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Exponentially Ramsey Sets[1]

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Abstract—We study chromatic numbers of spaces $\mathbb{R}_p^n = (\mathbb{R}^n, \ell_p)$ with forbidden monochromatic sets. For some sets, we for the first time obtain explicit exponentially growing lower bounds for the corresponding chromatic numbers; for some others, we substantially improve previously known bounds.

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1. INTRODUCTION

Problems addressed in the present paper originated from Nelson's famous question (see [1]) posed as early as in 1950: "What is the least number $\chi(\mathbb{R}^2)$ of colors sufficient for coloring points of a plain so that no two points at unit distance from each other have the same color?"

Though this question, at first glance, seems to be very simple, the answer is not presently known. It is only proved that

$$4 \le \chi(\mathbb{R}^2) \le 7 \tag{1}$$

(for a proof, see, e.g., [1])[2].

This classical Nelson's question admits numerous generalizations. For example, instead of coloring a plane one can try to color the n-dimensional Euclidean space $\mathbb{R}^n$ or even a metric space $\mathbb{R}_p^n$, which for all $p \ge 1$ (including $p = \infty$) is defined as a "usual" space $\mathbb{R}^n$ with a slightly "unusual" metric ℓ_p defines as follows:

$$\boldsymbol{x} = (x_1, \ldots, x_n),\ \boldsymbol{y} = (y_1,\ \ldots, y_n)\ \Rightarrow\ \ell_p(\boldsymbol{x}, \boldsymbol{y}) = \sqrt[p]{|x_1 - y_1|^p + \ldots + |x_n - y_n|^p}.$$

Many papers have been devoted to the study of the chromatic number $\chi(\mathbb{R}_p^n)$, and known facts that we need in the present paper are summarized in the following theorem.

Theorem 1. *The following four statements hold true*:

1. *For any* $p \ge 1$ *as* $n \to \infty$

$$\left(\frac{1+\sqrt{2}}{2} + o(1)\right)^n = (1.207\ldots + o(1))^n \le \chi(\mathbb{R}_p^n) \le (4 + o(1))^n;$$

2. *For* $p = 2$, $n \to \infty$,

$$(1.239\ldots + o(1))^n \le \chi(\mathbb{R}_2^n) \le (3 + o(1))^n;$$

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[2] While the paper was being prepared for publication, a note [2] appeared, where it was proved that $\chi(\mathbb{R}^2) \ge 5$.

3. *For* $p = 1$, $n \to \infty$,

$$\left(\frac{1+\sqrt{3}}{2} + o(1)\right)^n = (1{,}366\ldots + o(1))^n \le \chi(\mathbb{R}_1^n) \le (4 + o(1))^n;$$

4. *For* $p = \infty$ *and any* n *we have an explicit equality*

$$\chi(\mathbb{R}_\infty^n) = 2^n.$$

The proof of the lower and upper bounds of statement 1 can be found in [3,4], respectively; the proof of statement 2, in [5,6]; of statement 3, in [7, 4]; and the equality in statement 4 is classical, and its proof can be found, e.g., in [8].

Though we are already rather far from the original Nelson's question, the problem setting can be generalized further. The area which studies generalizations of this kind is referred to as *Euclidean Ramsey theory*. Numerous papers are devoted to this theory, among which we would like to mention the works [9–19]. In the framework of Euclidean Ramsey theory, instead of "forbidding" two points at unit distance from each other to have the same color, one may "forbid" other (more complicated) configurations to be monochromatic. Let us give a formal definition.

Let $\mathcal{X} = (X, d_X)$ and $\mathcal{Y} = (Y, d_Y)$ be arbitrary metric spaces. A subset $X' \subset X$ is called a *copy* of $\mathcal{Y}$ if there exists a bijection $f\colon Y \to X'$ such that for any $y_1, y_2 \in Y$ we have $d_Y(y_1, y_2) = d_X(f(y_1), f(y_2))$.

