is CR.*

This proposition immediately yields the following result.

Theorem 21 [69]. *We have the following*:

(i) *Any simple $T(n,w,t,\lambda)$-design with $w = t+1$ is CR*;
(ii) *A Steiner system $S(n,w,t)$ with $w = t+2$ is CR.*

We immediately conclude that the small Witt designs, i.e., the Steiner systems $S(12,6,5)$ and $S(11,5,4)$ are CR [69].

To check the complete regularity of the second large Witt design, i.e., the Steiner system $S(23,7,4)$, it is required to compute the numerical parameters $(\alpha_i, \beta_i, \gamma_i)$ for all $i \in \{0,\dots,\rho\}$ [69]. Since the first large Witt design $S(24,8,5)$ has covering radius 2, the second large Witt design $S(23,7,4)$ (obtained by 1-shortening) has covering radius $\rho = 3$ and minimum distance 4. Thus, we have to compute the corresponding parameters $(\alpha_i, \beta_i, \gamma_i)$ for $i \in \{0,1,2,3\}$. For this design we have from [69]

$$\begin{aligned}
&\alpha_0 = 0, && \beta_0 = 112, && \gamma_0 = 0,\\
&\alpha_1 = 21, && \beta_1 = 90, && \gamma_1 = 1,\\
&\alpha_2 = 108, && \beta_2 = 2, && \gamma_2 = 12,\\
&\alpha_3 = 7, && \beta_3 = 0, && \gamma_3 = 105.
\end{aligned}$$

Hence we conclude that the Steiner system $S(23,7,4)$ is CR. The third large Witt design, i.e., the Steiner system $S(22,6,3)$, is not CR [69]. The same negative result holds for the next Steiner system $S(21,5,2)$, i.e., for the projective plane $PG(2,4)$) of order 4 [69].

The following simple idea allows to obtain more CR designs from Steiner systems. This is a special case of a well-known more general design construction starting from known designs (see, e.g., [2]). Define the *k-shadow* of a design $T(n,w,t,\lambda)$ as the collection of all k-sets which are contained in some block of T. The resulting new design already belongs to $J(n,k)$. Godsil [71] noted the following fact.

Proposition 26 [69]. *The $(t+2)$-shadow of any Steiner system $S(n,w,t)$ is a CR design with covering radius $\rho = 2$.*

The next statement gives a necessary condition for the existence of a CR design.

Proposition 27 [69]. *If T is a CR design in $J(n,w)$ with minimum distance $d_J(T) < w$, then*

$$d_J(T) \le \frac{1}{3}(2w+1).$$

As a consequence, we have the following result.

Corollary 12 [69]. *If a Steiner system $S(n,w,t)$ is CR, then $w \le 3t-2$.*

The following remarkable result in [69] classified all possible completely regular symmetric designs $T(n,w,2,\lambda)$ (i.e., designs T for which $n = |T|$).

Theorem 22 [69]. *If T is a symmetric CR design $T(n,w,2,\lambda)$, then T is a projective plane (i.e., $\lambda = 1$) of order two or three (i.e., it is one of the two Steiner systems $S(7,3,2)$ or $S(13,4,2)$).*

Recall that a $T(n,w,t,\lambda)$-design with $t \ge 2$ is said to be *quasi-symmetric* if its degree is $s = 2$ (i.e., there are two different values of distances between distinct vectors of T). For designs with $t = 2$ that are not symmetric, Martin [69] obtained the following result.

Corollary 13 [69]. *We have the following necessary conditions*:

(i) *If T is a CR design $T(n,w,2,\lambda)$ with minimum distance $d_J(T) \ge 4$, then $\lambda \ge n-w$.*
(ii) *If T is a quasi-symmetric CR design in $J(n,w)$, then $d_J(T) \le 7$.*

For a given Steiner system $S(n,w,w-1)$ (which is CR according to Theorem 21), Martin [69] noted that the distance coloring of $J(n,w)$ is a perfect coloring of $J(n,w)$ with the matrix of parameters

$$A = \begin{bmatrix} 0 & w(n-w) \\ w & w(n-w-1) \end{bmatrix}.$$

This result was strengthened in [72] using the so-called *induced perfect coloring*, which comes from earlier results in [73] (*distance-biregular graphs*) and in [74] (*induced eigenfunctions*).

Proposition 28 [72]. *A Steiner system $S(n,w,w-1)$ induces a perfect 2-coloring of $J(n,w+1)$ with the matrix of parameters*

$$A = \begin{bmatrix} (w+1)(n-2w-1) & (w+1)w \\ w(n-2w) & w^2-2w-1+n \end{bmatrix}.$$

Since there are infinite families of triple and quadruple Steiner systems, i.e., $S(n,3,2)$ and $S(n+1,4,3)$ (which exist for any $n \equiv 1,3 \pmod 6$), we arrive at the following result.

Proposition 29 [72]. *The following perfect 2-colorings exist*:

(i) *For every $n \equiv 1,3 \pmod 6$ there exists a perfect 2-coloring of $J(n,4)$ with the matrix of parameters*

$$A = \begin{bmatrix} 4(n-7) & 12 \\ 3(n-8) & n+2 \end{bmatrix};$$

(ii) *For every $n \equiv 2,4 \pmod 6$ there exists a perfect 2-coloring of $J(n,5)$ with the matrix of parameters*

$$A = \begin{bmatrix} 5(n-5) & 20 \\ 4(n-8) & n+7 \end{bmatrix}.$$

Finally we formulate one more result from [72] on a coloring induced by a perfect 1-code.

Proposition 30 [72]. *Let C be a perfect 1-code in $J(n,w)$. Then there exists a perfect 2-coloring of $J(n,w+1)$ with the matrix of parameters*

$$A = \begin{bmatrix} (w+1)(n-w-2) & w+1 \\ w(n-w-1) & n-w-1 \end{bmatrix}.$$

Similarly to the CR codes in the Hamming scheme, which have a subclass of CT codes, there exists an analogous concept for CR codes in the Johnson scheme. A $T(n,w,t,\lambda)$ design whose distance partition is an orbit partition under some automorphism group of $J(n,w)$ (i.e., for any $i = 0,1,\ldots,\rho$ the set T_i is an orbit) is called a *completely transitive design* (see [49] for this interesting subclass of CR designs).

For other results on perfect e-codes, CR designs, and perfect colorings of $J(n,w)$, as well as for examples of such designs of small lengths, see [53, 54, 65–69, 72, 75].

5. CONSTRUCTIONS AND EXISTENCE OF CR CODES

In this section we give infinite families (numbered by (F.i)) and sporadic cases (numbered by (S.i)) of CR codes known to us. For all codes we present their intersection arrays (IA), defined—as well as all the terminology that we use—in Section 2.1. Recall that q is a prime power.

5.1. CR and CT Codes from Perfect Codes

From [1–3] we have the following well-known families of CR codes. In several cases we also indicate if they are CT.

(F.1) Any q-ary perfect $(n, q^n/(1+n(q-1)), 3; 1)_q$ code is a CR and CT (if it is linear) code with

$$\mathrm{IA} = \{(q-1)n; 1\}, \quad \text{where} \quad q \geq 2,\ n \geq 3.$$

(F.2) Any q-ary extended perfect $(n+1, q^n/(1+n(q-1)), 4; 2)_q$ code is a CR code (and CT in the binary linear case) with

$$\mathrm{IA} = \{(q-1)(n+1), (q-1)n; 1, 4\}, \quad \text{where} \quad q \geq 2,\ n \geq 4.$$

The code is also CT when $q = 4$ and $n = 5$.

(F.3) Any q-ary $(n-1, q^n/(1+n(q-1)), 2; 1)$ code obtained by puncturing any perfect $(n, q^n/(1+n(q-1)), 3; 1)$ code is a CR and a CT (if it is linear) code with

$$\mathrm{IA} = \{(q-1)(n-1); q)\}, \quad \text{where} \quad q, n \geq 3.$$

Let B be any q-ary perfect $(n, N, 3)_q$-code, and let C be any $(n, N/q, 4)$ subcode of B with n odd and having the following property: for any choice of the zero codeword in B, the set C_4 is a q-ary $T(n, 4, 2, \lambda)_q$-design with $\lambda = (n-3)/2$. Then from [32, 39] we have:

(F.4) For $q \geq 2$ and $n \geq 4$ the code C is a CR code with

$$\mathrm{IA} = \{n(q-1), (n-1)(q-1), 1; 1, (n-1), n(q-1)\}.$$

(F.5) In particular, for any prime power $q = 2^s$ where $s \in \{2, 3 \ldots\}$, there exists a CR $[q+1, q-2, 4; 3]_q$ code, a subcode of a perfect $[q+1, q-1, 3]_q$-code, with

$$\mathrm{IA} = \{q^2-1, q(q-1), 1; 1, q, q^2-1\}.$$

(F.6) Any q-ary $(n, N, 3; 2)_q$ code obtained by shortening a perfect $(n+1, qN, 3; 1)_q$ code C is a CR and CT (in the linear case) code with

$$\mathrm{IA} = \{(q-1)n, q-1; 1, (q-1)n\}.$$

Now we present several different halves of a binary perfect code giving CR codes with different intersection arrays.

Let C be a binary perfect $(n, N, 3)$ code. From [19], we have that the even or odd half of C is a CR code (included in the family (F.4)). Also from [19], we have that the punctured code of the even half of C is a CR code (included in the family (F.6)).

Other halves of binary Hamming codes can be obtained by following procedure. Let H_m denote the parity check matrix of a binary Hamming code of length $n = 2^m - 1$. For a given even $m \geq 4$ and any $i_1, i_2 \in \{0, 1, 2, 3\}$, where $i_1 \neq i_2$, denote by $\boldsymbol{v}_{i_1,i_2}$ the binary vector whose ith position v_i is a function of the value of the weight of the column $\boldsymbol{h}_i$:

$$v_i = \begin{cases} 1 & \text{if } \mathrm{wt}(\boldsymbol{h}_i) \equiv i_1 \text{ or } i_2 \pmod 4, \\ 0 & \text{otherwise.} \end{cases}$$

Let C_{i_1,i_2} be the binary linear $[n = 2^m - 1, k = n - m - 1, 3]$ code with m even and parity check matrix $H_m(\boldsymbol{v}_{i_1,i_2})$ obtained from H_m by adding one more row $\boldsymbol{v}_{i_1,i_2}$. Then for the resulting code C_{i_1,i_2} we have [38]:

(F.7) If $\{i_1, i_2\} = \{1, 2\}$ or $\{i_1, i_2\} = \{2, 3\}$, then C_{i_1,i_2} is an antipodal half of a Hamming code, which is a CR and CT $[n, n-m-1, 3; 3]$-code with

$$\mathrm{IA} = \{n, (n+1)/2, 1; 1, (n+1)/2, n\}.$$

(F.8) If $\{i_1, i_2\} = \{0, 1\}$ or $\{i_1, i_2\} = \{0, 3\}$, then C is a nonantipodal half of a Hamming code, which is a CR and CT $[n, n-m-1, 3; 3]$-code with

$$\mathrm{IA} = \{n, (n-3)/2, 1; 1, (n-3)/2, n\}.$$

As can be seen in [38], if $\{i_1, i_2\} = \{0, 2\}$, then C_{i_1,i_2} is the even half of a Hamming code, hence included in the family (F.4). If $\{i_1, i_2\} = \{1, 3\}$, then C_{i_1,i_2} is the Hamming code $[n, n-m, 3]$.

Extensions of the codes (F.7) are also CR [38]:

(F.9) The extension of the code (F.7) is an antipodal CR and CT $[n+1, n-m, 4; 4]$ code with

$$\mathrm{IA} = \{n+1, n, (n+1)/2, 1; 1, (n+1)/2, n, n+1\}.$$

From [32,76], we have several families obtained by shortening binary perfect or extended perfect codes. Let C^* be any binary extended perfect $(n^*, N^*, 4)$ code, $n^* = 2^m \geq 8$.

(F.10) Let C be the $(n = n^*-2, N = N^*/2, 2; 3)$ code obtained from C^* by $\{(00), (11)\}$-shortening. Then C is a CR code and also CT with

$$\mathrm{IA} = \{n, n-2, 2; 2, n-2, n\}, \quad \text{where} \quad n = 2^m - 2 \geq 6.$$

(F.11) Let C be the $(n = n^* - 3, N^*/4, 1; 2)$ code obtained from C^* by $\{(000), (111)\}$-shortening. Then C is a CR code and also CT with

$$\mathrm{IA} = \{n-1, 3; 1, n-1\}, \quad \text{where} \quad n = 2^m - 3 \geq 5.$$

(F.12) Let B be any binary perfect $(n, N, 3)$ code, $n = 2^m - 1 \geq 7$. Let C be the $(n-2, N/2, 1; 2)$ code obtained from B by $\{(00), (11)\}$-shortening. Then C is a CR code and also CT with

$$\mathrm{IA} = \{n-3, 2; 2, n-3\}.$$

Also from [32,76], we have a family obtained by shortening q-ary extended perfect codes:

(F.13) Let C^* be any q-ary extended perfect $[n^*, k^*, 4; 2]_q$ code where $q = 2^m \geq 4$, $n^* = q+2$ and $k^* = q-1$. Let C be the $[n = q, k = q-2, 2; 2]_q$ code obtained from C^* by S-shortening, where $S = \{(\alpha, \alpha) : \alpha \in \mathbb{F}_q\}$. Then C is a CR code with

$$\mathrm{IA} = \{q(q-1), (q-1)(q-2); 2, q\}, \quad \text{where} \quad q = 2^m \geq 4.$$

Now we give sporadic CR codes, which come from Golay codes [19,76]. The complete regularity and complete transitivity of the codes (S.5) and (S.6) were stated in [19].

(S.1) *Binary Golay code.* This perfect $[23, 12, 7; 3]$ code is CR and CT with

$$\mathrm{IA} = \{23, 22, 21; 1, 2, 3\}.$$

(S.2) *Binary punctured Golay code.* This $[22, 12, 6; 3]$ code is CR and CT with

$$\mathrm{IA} = \{22, 21, 20; 1, 2, 6\}.$$

(S.3) *Binary extended Golay code.* This $[24, 12, 8; 4]$ code is CR and CT with

$$\mathrm{IA} = \{24, 23, 22, 21; 1, 2, 3, 24\}.$$

(S.4) *Binary double punctured Golay code.* This $[21, 12, 5; 3]$ code is CR and CT with

$$\mathrm{IA} = \{21, 20, 16; 1, 2, 12\}.$$

(S.5) *Half of the binary Golay code.* This $[23, 11, 8; 7]$ code is CR and CT with

$$\mathrm{IA} = \{23, 22, 21, 20, 3, 2, 1; 1, 2, 3, 20, 21, 22, 23\}.$$

(S.6) *Punctured half of the binary Golay code.* This $[22, 11, 7; 6]$ code is CR and CT with

$$\mathrm{IA} = \{22, 21, 20, 3, 2, 1; 1, 2, 3, 20, 21, 22\}.$$

(S.7) $\{(00, 11)\}$*-shortened binary extended Golay code.* This $[22, 11, 6; 7]$ code is CR and CT with

$$\mathrm{IA} = \{22, 21, 20, 16, 6, 2, 1; 1, 2, 6, 16, 20, 21, 22\}.$$

(S.8) $\{(000, 111)\}$*-shortened binary extended Golay code.* This is a CR and CT $[21, 10, 5; 6]$ code with

$$\mathrm{IA} = \{21, 20, 16, 9, 2, 1; 1, 2, 3, 16, 20, 21\}.$$

(S.9) $\{(00, 11)\}$*-shortened binary Golay code.* This $[21, 11, 5; 6]$ code is CR and CT with

$$\mathrm{IA} = \{21, 20, 16, 6, 2, 1; 1, 2, 6, 16, 20, 21\}.$$

Let G denote the ternary perfect Golay $[11, 6, 5]_3$-code; denote by $G^{(0)}$ the subcode of G formed by all codewords of G with parity 0. It is easy to see that $G^{(0)}$ is the $[11, 5, 6]_3$ code formed by all codewords of weights 0, 6, and 9. We call this code the third part of the ternary Golay.

(S.10) *Ternary Golay code.* This perfect $[11, 6, 5; 2]_3$ code is CR and CT with

$$\mathrm{IA} = \{22, 20; 1, 2\}.$$

(S.11) *Ternary punctured Golay code.* This $[10, 6, 4; 2]_3$ code is CR and CT with

$$\mathrm{IA} = (20, 18; 1, 6).$$

(S.12) *Ternary extended Golay code.* This $[12, 6, 6; 3]_3$ code is CR and CT with

$$\mathrm{IA} = \{24, 22, 20; 1, 2, 12\}.$$

(S.13) *Third part of the ternary Golay code.* The ternary $[11, 5, 6; 5]_3$-code $G^{(0)}$ is CR and CT with

$$\mathrm{IA} = \{22, 20, 18, 2, 1; 1, 2, 9, 20, 22\}.$$

(S.14) *Punctured code of the third part of the ternary Golay code.* The ternary $[10, 5, 5; 4]_3$ code is CR and CT with

$$\mathrm{IA} = \{20, 18, 4, 1; 1, 2, 18, 20\}.$$

There are many codes considered in further sections, based in some way on perfect codes (see Sections 5.2, 5.4, 5.9, 5.10, and 5.12).

5.2. Nested Families of CR Codes

Recall that $\mathcal{H}_m$ is a binary Hamming code of length $n = 2^m - 1$. Let $m = 2u$, $q = 2^u$, $r = 2^u + 1$, and $\bar{r} = 2^u - 1$. We can think of the parity check matrix H_m of $\mathcal{H}_m$ as the binary representation of $[\alpha^0, \alpha^1, \ldots, \alpha^{n-1}]$, where $\alpha \in \mathbb{F}_{2^m}$ is a primitive element. Now represent the elements of $\mathbb{F}_{2^m}$ as elements in a quadratic extension of $\mathbb{F}_{2^u}$. Let $\beta = \alpha^r$ be a primitive element of $\mathbb{F}_{2^u}$, and let $\mathbb{F}_{2^m} = \mathbb{F}_{2^u}[\alpha]$. Let E_m be the binary representation of the matrix $[\alpha^0, \alpha^r, \ldots, \alpha^{(n-1)r}]$. Take the matrix P_m as the vertical join of H_m and E_m:

$$P_m = \begin{bmatrix} H_m \\ E_m \end{bmatrix}.$$

It is well known [40] that the code $C^{(u)}$ with parity check matrix P_m is a cyclic binary CR code with covering radius $\rho = 3$, minimum distance $d = 3$, and dimension $n - (m + u)$.