The *chromatic number* $\chi(\mathcal{X}; \mathcal{Y})$ of a space $\mathcal{X}$ with a forbidden space $\mathcal{Y}$ is defined as the minimum number of colors sufficient for coloring $\mathcal{X}$ in such a way that no copy $X' \subset X$ of $\mathcal{Y}$ is completely monochromatic. Of course, in this general setting one could hardly hope to obtain profound results, so in this paper we will consider $\mathbb{R}_p^n$ as a space $\mathcal{X}$ to be colored, though in the framework of Euclidean Ramsey theory the more particular case of $\mathcal{X} = \mathbb{R}_2^n$ is considered most often, which explains the name of the theory.

To illustrate this definition, note that if $\mathcal{Y}$ is a pair of points at a unit (or any other) distance from each other, then by definition $\chi(\mathbb{R}_p^n; \mathcal{Y}) = \chi(\mathbb{R}_p^n)$; i.e., our definition is indeed a generalization. As we have already seen in Theorem 1, in this case $\chi(\mathbb{R}_p^n; \mathcal{Y})$ for each $p \ge 1$ grows exponentially with n. It is found that this can be the case not only for two-point spaces $\mathcal{Y}$, which motivates the following definitions.

A metric space $\mathcal{Y}$ is said to be ℓ_p-*exponentially Ramsey* if there exists $c = c(p, \mathcal{Y}) > 1$ such that

$$\chi(\mathbb{R}_p^n; \mathcal{Y}) \ge (c + o(1))^n \tag{2}$$

as $n \to \infty$. The largest constant c with which inequality (2) is still valid will be referred to as $\chi_-(\ell_p; \mathcal{Y})$. Thus, more formally,

$$\chi_-(\ell_p; \mathcal{Y}) = \liminf_{n\to\infty} \chi(\mathbb{R}_p^n; \mathcal{Y})^{1/n}.$$

The quantity $\chi_+(\ell_p; \mathcal{Y})$ can be defined similarly, replacing the lim inf with lim sup.

Remark 1. Unfortunately, in all papers on this subject the term *exponentially Ramsey* is commonly referred to not a metric space $\mathcal{Y} = (Y, d_Y)$ but a set Y. Each time, of course, it is assumed that on a given Y there is defined some "natural" metric d_Y. For instance, on the "set of vertices of a regular triangle with unit side," a "natural" metric is assumed to be a function that assigns unit distance to each pair of distinct "vertices." On subsets of $\mathbb{R}_p^n$, a "natural" metric is assumed to be the metric ℓ_p induced from the "ambient" space.

Therefore, in this paper we will also sometimes speak about exponentially Ramsey sets, assuming, of course, exponentially Ramsey metric spaces.

Central questions that we would like to be able to answer for concrete p and $\mathcal{Y}$ are the following:
1. Is it true that $\mathcal{Y}$ is ℓ_p-exponentially Ramsey?
2. If so, what are the values of $\chi_-(\ell_p;\mathcal{Y})$ and $\chi_+(\ell_p;\mathcal{Y})$?

In the general case, answers to these questions are not yet known, but they are known in some particular cases. Concerning the first question, it is proved (see [9]) that in the case of $p=2$ necessary conditions for the exponential Ramsey property of $\mathcal{Y}$ is its finiteness and ℓ_2-sphericity (i.e., $\mathcal{Y}$ must be a finite subspace of an ℓ_2-sphere in some dimension). A popular conjecture says that these two conditions are not only necessary but also sufficient. However, at present the exponential Ramsey property is proved for a rather small number of sets, which will be listed immediately after we give some auxiliary definitions.

The *ℓ_p-Cartesian product* $\times_p$ of metric spaces $\mathcal{X}=(X,d_X)$ and $\mathcal{Y}=(Y,d_Y)$ is a metric space $\mathcal{X}\times_p\mathcal{Y}=(X\times Y,d)$ with metric d defined as follows:

$$\forall\, x_1,x_2\in X\quad \forall\, y_1,y_2\in Y\quad d((x_1,y_1),(x_2,y_2))=\sqrt[p]{d_X^p(x_1,x_2)+d_Y^p(y_1,y_2)}.$$

A *k-dimensional ℓ_p-rectangular parallelepiped* is the metric space which is the ℓ_p-Cartesian product of any k "segments" (two-point metric spaces).