It can be seen in [39] that the number of cosets $C^{(u)} + v$ of weight three is $\bar{r}$. Indeed, their syndromes $S(v)$ are the nonzero elements of $\mathbb{F}_{2^u}$. For every $i \in \{0, \dots, u\}$, by taking $u - i$ cosets $C^{(u)} + v^{(1)}, \dots, C^{(u)} + v^{(u-i)}$ with independent syndromes $S(v^{(1)}), \dots, S(v^{(u-i)})$ (*independent* means that they are independent binary vectors in $\mathbb{F}_2^u$), we can generate a linear binary code $C^{(i)}$ as the following linear span:

$$C^{(i)} = \langle C^{(u)}, v^{(1)}, \dots, v^{(u-i)} \rangle.$$

Let A_{u-i} be the linear subspace of $\mathbb{F}_2^u$ generated by the syndromes $S(v^{(1)})$, …, $S(v^{(u-i)})$.

The dimension of the code $C^{(i)}$ is $\dim(C^{(i)}) = u - i + \dim(C^{(u)})$, where $\dim(C^{(u)}) = n - m - u$. Note that the maximum number of independent syndromes that we can take is u, so the biggest code that we can obtain is of dimension $u + \dim(C^{(u)}) = n - m$, which is the Hamming code $C^{(0)} = \mathcal{H}_m$. All the constructed codes contain $C^{(u)}$; at the same time, they are contained in the Hamming code $C^{(0)}$.

The number of different codes $C^{(u-i)}$ equals the number of subspaces of dimension i that we can take in $\mathbb{F}_2^u$, i.e., the Gaussian binomial coefficient

$$|\{C^{(u-i)}\}| = \begin{bmatrix} u \\ i \end{bmatrix}_2 = \frac{(2^u - 1)(2^u - 2)\dots(2^u - 2^{i-1})}{(2^i - 1)(2^i - 2)\dots(2^i - 2^{i-1})}.$$

The number of different nested families of codes between $C^{(u)}$ and $C^{(0)} = \mathcal{H}_m$ that we can construct equals

$$\prod_{i=0}^{u-1} (2^{u-i} - 1).$$

The following property was stated in [40] for the code $C^{(u)}$, but it can be extended to all the codes $C^{(i)}$, for $i \in \{1, \dots, u\}$.

Proposition 31. *For every $i \in \{1, \dots, u\}$, the cosets of weight three of $C^{(i)}$ are at distance three from each other, and $C^{(i)} \cup C^{(i)}(3)$ is the Hamming code.*

Recall that $C^{(i)*}$ is obtained from $C^{(i)}$ by extension and that the CT property implies the CR property.

Theorem 23 [39]. *We have the following*:

(i) *$C^{(1)}$ is a CT code with covering radius* 3;
(ii) *$C^{(u)}$ is a CT code with covering radius* 3;
(iii) *For every $i \in \{0, \dots, u\}$, the code $C^{(i)*}$ is CT when $C^{(i)}$ is CT*;
(iv) *For every $i \in \{1, \dots, u\}$, the code $C^{(i)}$ is a subcode of $C^{(i-1)}$, and $C^{(i)*}$ is a subcode of $C^{(i-1)*}$.*

(F.14) *For every $i \in \{0, \dots, u\}$, the code $C^{(i)}$ is CR with*

$$\mathrm{IA} = \{2^m - 1, 2^m - 2^{m-i}, 1; 1, 2^{m-i}, 2^m - 1\};$$

(F.15) *For every $i \in \{0, \dots, u\}$, the extended code $C^{(i)*}$ is CR with*

$$\mathrm{IA} = \{2^m, 2^m - 1, 2^m - 2^{m-i}, 1; 1, 2^{m-i}, 2^m - 1, 2^m\}.$$

Note that, for $i \in \{0, 1, u\}$, the codes $C^{(i)}$ and $C^{(i)*}$ are CT. Furthermore, computations for $m = 6$ have shown that all codes $C^{(i)}$ and $C^{(i)*}$ are CT. In [77] it was proved that distance-regular coset graphs coming from such codes are distance-transitive if any of the following divisibility conditions holds: 2^m is a power of 2^i, or $2^m = 2^i$, or $2^i - 1$ divides $2m$. Therefore, we conjecture that when one of these divisibility conditions is satisfied, the extended code $C^{(i)*}$ is a CT code too. Moreover, in such cases, we conjecture that $C^{(i)}$ is also CT. However, the question about complete transitivity of codes $C^{(i)}$ and $C^{(i)*}$ for the cases of $i \notin \{0, 1, u\}$ is open and needs more attention.

5.3. Preparata-like and BCH Codes

From [2, 24] we have the following CR codes.

(F.16) Any (binary) Preparata-like code, i.e., a code with parameters $(n = 2^{2m}-1, M = 2^{n+1-4m}, 5)$, where $m \geq 2$, is a CR code with covering radius $\rho = 3$ [2] and

$$\mathrm{IA} = \{n, n-1, 1; 1, 2, 3\}.$$

(F.17) An extended Preparata-like code, i.e., a code with parameters $(n+1 = 2^{2m}, M = 2^{n+1-4m}, 6)$ is a CR code with covering radius $\rho = 4$ [2] and

$$\mathrm{IA} = \{n+1, n, n-1, 1; 1, 2, 3, n+1\}.$$

(F.18) Primitive binary BCH $(n = 2^{2m+1}-1, N = 2^{n-4m}, 5; 3)$ codes, $m \geq 2$, are CR with

$$\mathrm{IA} = \{n, n-1, (n+3)/2; 1, 2, (n-1)/2\}.$$

(F.19) Extended primitive BCH $(n+1 = 2^{2m+1}, N = 2^{n-4m}, 6; 4)$ codes are CR with

$$\mathrm{IA} = \{n+1, n, n-1, (n+3)/2; 1, 2, (n-1)/2, n+1\}.$$

5.4. Lifting Alphabets of Hamming Codes

CR codes can be obtained by lifting the alphabet of Hamming codes. Denote by H^q_m the parity check matrix of the Hamming code $C = C(H^q_m)$ of length $n = (q^m-1)/(q-1)$ over $\mathbb{F}_q$. Define a new linear code, denoted by $C_r(H^q_m)$, of length n over $\mathbb{F}_{q^r}$, $r \geq 2$, with this parity check matrix H^q_m.

Theorem 24 [35]. *We have the following*:

(F.20) *The code* $C_r(H^q_m)$ *is a CR* $[n, n-m, 3; \rho]_{q^r}$ *code with* $\rho = \min\{r, m\}$ *and intersection numbers*

$$b_i = \frac{(q^r-q^i)(q^m-q^i)}{(q-1)}, \quad i = 0, \ldots, \rho-1; \quad c_i = q^{i-1}\frac{q^i-1}{q-1}, \quad i = 1, \ldots, \rho.$$

When $r \neq m$, *the codes* $C_r(H^q_m)$ *and* $C_m(H^q_r)$ *are not equivalent but have the same intersection array.*

Note that Hamming codes are the only codes whose lifting gives CR codes.

Theorem 25 [35]. *Let* $C(H^q)$ *be a nontrivial code of length* n *over the field* $\mathbb{F}_q$ *with minimum distance* $d \geq 3$ *and covering radius* $\rho \geq 1$, *and let* $C_r(H^q)$ *be its lifting over* $\mathbb{F}_{q^r}$. *Then* $C_r(H^q)$ *is CR if and only if the initial code* $C(H^q)$ *is a Hamming code.*

According to Theorem 25, the codes obtained by lifting extended perfect codes never give CR codes. However, all these codes are UP in the wide sense [35].

The next statement generalizes results of [24, 33] to the nonbinary case.

Proposition 32 [35]. *Let* C *be the* q*-ary* $[n, n-m, 3]$ *Hamming code,* $n = (q^m-1)/(q-1)$, *and let* C^* *be its extended code. The code* C^* *is UP if and only if it has minimum distance* 4. *In other words, the code* C^* *is UP if and only if either* $q = 2$ *with* $m \geq 2$ *or* $q = 2^u \geq 4$ *with* $m = 2$.

Theorem 26 [35]. *Let* C *be the* $[n, n-m, 3]_q$ *Hamming code of length* $n = (q^m-1)/(q-1)$, *and let* C^* *be the extended code. The lifted code* C^*_r *is a UP code if and only if* C^* *is such. Hence, the lifted code* C^*_r *is a UP code if and only if either* $q = 2$ *with* $m \geq 2$ *or* $q = 2^u \geq 4$ *with* $m = 2$.

5.5. CR Codes by Kronecker Product

In [34] a Kronecker construction of CR codes has been investigated; later, in [78], the construction has been extended to the case of different alphabets in the component codes. This new construction was found to be connected with the lifting constructions of CR codes considered in the preceding subsection. Note the following interesting fact. The new construction yields a growing number of CR codes with different parameters (and over different alphabets) but with identical intersection arrays (i.e., all resulting distance-regular graphs defined on cosets of these CR codes are isomorphic, though being constructed from different CR codes with different parameters).

Recall that the Kronecker product of two matrices $A = [a_{r,s}]$ over $\mathbb{F}_{q^u}$ and B over $\mathbb{F}_q$ is a new matrix $H = A \otimes B$ obtained by replacing every element $a_{r,s}$ in A by the matrix $a_{r,s}B$.

Theorem 27 [34, 78]. *Let $C(H^{q^u}_{m_a})$ and $C(H^q_{m_b})$ be two Hamming codes with parameters $[n_a, n_a - m_a, 3]_{q^u}$ and $[n_b, n_b - m_b, 3]_q$, respectively, where q is a prime power,*

$$n_a = (q^{um_a} - 1)/(q^u - 1), \qquad n_b = (q^{m_b} - 1)/(q - 1),$$

$m_a, m_b \geq 2$, and $u \geq 1$.