Now we are in position to list all presently known exponentially Ramsey sets. In [13] it was shown that for any $p\ge 1$ the set of vertices of any ℓ_p-rectangular parallelepiped is ℓ_p-exponentially Ramsey. In the same paper, for $p=2$ there was established the ℓ_2-exponential Ramsey property for the set of vertices of any nondegenerate simplex. Besides that, it is clear that any subset of an exponentially Ramsey set is exponentially Ramsey itself. No other examples of exponentially Ramsey sets are known.

Note that for many other sets their *ℓ_p-Ramsey property* (weaker than the ℓ_p-exponential Ramsey property) is proved, which means that the corresponding chromatic numbers $\chi(\mathbb{R}_p^n;\mathcal{Y})$ tend to infinity as n grows. For instance, in [15] the ℓ_2-Ramsey property for an arbitrary regular k-gon is proved for any k. In [16], the ℓ_2-Ramsey property for an arbitrary trapezoid, and in [18], the ℓ_2-Ramsey property for the sets of vertices of an arbitrary regular polytope in any dimension are proved.

The second of the posed questions seems to be more complicated than the first, even if instead of finding exact values of $\chi_-(\ell_p;\mathcal{Y})$ and $\chi_+(\ell_p;\mathcal{Y})$ we are only interested in finding some estimates for them. For example, essentially there are only two results on lower bounds for $\chi_-(\ell_p;\mathcal{Y})$ (of course, we are speaking about "nontrivial" lower bounds, i.e., those greater than 1). The first of them is Theorem 1 formulated above, and to state the second, we need an extra notation.

For every integer $k\ge 1$ define a metric space $\mathcal{S}_k$ as the set of vertices of a regular k-dimensional simplex, i.e., $\mathcal{S}_k$ consists of $k+1$ points with the same distance between any two of them. Of course, this definition is the most natural in the ℓ_2 metric, since if we ask the question of what is the largest number of points in $\mathbb{R}_2^k$ with all pairwise distances the same, the answer is exactly $k+1$. From this point of view, in the ℓ_1 metric it would be natural to call $\mathcal{S}_{2k-1}$ a k-dimensional simplex, and in the ℓ_∞ metric, $\mathcal{S}_{2^k-1}$. However, we would not pay attention to this terminological problem.

So, the second result related to lower bounds on $\chi_-(\ell_p;\mathcal{Y})$ is the following theorem, finally proved in a series of works [19–25].

Theorem 2. *For every $p\ge 1$ the following two statements hold true*:
1. $\chi_-(\ell_p;\mathcal{S}_2)\ge 1.00085\ldots$;
2. $\forall\, k\ge 1\quad \chi_-(\ell_p;\mathcal{S}_k)\ge 1+\dfrac{1}{2^{2^{k+4}}}$.

Note that the exponential Ramsey property for the sets $\mathcal{S}_k$ was also known before this theorem, since $\mathcal{S}_k$ is clearly a subset of the set of vertices of some $(k+1)$-dimensional ℓ_p-rectangular parallelepiped for any p.

It is seen that the estimates in Theorem 2 are extremely close to 1. However, nothing better has been proved up to now. In this paper we not only considerably improve the estimates of Theorem 2 but also obtain a variety of absolutely new explicit bounds for other sets whose ℓ_p-exponential Ramsey property was earlier proved implicitly.

Formulations of our theorems are given in Section 2, and subsequent sections are devoted to their step-by-step proofs.

In conclusion we note that related problems of combinatorial geometry and Euclidean Ramsey theory were considered in [9, 26–32].

2. FORMULATION OF RESULTS

Here we present the main results of the paper. We have obtained lower bounds on $\chi_-(\ell_p; \mathcal{Y})$ for many fixed $\mathcal{Y}$ for all values of $p \geq 1$. These results are grouped into theorems according to the following principle. Theorem 3 contains bounds that are valid for all $p \geq 1$, and in this sense it is an analog of statement 1 of Theorem 1. Theorems 4–6, on one hand, contain stronger estimates, and on the other hand, are valid only for $p = 2$, 1, and ∞, respectively, and in this sense they are analogs of statements 2–4 of Theorem 1.

Note that the exponential Ramsey property of all sets $\mathcal{Y}$ mentioned in Theorems 3–6 was previously known, but no explicit exponentially growing estimates for the chromatic numbers were given other than Theorems 1 and 2. In this sense, our theorems not only fill this gap but also considerably improve previously known bounds from Theorem 2.

Before formulating our results, we need to formally describe spaces $\mathcal{Y}$ to be considered.

Let $0 < a_1 \leq a_2 \leq \ldots \leq a_k$ be arbitrary positive numbers. Let $\mathcal{I}(a_i)$ be a two-point metric space with the only nonzero distance a_i $(1 \leq i \leq k)$. For each $p \geq 1$ define $\mathcal{I}_p^k = \mathcal{I}_p^k(a_1, \ldots, a_k)$ to be the ℓ_p-rectangular parallelepiped with sides $a_1, \ldots, a_k$, i.e.,

$$\mathcal{I}_p^k = \mathcal{I}(a_1) \times_p \ldots \times_p \mathcal{I}(a_k).$$

We will also need various spaces corresponding to sets of triangle vertices. Let positive numbers a, b, c be such that $a \leq b \leq c \leq a + b$. Define a metric space $\mathcal{T} = \mathcal{T}(a, b, c)$ as a three-element set T with metric taking values a, b, and c at three pairs of distinct elements of T. A triangle $\mathcal{T}(a, b, c)$ well be called *ℓ_p-right* if $a^p + b^p = c^p$, and *ℓ_p-acute* if $a^p + b^p > c^p$.

We will also be interested in spaces $\mathcal{S}_k$ corresponding to sets of vertices of regular simplices, which we have already defined before formulating Theorem 2.

Finally, define the space $\mathcal{O}_k$ as the set of vertices of a k-dimensional cross-polytope, or, more formally, $\mathcal{O}_k = (\{x_1, \ldots, x_{2k}\}, d)$, where

$$d(x_i, x_j) = \begin{cases} 0 & \text{if } i = j, \\ 2 & \text{if } |i - j| = k, \\ \sqrt{2} & \text{otherwise.} \end{cases}$$

Note that we will be interested in cross-polytopes only in the context of the ℓ_2 metric, since in this case they together with the above-mentioned regular simplices $\mathcal{S}_k$ and hypercubes $\mathcal{I}_2^k(1, \ldots, 1)$ cover all regular polytopes in higher dimensions (starting from dimension 5).

Now we are ready to formulate the obtained theorems.

Theorem 3. *Let $p \geq 1$, and let $\mathcal{T}_r$ and $\mathcal{T}_a$ be an ℓ_p-right and ℓ_p-acute triangle, respectively. Then the following inequalities hold true*:

1. $\chi_-(\ell_p; \mathcal{I}_p^2) \geq 1.0428\ldots$;

2. $\chi_-(\ell_p; \mathcal{I}_p^3) \geq 1.0126\ldots$;
3. $\chi_-(\ell_p; \mathcal{I}_p^k) \geq 1 + \dfrac{1}{(2+o(1))^k}$ *as* $k \to \infty$;
4. $\chi_-(\ell_p; \mathcal{S}_2) \geq \chi_-(\ell_p; \mathcal{S}_3) \geq 1.0126\ldots$;
5. $\chi_-(\ell_p; \mathcal{S}_k) \geq 1 + \dfrac{1}{(2+o(1))^k}$ *as* $k \to \infty$;
6. $\chi_-(\ell_p; \mathcal{T}_r) \geq 1.0428\ldots$;
7. $\chi_-(\ell_p; \mathcal{T}_a) \geq 1.0126\ldots$.