(F.21) *The code $C = C(H)$ with parity check matrix $H = H^{q^u}_{m_a} \otimes H^q_{m_b}$, the Kronecker product of $H^{q^u}_{m_a}$ and $H^q_{m_b}$, is a CT and hence CR code with parameters $[n, k, d; \rho]_{q^u}$, where*

$$n = n_a n_b, \qquad k = n - m_a m_b, \qquad d = 3, \qquad \rho = \min\{um_a, m_b\}, \tag{8}$$

and with intersection numbers

$$b_\ell = \frac{(q^{u\,m_a} - q^\ell)(q^{m_b} - q^\ell)}{(q-1)}, \quad \ell = 0, 1, \ldots, \rho - 1,$$
$$c_\ell = q^{\ell-1}\frac{q^\ell - 1}{q - 1}, \quad \ell = 1, 2, \ldots, \rho.$$

The lifted code $C_{m_b}(H^q_{um_a})$ is CR with the same intersection array as C.

Note that in Theorem 27 above we cannot take the code $C_{m_b}(H^{q^u}_{m_a})$ (instead of $C_{m_b}(H^q_{um_a})$), which seems to be natural. We emphasize that the codes $C_{m_b}(H^q_{um_a})$ and $C_{m_b}(H^{q^u}_{m_a})$ are not only different CR codes, but they induce different distance-regular graphs with different intersection arrays. Thus, the code $C_{m_b}(H^q_{um_a})$ suits the codes in the family (F.21) in the sense that it has the same intersection array. For example, the code $C_2(H^{2^2}_3)$ induces a distance-regular graph with intersection array $\{315, 240; 1, 20\}$, and the code $C_2(H^2_6)$ gives a distance-regular graph with intersection array $\{189, 124; 1, 6\}$. Note that, to obtain these results, we use the same Theorem 24 in both cases.

Theorem 27 cannot be extended to a more general case where the alphabets $\mathbb{F}_{q^a}$ and $\mathbb{F}_{q^b}$ of the component codes C_A and C_B, respectively, are fields of arbitrary orders and the same characteristic. We illustrate this by considering the following smallest nontrivial example. Take two Hamming codes, the $[5, 3, 3]$ code C_A over $\mathbb{F}_{2^2}$ with parity check matrix $H^{2^2}_2$, and the $[9, 7, 3]$ code C_B over $\mathbb{F}_{2^3}$ with parity check matrix $H^{2^3}_2$. Then the resulting $[45, 41, 3]$ code $C = C(H^{2^2}_2 \otimes H^{2^3}_2)$ over $\mathbb{F}_{2^6}$ is not even a UP code in the wide sense, since it has covering radius $\rho = 3$ and external distance $s = 7$, which can easily be checked by considering the parity check matrix of C.

Theorem 28 [78]. *Let q be any prime number, and let a, b, and u be any natural numbers. Then there exist the following CR codes with different parameters $[n, k, d; \rho]_{q'}$, where q' is a power of q, the minimum distance is $d = 3$, and the covering radius is $\rho = \min\{ua, b\}$:*

(F.22) $C_{ua}(H_b^q)$ *over* $\mathbb{F}_{q^{ua}}$ *of length* $n = \dfrac{q^b-1}{q-1}$, $k = n - b$;

(F.23) $C_b(H_{ua}^q)$ *over* $\mathbb{F}_{q^b}$ *of length* $n = \dfrac{q^{ua}-1}{q-1}$, $k = n - ua$;

(F.24) $C(H_b^q \otimes H_{ua}^q)$ *over* $\mathbb{F}_q$ *of length* $n = \dfrac{q^b-1}{q-1}\dfrac{q^{ua}-1}{q-1}$, $k = n - bua$;

(F.25) $C(H_b^q \otimes H_u^{q^a})$ *over* $\mathbb{F}_{q^a}$ *of length* $n = \dfrac{q^b-1}{q-1}\dfrac{q^{ua}-1}{q^a-1}$, $k = n - bu$;

(F.26) $C(H_b^q \otimes H_a^{q^u})$ *over* $\mathbb{F}_{q^u}$ *of length* $n = \dfrac{q^b-1}{q-1}\dfrac{q^{ua}-1}{q^u-1}$, $k = n - ba$;

All the above codes have the same intersection numbers:

$$b_\ell = \frac{(q^b - q^\ell)(q^{ua} - q^\ell)}{(q-1)}, \quad \ell = 0, \ldots, \rho - 1; \quad c_\ell = q^{\ell-1}\frac{q^\ell - 1}{q-1}, \quad \ell = 1, \ldots, \rho.$$

All of the above codes that come from Kronecker constructions (*i.e., the families* (F.24)–(F.26)) *are CT.*

Denote by $\tau(n)$ the number of different divisors of n.

Corollary 14 [78]. *Given a prime power* q, *choose any two natural numbers* $a, b > 1$. *Then we can build* $\tau(a) + \tau(b)$ *different CR codes with an identical intersection array and covering radius* $\rho = \min\{a, b\}$:

(i) *CT codes* $C(H_r^{q^{r^*}} \otimes H_b^q)$ *over* $\mathbb{F}_{q^{r^*}}$, *for any proper divisor* r *of* a, *with* $rr^* = a$;

(ii) *CT codes* $C(H_a^q \otimes H_r^{q^{r^*}})$ *over* $\mathbb{F}_{q^{r^*}}$, *for any proper divisor* r *of* b, *with* $rr^* = b$;

(iii) *CR codes* $C_a(H_b^q)$ *over* $\mathbb{F}_{q^a}$ *and CR codes* $C_b(H_a^q)$ *over* $\mathbb{F}_{q^b}$.

This construction gives also UP codes in the wide sense that are not CR. Let $R_t = [I_{t-1} \,|\, \mathbf{1}]$, where I_{t-1} is a binary diagonal matrix of order $t-1$, and $\mathbf{1}$ is the all-one column vector.

Theorem 29 [78]. *Let* $C(H_m^{q^u})$ *be the* q^u*-ary* $[n, k, 3]_{q^u}$ *Hamming code of length* $n_a = (q^{um} - 1)/(q^u - 1)$, *and let* $C(R_{n_b})$ *be the* $[n_b, 1, n_b]_q$ *repetition code, where* $4 \le n_b \le (q^u - 1)n_a + 1$, q *is a prime power,* $u \ge 1$, *and* $m \ge 2$.

(i) *The code* $C = C(H_m^{q^u} \otimes R_{n_b})$ *is a* q^u*-ary UP* (*in the wide sense*) $[n, k, d]_{q^u}$ *code with covering radius* $\rho = n_b - 1$ *and parameters*

$$n = n_a n_b, \qquad k = n - m(n_b - 1), \qquad d = 3. \tag{9}$$

(ii) *The code* C *is not CR.*

5.6. Binomial CR Codes

Denote by $H^{(m,\ell)}$ the binary matrix of size $m \times \dbinom{m}{\ell}$ whose columns are all different vectors of length m and weight ℓ. Define the binary linear code $C^{(m,\ell)}$, which we call binomial, with parity check matrix $H^{(m,\ell)}$.

Theorem 30 [79]. *Let* m *and* ℓ *be two natural numbers such that* $2 \le \ell \le m - 2$. *The code* $C^{(m,\ell)}$ *is CT* (*and therefore CR*) *in exactly the following four cases*:

(F.27) *For any* $m \ge 4$, *the code* $C^{(m,2)}$ *is an* $[n, k, d; \rho]$ *code with parameters*

$$n = \binom{m}{2}, \qquad k = n - m + 1, \qquad d = 3, \qquad \rho = \lfloor m/2 \rfloor$$

and the following intersection numbers: *for* $i = 0, \ldots, \rho$,

$$a_i = 2i(m - 2i), \qquad b_i = \binom{m-2i}{2}, \qquad c_i = \binom{2i}{2};$$

(S.15) *The code* $C^{(5,3)}$ *is a* $[10, 5, 4; 3]$ *code with*

$$\mathrm{IA} = \{10, 9, 4; 1, 6, 10\};$$

(S.16) *The code* $C^{(6,4)}$ *is a* $[15, 10, 3; 3]$ *code with*

$$\mathrm{IA} = \{15, 8, 1; 1, 8, 15\};$$

(S.17) *The code* $C^{(7,4)}$ *is a* $[35, 29, 3; 2]$ *code with*

$$\mathrm{IA} = \{35, 16; 1, 20\}.$$

In fact, from the codes with $\ell = 2$ we have some more new CR codes. First, we divide these codes into two families [79]. Denote $C^{(m)} = C^{(m,2)}$.

Theorem 31 [36, 79]. *Let* m *be a natural number*, $m \geq 3$. *The code* $C^{(m)}$ *is antipodal if* m *is odd and nonantipodal if* m *is even.*

Since for even m the code $C^{(m)}$ is nonantipodal, its covering set $C^{(m)}(\rho)$ is a translate of $C^{(m)}$ (Theorem 6). Consider the new (linear) code $C^{[m]} = C^{(m)} \cup C^{(m)}(\rho)$. The generator matrix $G^{[m]}$ of this code has a highly symmetric structure:

$$G^{[m]} = \left[\begin{array}{c|c} I_{k-1} & H^t_{m-1} \\ \hline 0\ldots0 & 1\ldots1 \end{array}\right],$$

where H^t_{m-1} is the transpose of $H^{(m-1,2)}$. Using that $C^{(m)}(\rho) = C^{(m)} + (1, 1, \ldots, 1)$ (Theorem 6), we obtain the following.