After formulating the theorem, it is worth making several remarks. First, though "true" chromatic numbers from this theorem may depend on sides of the considered triangles and parallelepipeds, the obtained lower bounds do not depend on them at all. The same situation will be observed in the theorems below. Second, our method allows to write an explicit estimate for $\chi_-(\ell_p; \mathcal{I}_p^k)$ for each fixed k. However, we decided to do this only for $k = 2$ and 3 and then immediately pass to the "asymptotic" case. Third, it is seen that statements 4 and 5 of Theorem 3 provide a substantial improvement of Theorem 2 announced above.

Theorem 4. *Let* $p = 2$, *and let* $\mathcal{T}_r$ *and* $\mathcal{T}_a$ *be an* ℓ_2*-right and* ℓ_2*-acute triangle, respectively. Then the following inequalities hold true*:

1. $\chi_-(\ell_2; \mathcal{I}_2^2) \geq 1.0471\ldots$;
2. $\chi_-(\ell_2; \mathcal{I}_2^3) \geq 1.0136\ldots$;
3. $\chi_-(\ell_2; \mathcal{I}_2^4) \geq 1.00459\ldots$;
4. $\chi_-(\ell_2; \mathcal{I}_2^k) \geq 1 + \dfrac{1}{(2+o(1))^k}$ *as* $k \to \infty$;
5. $\chi_-(\ell_2; \mathcal{S}_2) \geq \chi_-(\ell_2; \mathcal{S}_3) \geq 1.0136\ldots$;
6. $\chi_-(\ell_2; \mathcal{S}_k) \geq 1 + \dfrac{1}{(2+o(1))^k}$ *as* $k \to \infty$;
7. $\chi_-(\ell_2; \mathcal{O}_3) \geq \chi_-(\ell_2; \mathcal{O}_4) \geq 1.00459\ldots$;
8. $\chi_-(\ell_2; \mathcal{O}_k) \geq 1 + \dfrac{1}{(2+o(1))^k}$ *as* $k \to \infty$;
9. $\chi_-(\ell_2; \mathcal{T}_r) \geq 1.0471\ldots$;
10. $\chi_-(\ell_2; \mathcal{T}_a) \geq 1.0136\ldots$.

Theorem 5. *Let* $p = 1$, *and let* $\mathcal{T}_r$ *and* $\mathcal{T}_a$ *be an* ℓ_1*-right and* ℓ_1*-acute triangle, respectively. Then the following inequalities hold true*:

1. $\chi_-(\ell_1; \mathcal{I}_1^2) \geq 1.07389\ldots$;
2. $\chi_-(\ell_1; \mathcal{I}_1^3) \geq 1.02188\ldots$;
3. $\chi_-(\ell_1; \mathcal{I}_1^k) \geq 1 + \dfrac{1}{(2+o(1))^k}$ *as* $k \to \infty$;
4. $\chi_-(\ell_1; \mathcal{S}_2) \geq \chi_-(\ell_1; \mathcal{S}_3) \geq 1.02188\ldots$;
5. $\chi_-(\ell_1; \mathcal{S}_k) \geq 1 + \dfrac{1}{(2+o(1))^k}$ *as* $k \to \infty$;
6. $\chi_-(\ell_1; \mathcal{T}_r) \geq 1.07389\ldots$;
7. $\chi_-(\ell_1; \mathcal{T}_a) \geq 1.02188\ldots$.

In the last theorem of this section we formulate analogous results for the case $p = \infty$. However, it was found that our estimates for $\chi_-(\ell_\infty; \mathcal{I}_\infty^k)$ have a very simple structure, so we do not need to consider the cases of low dimensions and growing dimension separately: if $p = \infty$, we can give a unified bound. The same concerns regular simplices $\mathcal{S}_k$. Furthermore, in the case $p = \infty$ there are no acute triangles at all, and right triangles degenerate into isosceles ones, so we decided to skip investigating the values of $\chi_-(\ell_\infty;\mathcal{T})$.

We should also note the fact that this theorem gives an *exact* value of the $\chi_-(\ell_\infty;\mathcal{S}_k)$, whereas in all other cases we only gave lower bounds. At present, no other exact values of $\chi_-(\ell_p;\mathcal{Y})$ are known.

Theorem 6. *Let $p=\infty$. Then the following two statements hold true*:

1. *Let among positive numbers $a_1,\ldots,a_k$ there be exactly s different ones. Then*
$$\chi_-(\ell_\infty;\mathcal{I}^k_\infty(a_1,\ldots,a_k))\ge 2^{1/s}.$$
In particular,
$$\chi_-(\ell_\infty;\mathcal{I}^k_\infty)\ge 2^{1/k}.$$

2. *For any integer $k\ge 1$ we have an exact equality*
$$\chi_-(\ell_\infty;\mathcal{S}_k)=2.$$

It is seen that the lower bound on $\chi_-(\ell_\infty;\mathcal{I}^k_\infty)$ given in Theorem 6 for large k is much better than the similar bounds from Theorems 3–5. Indeed,
$$2^{1/k}=1+\frac{\ln 2}{k}+O\left(\frac{1}{k^2}\right),$$
which tends to 1 much slower than
$$1+\frac{1}{(2+o(1))^k}.$$

The rest of the paper is organized as follows. In Section 3 we formulate and prove a rather general auxiliary proposition, which will be used in Sections 4–6 to successively prove Theorems 3–5, respectively. In Section 7 we slightly improve this proposition in a particular case, which allows us to prove Theorem 6.

3. AUXILIARY PROPOSITION

Here we prove a proposition which will be highly important to justify all other results of this paper and which, perhaps, can also be of independent interest. A weaker analog of this proposition was given as early as in [13], but the authors of that paper were only interested in proving that $\chi_-(\ell_p;\mathcal{Y})>1$ but not in obtaining maximally strong lower bounds, so computations in [13] were sometimes very rough. Here we tried to carry out all computations as accurately as possible, and so the reasoning has become much more complicated than its analog from [13]. But thanks to this, we managed to prove rather strong explicit bounds formulated in Section 2.

Before formulating and proving this auxiliary proposition, we give several definitions and lemmas. Let $\mathcal{X}=(X,d_X)$ and $\mathcal{Y}=(Y,d_Y)$ be arbitrary metric spaces. The *independence number* $\operatorname{Ind}(\mathcal{X};\mathcal{Y})$ is the largest n for which there exists an n-element subset $X'\subset X$ containing no copies of $\mathcal{Y}$. Denote the total number of distinct copies of $\mathcal{Y}$ in $\mathcal{X}$ by $C(\mathcal{X};\mathcal{Y})$.

Lemma 1. *Let $\mathcal{X}=(X,d_X)$ and $\mathcal{Y}=(Y,d_Y)$ be finite metric spaces, $|Y|>1$. Let $|X|=x$ and $\operatorname{Ind}(\mathcal{X};\mathcal{Y})=y$. Then*
$$C(\mathcal{X};\mathcal{Y})\ge kx-\frac{k(k+1)}{2}y,$$
where $k=\left\lfloor\dfrac{x}{y}\right\rfloor$.

Proof. Let $W_1 \subset X$ be the largest subset containing no copies of $\mathcal{Y}$, $|W_1| = w_1$. By the condition we know that $w_1 \le y$. By adding to W_1 any new point, we find a copy of $\mathcal{Y}$. Since this new point can be added in $x - w_1$ ways, we thus can find $x - w_1 \ge x - y$ different copies of $\mathcal{Y}$.

Now let $W_2 \subset X \backslash W_1$ be the largest subset containing no copies of $\mathcal{Y}$, $|W_2| = w_2 \le y$. Proceeding in the same way as above, using this set we can find $x - w_1 - w_2 \ge x - 2y$ new copies of $\mathcal{Y}$.