Theorem 32 [36]. *Let* m *be even*, $m \geq 6$.

(F.28) *The code* $C^{[m]}$ *is CT* (*and hence CR*) *with parameters*

$$n = m(m-1)/2, \qquad k = n - m + 2, \qquad d = 3, \qquad \rho = \lfloor m/4 \rfloor$$

and the following intersection numbers: *for* $m \equiv 0 \pmod 4$ *and* $\rho = m/4$,

$$b_i = \binom{m-2i}{2}, \qquad c_i = \binom{2i}{2}, \quad i = 0, 1, \ldots, \rho - 1, \qquad c_\rho = 2\binom{2\rho}{2},$$

and for $m \equiv 2 \pmod 4$ *and* $\rho = (m-2)/4$,

$$b_i = \binom{m-2i}{2}, \qquad c_i = \binom{2i}{2}, \quad i = 0, 1, \ldots, \rho.$$

5.7. CR Codes by Direct Sum Construction

Let C_1 and C_2 be two codes, not necessarily linear, of the same length n. The direct sum of C_1 and C_2 is a code defined as follows:

$$C_1 \oplus C_2 = \{(c_1, c_2) \mid c_1 \in C_1,\ c_2 \in C_2\}.$$

Theorem 33 [6, 24]. *Let r be any positive integer, and let C_i, $i = 1, 2, \ldots, r$, be q-ary CR $(n, N, d; 1)_q$ codes with the same intersection array $(b_0; c_1)$.*

(F.29) *Then for any $r \geq 1$ their direct sum*

$$C = C_1 \oplus \ldots \oplus C_r$$

is a CR $(nr, N^r, d; r)_q$ code with intersection numbers

$$a(r)_i = n(q-1) - (r-i)b_0 - ic_1, \qquad b(r)_i = (r-i)b_0, \qquad c(r)_i = ic_1,$$

$i = 0, 1, \ldots, r$. Moreover, if all codes C_i are CT and permutation equivalent to each other, then C is also CT.

We note that the construction described in Theorem 33 was considered in [6] for the particular case where all codes C_i are binary perfect codes.

5.8. CR Codes from Combinatorial Configurations

The following constructions are known.

(F.30) *One Latin square codes.* For any $q \geq 2$ a q-ary MDS $(3, q^2, 2; 1)_q$ code is CR with

$$\mathrm{IA} = \{3(q-1); 3\}.$$

In this case the set $C(\rho)$ is the rest of $\mathbb{F}^3$, i.e., $C(\rho) = F^3 \setminus C$.

(F.31) *Two Latin square codes.* For any $q \geq 3$ and $q \neq 6$ a q-ary MDS $(4, q^2, 3; 2)_q$ code is CR with

$$\mathrm{IA} = \{4(q-1), 3(q-3); 1, 12\}.$$

(S.18) *Three Latin square code.* Three mutually orthogonal Latin squares of order 4 form an equidistant $[5, 2, 4; 3]_4$ code (i.e., any two different codewords are at distance d from each other). This code is CR with

$$\mathrm{IA} = \{15, 12, 3; 1, 4, 15\}.$$

The punctured $[4, 2, 3]_4$ code obtained from the above code by deleting any single position is also a CR code and it belongs to the family (F.31).

(S.19) *Four Latin squares code.* Four mutually orthogonal Latin squares of order 5 form an equidistant $[6, 2, 5; 3]_5$ code. This code is CR with

$$\mathrm{IA} = \{24, 20, 13; 1, 2, 6\}.$$

The $[5, 2, 4]_5$ code $C^{(i)}$ obtained by puncturing the above code is not CR. The subset C_4 is not a 2-design. But the $[4, 2, 3; 2]_5$ code obtained by double puncturing is CR and belongs to the family (F.31).

(S.20) *The Hadamard code.* The unique (up to equivalence) Hadamard matrix of order 12 induces a binary CR $(11, 24, 5; 3)$ code H (see [4]) with

$$\mathrm{IA} = \{11, 10, 3; 1, 2, 9\}.$$

In [37] it was proved that H is a CT code and was also shown that H is unique in the class of CR codes with $(n, d) = (11, 5)$ (see Proposition 8).

(S.21) *Extended Hadamard code.* The extension H^* of the code H considered above is a $(12, 24, 6; 4)$ code. According to [33], H^* is UP and therefore a CR code with

$$\mathrm{IA} = \{12, 11, 10, 3; 1, 2, 9, 12\}.$$

In [37] it was established that H^* is also a CT code and is unique in the class of CR codes with $(n, d) = (12, 6)$ (see Proposition 8).

(F.32) *Johnson scheme in Hamming scheme.* For any natural w, $w \geq 1$, the trivial constant weight $(n, N, d; \rho)$ code (or the Johnson scheme $J(n, w)$) with parameters

$$n = 2w, \qquad N = \binom{2w}{w}, \qquad d = 2, \qquad \rho = w$$

is CR with

$$\mathrm{IA} = \{2w, 2w-1, 2w-2, \ldots, 1; w+1, w+2, \ldots, 2w\}.$$

Clearly, this code is CT, since its automorphism group is the full symmetric group S_n. In Section 4 we discuss CR codes in Johnson schemes. Thus, the Johnson scheme $J(2w, w)$ is CR and CT in the Hamming scheme $H(2, 2w)$.

5.9. CR Codes by Concatenation Constructions

In this subsection we collect some results from [80] dealing with CR codes constructed using concatenations of Hamming codes.

Let H be a cyclic parity check matrix of a q-ary Hamming code of length $n = (q^m - 1)/(q - 1)$ (i.e., we assume that $\gcd(n, q - 1) = 1$). Thus, the simplex code generated by H is also a cyclic code. For any $c \in \{2, \ldots, n\}$, consider the code C with parity check matrix

$$\begin{bmatrix} H & H & \ldots & H \\ H_1 & H_2 & \ldots & H_c \end{bmatrix}, \tag{10}$$

where H_i is the matrix H after cyclically shifting its columns i times to the right.

(F.33) The code C with the parity check matrix given in (10) is a CR code with parameters $[nc, nc - 2m, 3; 2]_q$ and intersection array

$$\mathrm{IA} = \{(q-1)nc, ((q-1)n - c + 2)(c-1); 1, c(c-1)\}.$$

Almost all codes described in (F.33) are not CT. However, in the binary case and for any value of m, i.e., for any length $n = 2^m - 1$, the CT codes are those with $c \in \{2, 3, n-1, n\}$. In the q-ary case, the CT codes are those with $c = 2$.

By extending the codes given in (F.33), we do not obtain CR codes, except for the binary case with $c = 2^{m-1} + 1$. In this case, the resulting extended $[n(2^{m-1} + 1) + 1, n(2^{m-1} + 1) - 2m, 4; 3]$ code coincides with the codes of the family (F.35) described below.

Define now a code $A^{(c)}$ of length $n(c + 3)$ with the parity check matrix

$$H_a(c) = \begin{bmatrix} H & 0 & H & H & \ldots & H \\ 0 & H & H & H_1 & \ldots & H_c \end{bmatrix}, \tag{11}$$

where 0 denotes the zero matrix (of the same size as H). In [80] it was proved that all codes $A^{(c)}$ are CR.

(F.34) For $c \le n-1$, the code $A^{(c)}$ with the parity check matrix given in (11) is a CR code with parameters $[(c+3)n, (c+3)n-2m, 3; 2]_q$ and intersection array

$$\mathrm{IA} = \{(c+3)(q-1)n, (c+2)((q-1)n-1-c); 1, (c+2)(c+3)\}.$$

In the binary case with $c = n-1$, the code $A^{(n-1)}$ coincides with the $[2^{2m}-1, 2^{2m}-1-2m, 3; 1]$ Hamming code.

Almost all codes in the family (F.34) are not CT. However, in the binary case and for any value of $m > 2$, the codes $A^{(c)}$ are CT for $c \in \{2^m-5, 2^m-4, 2^m-3\}$.

Extensions of the codes $A^{(c)}$ given in (F.34) lead to new CR codes only in the binary case for two values of c, namely $c \in \{2^{m-1}-2, 2^m-2\}$.

(F.35) Let $A^{(c)}$ be a code with the parity check matrix $H_a(c)$, and let $A^{(c)*}$ be its extended code. For $c = 2^{m-1}-2$, the code $A^{(c)*}$ is a $[(c+3)n+1, (c+3)n-2m, 4; 3]$ CR code with

$$\mathrm{IA} = \{(c+3)n+1, (c+3)n, 2^{2m-2}; 1, (c+2)(c+3), (c+3)n+1\}.$$

For $c = 2^m-2$, the code $A^{(c)*}$ is a $[(c+3)n+1, (c+3)n-2m, 4; 2]$ CR code which coincides with the extended $[2^{2m}, 2^{2m}-2m, 4; 2]$ Hamming code. For $c \notin \{2^{m-1}-2, 2^m-2\}$, the code $A^{(c)*}$ is not CR.