This process can be repeated at least $k = \left\lfloor \dfrac{x}{y} \right\rfloor$ times to find thereupon at least

$$(x - y) + (x - 2y) + \ldots + (x - ky) = kx - \frac{k(k+1)}{2}y$$

different copies of $\mathcal{Y}$. △

To estimate the number of copies of $\mathcal{Y}$ in $\mathcal{X}$ obtained in Lemma 1, it is convenient to introduce a short notation. Let

$$f(x; y) = \left\lfloor \frac{x}{y} \right\rfloor x - \frac{\left\lfloor \frac{x}{y} \right\rfloor \left(\left\lfloor \frac{x}{y} \right\rfloor + 1 \right)}{2} y.$$

For any tuple $\boldsymbol{z} = (z_1, \ldots, z_n)$ of nonnegative integers, define

$$f(\boldsymbol{z}; y) = \sum_{i=1}^{n} f(z_i; y).$$

Lemma 2. *Let a tuple $\boldsymbol{x} = (x_1, \ldots, x_n)$ minimize the function $f(\cdot\,; y)$ over all tuples of nonnegative integers $\boldsymbol{z} = (z_1, \ldots, z_n)$ such that $z_1 + \ldots + z_n = x_1 + \ldots + x_n$. Put $N = x_1 + \ldots + x_n$. Then the following inequalities hold true*:

1. $|x_i - x_j| \le y$, *for all* $1 \le i \le j \le n$;
2. $\left| x_i - \frac{N}{n} \right| \le y$, *for all* $1 \le i \le n$.

Proof. Assume that statement 1 is wrong. Then without loss of generality we may assume that $x_1 - x_2 > y$. Consider the tuple $\boldsymbol{x}' = (x_1 - y, x_2 + y, x_3, \ldots, x_n)$. One can easily check that

$$f(\boldsymbol{x}; y) - f(\boldsymbol{x}'; y) = f(x_1; y) - f(x_1 - y; y) + f(x_2; y) - f(x_2 + y; y) = x_1 - x_2 - y > 0,$$

which contradicts the minimality of $\boldsymbol{x}$ and thus completes the proof of statement 1.

Let us show that statement 2 is a rather trivial consequence of the former. Indeed, for each i there exists j such that x_i and x_j lie on different sides of the arithmetic mean of all coordinates, which equals $\dfrac{N}{n}$. Hence,

$$\left| x_i - \frac{N}{n} \right| \le |x_i - x_j| \le y,$$

and Lemma 2 is completely proved. △

Following [13], introduce one more definition. Let $\mathcal{A} = (A, d_A)$ be a finite metric space consisting of at least two points, and let $p \ge 1$ and $F_{\mathcal{A}} \ge \chi_{\mathcal{A}} > 1$ be arbitrary numbers. The space $\mathcal{A}$ will be called $(\ell_p; F_{\mathcal{A}}, \chi_{\mathcal{A}})$-*super-Ramsey* if there exists a sequence of sets $V_{\mathcal{A}}(n) \subset \mathbb{R}^n$ such that

$$|V_{\mathcal{A}}(n)| \le (F_{\mathcal{A}} + o(1))^n \quad \text{and} \quad \frac{|V_{\mathcal{A}}(n)|}{\operatorname{Ind}(\mathcal{V}_{\mathcal{A}}(n); \mathcal{A})} \ge (\chi_{\mathcal{A}} + o(1))^n \quad \text{as} \quad n \to \infty, \tag{3}$$

where the metric space $\mathcal{V}_{\mathcal{A}}(n)$ is defined as the set $V_{\mathcal{A}}(n)$ with the metric induced from the ambient space $\mathbb{R}_p^n$.

In fact, there is nothing unnatural in this definition. One can easily check that $\chi(\mathbb{R}_p^n; \mathcal{A}) \ge (\chi_{\mathcal{A}} + o(1))^n$ and therefore $\chi_-(\ell_p; \mathcal{A}) \ge \chi_{\mathcal{A}}$. Thus, any $(\ell_p; F_{\mathcal{A}}, \chi_{\mathcal{A}})$-super-Ramsey set is in the same time ℓ_p-exponentially Ramsey. At present, no other ways are known to prove the exponential Ramsey property of a set.