Let m and c be natural numbers such that $1 \le c \le n-1 = 2^m-2$. Let $C = A^{(n-1)}$, and let H_c be its parity check matrix. The code C is a Hamming code of length $2^{2m}-1$. Let $B^{(n-1-c)}$ denote the code with parity check matrix (10) after removing the columns in $H_a(c)$. Let $H_b(n-1-c)$ denote the parity check matrix of the code $B^{(n-1-c)}$. The above three codes $A^{(c)}$, $B^{(n-1-c)}$, and C have lengths $n_a = (c+3)(2^m-1)$, $n_b = (n-1-c)(2^m-1)$, and $n_c = n_a + n_b = 2^{2m}-1$, respectively. Therefore, the matrix H_c can be presented as follows:

$$H_c = \Big[H_a(c) \mid H_b(n-1-c)\Big].$$

The code $B^{(n-1-c)}$ is equivalent to the code with parity check matrix as in (10), where we replace c with $n-1-c$. Therefore, the codes $A^{(c)}$ and $B^{(n-1-c)}$ are CR. Moreover, both codes are either CT or not simultaneously.

Let $\mathrm{Aut}(B)$ be the automorphism group of the code $B^{(n-1-c)}$ and $\mathrm{Aut}(A)$ the automorphism group of the code $A^{(c)}$, for every $1 \le c \le n-1$. Then the groups $\mathrm{Aut}(A)$ and $\mathrm{Aut}(B)$ are isomorphic [81].

Let $\mathcal{C}_t$ be the cyclic group of order t, $\mathcal{S}_t$ the symmetric group of order t, and let $\mathrm{GL}(m,2)$ denote the binary general linear group. In [81] it was obtained that

- $\mathrm{Aut}(B) = \mathrm{GL}(m,2) \times \mathrm{GL}(m,2) \times \mathcal{C}_2$ *for* $n-1-c = 2$;
- $\mathrm{Aut}(B) = \mathrm{GL}(m,2) \times \mathcal{S}_3$ *for* $n-1-c = 3$;
- *The codes* $A^{(c)}$ *and* $B^{(n-1-c)}$ *are CT codes if and only if* $n-1-c \in \{1,2,3\}$.

Note that in the case of $n-1-c = 1$ the code $B^{(c)}$ is a Hamming code of length 2^m-1.

To conclude this section, we give a few sporadic examples of CR codes constructed by using concatenation methods.

(S.22) Let $H = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ be the parity check matrix of the binary Hamming code of length 3; denote by H_1 (respectively, H_2) the matrices obtained by one cyclic shift of the columns of $H = H_0$ (respectively, by two cyclic shifts). The binary $[15, 9, 3; 3]$-code C with the parity check matrix

$$\begin{bmatrix} H & 0 & 0 & H & H \\ 0 & H & 0 & H & H_1 \\ 0 & 0 & H & H & H_2 \end{bmatrix}$$

is CR with

$$\mathrm{IA} = \{15, 12, 1; 1, 4, 15\}.$$

(S.23) The binary $[16, 9, 4; 4]$-code obtained by extension of the above-mentioned $[15, 9, 3]$ code is also CR with

$$\mathrm{IA} = \{16, 15, 12, 1; 1, 4, 15, 16\}.$$

Denote by $D(u, q)$ a difference matrix (see [25]), i.e., a square matrix of the order qu over an additive group of order q such that the componentwise difference of any two rows contains any element of the group exactly u times. Take the difference matrix $D = D(2, 3)$:

$$D = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 2 & 1 \\ 0 & 1 & 0 & 1 & 2 & 2 \\ 0 & 2 & 1 & 0 & 1 & 2 \\ 0 & 2 & 2 & 1 & 0 & 1 \\ 0 & 1 & 2 & 2 & 1 & 0 \end{bmatrix}.$$

(S.24) Let H be a binary (12×18) matrix obtained from D by replacing each element i with the matrix H_i (which is defined in (S.22)). Then the $[18, 12, 3; 2]$ code with parity check matrix H is a CR code with $\mathrm{IA} = \{18, 15; 1, 6\}$.

(S.25) Use the same construction as in (S.24) for the matrix D^*, which is the difference matrix $D(2, 3)$ without the trivial column. The resulting $[15, 9, 3; 3]$ code is CR with $\mathrm{IA} = \{15, 12, 1; 1, 4, 15\}$. This code coincides with the code described in (S.22).

5.10. q-ary Linear CR Codes with $\rho = 1$

Linear q-ary CR codes with $\rho = 1$ are completely classified in [82] by using the following two simple constructions.

Construction $I(u)$. Let C be an $[n, k, d]_q$ code with a parity check matrix H. Define a new code C^{+u} with parameters $[n + u, k + u, 1]_q$ as the code with the parity check matrix H^{+u} obtained by adding $u > 0$ zero columns to H.

Theorem 34 [82]. *The codes C and C^{+u} have the same covering radius; moreover, C^{+u} is CR if and only if C is CR. In this case, both codes have the same intersection numbers $b_i = b'_i$ and $c_i = c'_i$, i.e.,*

$$a'_i = a_i + (q-1)u, \qquad b'_i = b_i, \qquad c'_i = c_i, \quad i = 0, 1, \ldots, \rho$$

(here a_i, b_i, c_i are the intersection numbers of C).

Construction $II(\ell)$. Let C be an $[n, k, d]_q$ code with a parity check matrix H. Let $C^{\times \ell}$ be the code with parameters $[n\,\ell, k + (\ell - 1)n, 2]_q$, whose parity check matrix, denoted by $H^{\times \ell}$, is the ℓ-times repetition of H (or matrices monomially equivalent to H), i.e.,

$$H^{\times \ell} = \left[H^{(1)} \,|\, H^{(2)} \,|\, \ldots \,|\, H^{(\ell)} \right],$$

where, for all $i = 1, \ldots, \ell$, $H^{(i)}$ is a parity check matrix of a code equivalent to C.

Theorem 35 [82]. *An $[n, k, d]_q$ code $C^{\times \ell}$ is CR with covering radius $\rho = 1$ if and only if C is CR with covering radius $\rho' = 1$.*

Now for linear CR codes with $\rho = 1$ we have the following result.

Theorem 36 [82]. *Let $C = C(H)$ be a nontrivial $[n, k, d]_q$ code with covering radius $\rho = 1$ and with parity check matrix H.*

(F.36) *The code* C *is an* $[n, n-m, d; 1]_q$ *CT (and CR) code, where* $n = n_m\ell + u$ *and* $n_m = (q^m-1)/(q-1)$, *if and only if the matrix* H *is of the following form (up to monomial equivalence):*

$$H = \left((H_m^q)^{\times \ell}\right)^{+u},$$

where H_m^q *is a parity check matrix of a Hamming code of length* n_m *over* $\mathbb{F}_q$.

Furthermore,

$d = 3$ *if and only if* $u = 0$, $\ell = 1$, $n = n_m$, *and* C *is a Hamming code, in which case these codes are included in the family* (F.1);

(F.37) $d = 2$ *if and only if* $u = 0$, $\ell \geq 2$, $n = n_m\ell$;

(F.38) $d = 1$ *if and only if* $u > 0$, $\ell \geq 1$.

In all the cases the code C *has*

$$\mathrm{IA} = \{(q-1)\ell\, n_{n-k}; \ell\}$$

Similar result was also obtained in [13] in terms of arithmetic CR codes.

5.11. q-ary Nonlinear CR Codes with $\rho = 1$

A coset of a linear CR code with $\rho = 1$ obviously gives a nonlinear such code with the same $\rho = 1$. Apart from these trivial codes, very little is known for this case. There are some results for the binary case, mostly due to Fon-Der-Flaas [83–85]. An equivalent definition for the construction of CR codes with a given ρ is the so-called *perfect* $(\rho + 1)$*-coloring of a hypercube*. Especially simple perfect coloring are defined for the case $\rho = 1$, i.e., for 2-colorings. Let $H(2, n)$ be a binary hypercube of dimension n (or a binary Hamming scheme). Its vertices are binary vectors of length n, and two vertices are neighbors if the corresponding vectors are at distance 1 from each other. A coloring of its vertices into (usually) white and black colors is called a perfect 2-coloring with intersection matrix

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

if every black vertex has a black and b white neighbors, and every white vertex has c black and d white neighbors. Clearly, $a + b = c + d = n$. In our terminology,

$$a_0 = a, \qquad b_0 = b, \qquad c_1 = c, \qquad a_1 = d;$$

hence, this is a nonlinear CR code with $\mathrm{IA} = \{b; c\}$. The following result gives a lower bound for the value $a = n - b$ (this is the best known bound for the *correlation immunity*; see references in [83]).

Theorem 37 [83]. *Let* C *be a binary CR code of length* n *and* $\rho = 1$ *with* $\mathrm{IA} = \{b; c\}$. *If* $b \neq c$, *then*

$$c - a \leq \frac{n}{3}. \tag{12}$$

Another necessary condition is the following result from [84].

Theorem 38 [84]. *Let* C *be a binary CR code of length* n *with* $\rho = 1$ *and intersection array* $\{b; c\}$. *Then* $b \neq 0 \neq c$, *and*

$$\frac{b + c}{\gcd(b, c)} \text{ is a power of } 2, \tag{13}$$

where $\gcd(b, c)$ *is the greatest common divisor.*

Both constructions for linear codes with $\rho = 1$ considered in the preceding subsection can be used for nonlinear codes too. The following statement generalizes the corresponding results from [82] for the nonlinear case and from [84] for the nonbinary case.

Proposition 33. *We have the following*:

(i) *For every* $n = (q^m - 1)/(q-1)$, $m \geq 2$, *and any* k, $1 \leq k \leq (q-1)n$, *there exists a* q*-ary CR code* C *with* $\rho = 1$ *and* $\mathrm{IA} = \{(q-1)n - k + 1; k\}$ *formed by* k *arbitrary translates of* q*-ary perfect code of length* n *and minimum distance* $d = 3$;

(ii) *The existence of a* q*-ary CR code* C *of length* n *with* $\mathrm{IA} = \{b; c\}$ *implies the existence of a CR code* C^{+k} *with* $\mathrm{IA} = \{b + k(q-1); c + k(q-1)\}$, *for any* $k \geq 1$, *formed by replacing every codeword* $c \in C$ *with* q^k *codewords of the form* $(c \,|\, x)$ *where* x *runs over* $\mathbb{F}_q^n$;

(iii) *The existence of a CR code* C *of length* n *with* $\mathrm{IA} = \{b; c\}$ *implies the existence, for any* $k \geq 1$, *of a CR code* $C^{\times k}$ *with* $\mathrm{IA} = \{(kb; kc\}$. *Instead of every codeword* $v = (v_1, \ldots, v_n) \in C$, *we take all vectors of the form*

$$(v_{1,1}, v_{1,2}, \ldots, v_{1,k}, \ldots, v_{1,1}, v_{1,2}, \ldots, v_{1,k}) \in \mathbb{F}_q^{kn}$$

such that

$$\sum_{j=1}^{k} v_{i,j} = v_i.$$

Certainly, codes that meet the bound given in (12) are the most interesting. Such codes of length $n = 3u$ should have $\mathrm{IA} = \{3u - a; a + u\}$ where $a \geq 0$. Two infinite families of codes with such intersection arrays are known (see [83] and references therein): $\{3k; k\}$ and $\{5k; 3k\}$. These families are obtained from codes with intersection arrays $\{3; 1\}$ and $\{5; 3\}$ by applying Proposition 33. The first code is the trivial binary perfect code of length 3, and the second code of length 6 (constructed by Tarannikov [86]) can be constructed from the first using the two following lemmas due to Fon-Der-Flaass [84].

Lemma 4 [84]. *Let* C *be a CR code with* $\rho = 1$ *and intersection matrix*

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Assume that C *can be partitioned into disjoint* k*-faces*, $0 \leq k \leq a$. *Then there exists a CR code with* $\rho = 2$ *and intersection matrix*

$$\begin{bmatrix} a-k & a+k & 2b \\ a+k & a-k & 2b \\ c & c & 2d \end{bmatrix}.$$

Lemma 5 [84]. *Let* C *be a CR code with* $\rho = 2$ *and intersection matrix*

$$\begin{bmatrix} a-k & a+k & 2b \\ a+k & a-k & 2b \\ c & c & 2d \end{bmatrix},$$

where $c \geq a + k$. *Then there exists a CR code with* $\rho = 1$ *and intersection matrix*

$$\begin{bmatrix} a-k & 2b+c \\ c & 2d+2c-a-k \end{bmatrix}.$$

In connection with condition (13), a natural question arises [84]: *Find the value $a^*(b,c)$ for all pairs b,c satisfying condition* (13). Here by $a^*(b,c)$ we mean the minimum integer a such that there exists a CR code with IA $= \{b;c\}$ if and only if $a \geq a^*(b,c)$. There are lower and upper bounds for the value $a^*(b,c)$ (see [84] and references therein). The next statements give the best known such bounds.

Theorem 39 [84]. *We have the following lower bounds*:

(i) *If* $c < b < 2c$, *then*

$$a^*(b,c) \geq \frac{1}{2} + \sqrt{c(b-c) + \frac{1}{4}} - (b-c);$$

(ii) *If* $2c < b < 2c + \sqrt{3c-2}$, *then* $a^*(b,c) \geq 1$.

For given positive integers x and y such that $x + y = 2^k - 1$ with one of them being odd with ℓ consecutive ones in its binary expansion (where $1 \leq \ell < k$), define the function $z(x,y) = k - 1$.

Theorem 40 [84]. *We have the following*:

(i) $a^*(2b+c,c) \leq \max(0, a^*(b,c) - 1)$;

(ii) *Let b and c satisfy condition* (13). *Then*

$$a^*(c,b) = a^*(b,c) + b - c;$$

(iii) *Let b and c satisfy condition* (13) *and* $b \geq c$. *Let* $m = \gcd(b,c)$, $b = m(2b'+1)$, *and* $c = m(2c'+1)$. *If* $c' = 0$, *then* $a^*(b,c) = 0$. *Otherwise*,

$$a^*(b,c) \leq \max(0, c - m(z(b',c') + 1)).$$

An optimal binary CR code of length $n = 12$ with IA $= \{9;7\}$ meeting the bound (12) was constructed in [85]. In the same paper it was also proved that a putative CR code of length $n = 12$ with IA $= \{11;5\}$ does not exist.

5.12. q-ary Linear CR Codes with $\rho = 2$

According to Delsarte [3] (see Theorem 3), the dual code of any linear two-weight code should be a CR code with $\rho = 2$. There are many different families of such codes, and their classification is not completed (see [87] for an overview of such codes).

The classification of linear CR codes with covering radius $\rho = 1$ allows to describe the structure of linear CR codes with covering radius $\rho = 2$ whose dual codes are antipodal.

Theorem 41 [82]. *Let $C = C(H)$ be a nontrivial $[n,k,d]_q$ code. Then C is CR with covering radius $\rho = 2$, and the dual code $C^{\perp}$ is antipodal if and only if its parity check matrix H is, up to equivalence, of the following form*:

$$H = \begin{bmatrix} 1 & \dots & 1 \\ & M & \end{bmatrix},$$

where M generates an equidistant code E with minimum distance d' satisfying the following property: *for any nonzero codeword $v \in E$, every symbol $\alpha \in \mathbb{F}_q$ which occurs in a coordinate position of v occurs in this codeword exactly $n - d'$ times. Moreover, up to equivalence, C is an extension of a CR code C' with covering radius $\rho' = 1$.*

From [82] we can list known CR codes with covering radius $\rho = 2$ whose dual codes are antipodal. Let C_1 and C_2 be linear q-ary codes of the same cardinality, with generator matrices G_1 and G_2, where q is any prime power. We say that C_1 and C_2 are *complementary codes* if the matrix

$G = [G_1 \,|\, G_2]$ (up to permutation of rows of G_2) generates the *simplex code*, i.e., a commonly known code with the following parameters:

$$n = (q^m - 1)/(q - 1), \qquad k = m, \qquad d = q^{m-1}$$

(which is the dual of the q-ary Hamming code $\mathcal{H}_m$ of length n).

Corollary 15 [82]. *The following codes, whose dual codes are antipodal, are CR with covering radius* $\rho = 2$:

(F.39) *The binary extended perfect* $[n, k, 4; 2]_2$ *code* $\mathcal{H}^*_m$ *of length* $n = 2^m$, *where* $k = n - m - 1$ *and* $m \geq 2$, *with*

$$\mathrm{IA} = \{n, n-1; 1, n\};$$

(F.40) *The extended q-ary perfect* $[n, k, 4; 2]_q$ *code* $\mathcal{H}^*_m$ *of length* $n = q + 2$ *with* $k = q - 1$, *where* $q = 2^r \geq 4$, *and* $m = 2$ [88, 89] (*the family* TF1 *in* [87]) *with*

$$\mathrm{IA} = \{(q+2)(q-1), q^2 - 1; 1, q+2\};$$

(F.41) *The* $[n, k, 3; 2]_q$ *code dual to the code with difference matrix* $D_m = D(n/q, q)$ (*see the definition after the code* (S.23)), *with*

$$\mathrm{IA} = \{n(q-1), n-1; 1, n(q-1)\}.$$

It has length $n = q^m$, *dimension* $k = n - (m+1)$, *and parity check matrix* D_m, *where* $m \geq 1$ *and* $q \geq 3$ *is any prime power* (*all codes of such type generated by the matrix* D_m *have been constructed in* [90]). *The complementary code of this code is the Hamming code* $\mathcal{H}_m$;

(F.42) *The* $[n, n-2, 3; 2]_q$ *dual of an MDS code of length* n, *where* $3 \leq n \leq q$ *and* $q \geq 3$ *is any prime power* [89]. *The intersection array is*

$$\mathrm{IA} = \{n(q-1), (q-n+1)(n-1); 1, n(n-1)\};$$

(F.43) *An* $[n = q(q-1)/2,\ k = n-3,\ 4;\ 2]_q$ *code for* $q = 2^r \geq 4$ [89] *with*

$$\mathrm{IA} = \{(q-1)n, (q-2)(q+1)(q+2)/4; 1, q(q-1)(q-2)/4\}.$$

The complementary of this code belongs to the family $\mathrm{TF1}^d$, *i.e., it is the projective dual code of a code in the family* TF1 *in* [87];

(F.44) *A code with parameters* $[n = 1 + (q+1)(h-1), k = n-3, 4]_q$, *where* $1 < h < q$ *and* h *divides* q, *for* $q = 2^r \geq 4$ (*the family* TF2 *in* [87]), *with*

$$\mathrm{IA} = \{(q-1)n, (q+1)(h-1)(q-h+1); 1, (h-1)n\};$$

(F.45) *A code with parameters* $[n = q(q-h+1)/h, k = n-3, 4]_q$, *where* $1 < h < q$ *and* h *divides* q, *for* $q = 2^r \geq 4$, *with*

$$\mathrm{IA} = \{(q-1)n, (q+1)(q-h)(q(h-1)+h)/h^2; 1, q(q-h)(q-h+1)/h^2\}.$$

The complementary of this code belongs to the family $\mathrm{TF2}^d$ *in* [87].

Some codes of this kind are self-dual [82].

Theorem 42 [82]. *Let* $C' \subset \mathbb{F}^n_{q^r}$ *be the lifted code of a perfect* $[n, n-m, 3]_q$ *Hamming code. Then*

(F.46) *For any* r, *the code* C' *is CR. Moreover,* C' *is self-dual if and only if* $\mathcal{H}_m$ *is a ternary* $[4, 2, 3]_3$ *Hamming code.*

Theorem 43 [82]. *Let* $\{0, 1, \xi_2, \ldots, \xi_{q-1}\}$ *be the set of elements of the field* $\mathbb{F}_q$*, where* $q \geq 4$ *is any prime power. Let the matrix*

$$D_1 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & \xi_i & \xi_j \end{bmatrix}$$

be a parity check matrix for the code C *and a generator matrix for the code* $C^\perp$*, where* $\xi_i, \xi_j \in \mathbb{F}_q^*$ *are two different elements such that* $\xi_i + \xi_j + 1 = 0$*. Then*

(F.47) *The code* C*, as well as* $C^\perp$*, is a linear antipodal* $[4, 2, 3; 2]_q$ *CR code with*

$$\mathrm{IA} = \{4\,(q-1), 3\,(q-3); 1, 12\};$$

(F.48) *For the case* $q = 2^r \geq 4$*, these two equivalent codes coincide*: $C = C^\perp$*, i.e.,* C *is self-dual.*

We conclude this subsection with giving one more family of codes, whose dual are the well-known codes described as family SU1 in [87]. Let H_m be the parity check matrix of a q-ary Hamming code $\mathcal{H}_m$ of length $n = (q^m-1)/(q-1)$. For a given prime power q and natural numbers u and m with $u < m$, define a code $C_{u,m}$ whose parity check matrix is obtained from H_m by deleting all $(q^u-1)/(q-1)$ columns of the parity check matrix H_u. The code $C_{u,m}$ has parameters

$$[n = (q^m - q^u)/(q-1), k = (q^m - q^u)/(q-1) - m, d; 2]_q,$$

where

$$d = \begin{cases} 4 & \text{if } u = m-1,\ q = 2, \\ 3 & \text{otherwise.} \end{cases}$$

Theorem 44. *We have the following*:

(F.49) *For any prime power* q *and natural* $m > 3$ *and* $u < m$*, the code* $C_{u,m}$ *is CT and hence CR with* $\rho = 2$ *and*

$$\mathrm{IA} = \{q^m - q^u, q^u - 1; 1, q^m - q^u\}.$$

5.13. Binary Linear CR Codes with $\rho = 3$ *and* $\rho = 4$ *from Bent and AB Functions*

Let F be any function from $\mathbb{F}_2^m$ to $\mathbb{F}_2^m$. For any $(a, b) \in (\mathbb{F}_2^m)^2$, define the Fourier transform of F as

$$\mu_F(a, b) = \sum_{\boldsymbol{x} \in \mathbb{F}_2^m} (-1)^{\langle b \cdot F(x) \rangle + \langle a \cdot x \rangle}, \tag{14}$$

where $\langle \cdot \rangle$ is the inner product in $\mathbb{F}_2^m$.

For even m, a function F over $\mathbb{F}_2^m$ is a *bent function* if its Fourier transform is $\mu_F(a, b) = \pm 2^{m/2}$, for all $a, b \in \mathbb{F}_2^m$ with $b \neq 0$. For odd m, a function F over $\mathbb{F}_2^m$ is an *almost bent* (AB) *function* if its Fourier transform is $\mu_F(a, b) \in \{0, \pm 2^{(m-1)/2}\}$, for all $a, b \in \mathbb{F}_2^m$ with $b \neq 0$.

Let F be any function from $\mathbb{F}_q$ to $\mathbb{F}_q$ with $q = 2^m$ such that $F(0) = 0$. Define

$$\Omega_m = \begin{cases} \{2^{m-1}, 2^{m-1} \pm 2^{m/2}\}, & m \text{ even}, \\ \{2^{m-1}, 2^{m-1} \pm 2^{(m-1)/2}\}, & m \text{ odd}. \end{cases}$$

For a linear binary code C, define the set of integers W_C as a set of all weights of its nonzero codewords:

$$W_C = \{\mathrm{wt}(c) :\ c \in C,\ c \neq \mathbf{0}\}.$$

For any function F define the matrix

$$H_F = \begin{bmatrix} 1 & \alpha & \alpha^2 & \ldots & \alpha^{n-1} \\ F(1) & F(\alpha) & F(\alpha^2) & \ldots & F(\alpha^{n-1}) \end{bmatrix}. \tag{15}$$

The statements of the next theorem can be found in [91].

Theorem 45 [91]. *For a given function F, let $C = C_F$ be the $[n = 2^m - 1, k, d]$ code defined by the parity check matrix H_F of the form* (15)*.*

(i) *If m is odd, the function F is AB if and only if $W_{C^\perp} = \Omega_m$;*
(ii) *If m is even, the function F is bent if and only if $W_{C^\perp} = \Omega_m$;*
(iii) *The code C_F is UP in the wide sense if and only if $|W_{C^\perp}| = 3$;*

(F.50) *For m even, C_F is CR if F is bent;*

(F.51) *For m odd, C_F is CR if F is AB.*

Now, using the corresponding results from [38, 39], we obtain the following.

Proposition 34. *Let a code $C = C_F$ of length $n = 2^m - 1$, where m is odd, be defined by the parity check matrix* (15)*, and let $C^\perp$ be its dual code with the set of weights $W_{C^\perp}$. Then*

(i) *C^* is UP if and only if C is UP and $W_{C^\perp} = \Omega_m$;*
(ii) *C^* is CR if and only if C is CR with minimum distance $d \in \{3, 5\}$ and C^* is UP.*

Proof. The first statement comes directly from [33]. For the second statement, the case $d = 3$ is known [39]. Consider the case $d = 5$. Since C is CR, its covering radius is $\rho = 3$. Hence, C is quasi-perfect. Now the result follows from Proposition 4. △

(F.52) Taking a power function $F(x) = x^\ell$ in (15) for m odd, we obtain binary primitive cyclic $[n = 2^m - 1, n - 2m, d; \rho]$ codes with generator polynomials $g(x) = m_1(x)m_\ell$. In the case $d = 5$ they are CR [91] with $\rho = 3$ and

$$\mathrm{IA} = \{n, n-1, (n+3)/2; 1, 2, (n-1)/2\}.$$

In the literature, we found the following cases:

(i) BCH codes with $d = 5$ of length $n = 2^{2m+1} - 1$ [24] (see Section 5.3), $\ell = 3$;
(ii) Gold codes [92], $\ell = 2^t + 1$, $\gcd(t, m) = 1$;
(iii) Kasami codes [93], $\ell = 2^{2t} - 2^t + 1$, $\gcd(t, m) = 1$;
(iv) Welch codes [94] (see also [95, 96]), $\ell = 2^{\frac{m-1}{2}} + 3$;
(v) Niho codes [97]:

$$\ell = \begin{cases} 2^t + 2^{t/2} - 1, & t \text{ even}, \\ 2^t + 2^{(3t+1)/2} - 1, & t \text{ odd}; \end{cases}$$

(vi) Inverse codes [98], $\ell = 2^{2t} - 1$, $m = 2t + 1$;
(vii) Dobbertin codes [99], $\ell = 2^{4t} + 2^{3t} + 2^{2t} + 2^t - 1$, $m = 5t$.

The existence of the aforementioned codes come from known AB functions. Some new AB functions which are not power functions and provide new CR codes can be found, e.g., in [100–102]. The question on complete transitivity of all these codes needs additional attention.

(F.53) The extended codes of all the above binary codes are CR with $\rho = 4$ and

$$\mathrm{IA} = \{n+1, n, n-1, (n+3)/2; 1, 2, (n-1)/2, n+1\}.$$

5.14. New UP Code of Length 11 and CR Code of Length 33

In [24], by computer search for UP codes, the following parameters were found for a putative family of quaternary UP codes:

$$q = 4, \qquad n = \frac{2^{2m+1} + 1}{3}, \qquad k = n - 2m - 1, \qquad d = 5, \qquad \mu = \lambda + 1 = \frac{2^{2m} - 1}{3},$$
$m = 2, 3, \ldots.$

Linear codes over $\mathbb{F}_4$ with these parameters (which are found to be not UP) were constructed in [103]. Later, a nonlinear additive 4-ary code over $\mathbb{F}_4$ with parameters $(12, 4^6, 6)_4$ was constructed, referred to as a *dodecacode* (see a reference in [104]). Puncturing this code at any position yields an $(11, 4^6, 5)_4$ code, which is found to be UP, thus answering a question posed in [24]. Following [104], we describe the construction of this code:

(S.26) Delete one position of the dodecacode. Let G be a generator matrix of the resulting code of length 11, and let α be a primitive element of $\mathbb{F}_4$. Now construct from G a binary matrix H by replacing each element of $\mathbb{F}_4$ with one of two binary vectors of length 3 as follows:

$$\begin{aligned} 0 &\to 0\,0\,0 \quad \text{or} \quad 1\,1\,1, \\ 1 &\to 1\,0\,0 \quad \text{or} \quad 0\,1\,1, \\ \alpha &\to 0\,0\,1 \quad \text{or} \quad 1\,1\,0, \\ \alpha^2 &\to 0\,1\,0 \quad \text{or} \quad 1\,0\,1. \end{aligned}$$

This means that every row in G gives rise to 2^{11} binary rows in H. In this way, the matrix H generates a linear binary $[33, 23, 3; 3]$ code, which is CR (but not CT) with

$$\text{IA} := \{33, 30, 15; 1, 2, 15\}.$$

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