For each $(\ell_p; F_{\mathcal{A}}, \chi_{\mathcal{A}})$-super-Ramsey set $\mathcal{A}$, define

$$c_{\mathcal{A}} = \limsup_{n\to\infty} \left(\frac{C(\mathcal{V}_{\mathcal{A}}(n);\mathcal{A})}{|V_{\mathcal{A}}(n)|}\right)^{1/n};$$

i.e., $c_{\mathcal{A}}$ is the smallest constant c for which $C(\mathcal{V}_{\mathcal{A}}(n);\mathcal{A}) \le |V_{\mathcal{A}}(n)|(c+o(1))^n$.

Lemma 3. *Let $\mathcal{A}$ be $(\ell_p; F_{\mathcal{A}}, \chi_{\mathcal{A}})$-super-Ramsey. Then $c_{\mathcal{A}} \ge \chi_{\mathcal{A}}$.*

Proof. The desired inequality easily follows from the fact that

$$C(\mathcal{V}_{\mathcal{A}}(n);\mathcal{A}) \ge |V_{\mathcal{A}}(n)|(\chi_{\mathcal{A}}+o(1))^n,$$

which, in turn, results from applying Lemma 1 to inequality (3). △

Now we are in position to formulate and prove our auxiliary proposition.

Proposition. *Let finite metric spaces $\mathcal{A} = (A, d_A)$ and $\mathcal{B} = (B, d_B)$ consist of a and b points, respectively. Let also for some $p \ge 1$ the former be $(\ell_p; F_{\mathcal{A}}, \chi_{\mathcal{A}})$-super-Ramsey and the latter be $(\ell_p; F_{\mathcal{B}}, \chi_{\mathcal{B}})$-super-Ramsey. Then $\mathcal{A} \times_p \mathcal{B}$ is also $(\ell_p; F, \chi)$-super-Ramsey with some values of the parameters F and χ. These values can be chosen, e.g., as follows:*

$$F = F_{\mathcal{A}}^{1-\beta} F_{\mathcal{B}}^{\beta}, \qquad \chi = \chi_{\mathcal{B}}^{\beta}, \qquad \textit{where} \quad \beta = \frac{\ln \chi_{\mathcal{A}}}{\ln \chi_{\mathcal{A}} + \ln \chi_{\mathcal{B}} + \ln c_{\mathcal{B}}}.$$

Proof. Since Lemma 3 implies that $c_{\mathcal{B}} > 1$, it is easily seen that $0 < \beta < 1$. Consider an arbitrary natural-valued function $\beta(n)$ satisfying the limit relation $\lim\limits_{n\to\infty} \dfrac{\beta(n)}{n} = \beta$. Set $\alpha(n) = n - \beta(n)$. Clearly, $\lim\limits_{n\to\infty} \dfrac{\alpha(n)}{n} = \alpha$, where $\alpha = 1-\beta$.

Let $\mathfrak{A} = V_{\mathcal{A}}(\alpha(n))$ and $\mathfrak{B} = V_{\mathcal{B}}(\beta(n))$. Put $V = \mathfrak{A} \times \mathfrak{B} \subset \mathbb{R}^n$. Clearly, $|V| \le (F+o(1))^n$, where F is specified in the assertion of the proposition. Let W be the largest subset of V which nevertheless contains no copy of $\mathcal{A} \times_p \mathcal{B}$. It is clear that $|W| = \mathrm{Ind}((V, \ell_p); \mathcal{A} \times_p \mathcal{B})$. If we also have

$$\frac{|V|}{|W|} \ge (\chi + o(1))^n, \tag{4}$$

then the proposition is proved.

Assume that inequality (4) is wrong and arrive at a contradiction. Since (4) is wrong, we have

$$\liminf_{n\to\infty} \left(\frac{|V|}{|W|}\right)^{1/n} < \chi = \chi_{\mathcal{B}}^{\beta},$$

which in turn implies that for some $\delta > 0$, for infinitely many n we have the inequality

$$\frac{|V|}{|W|} \le (\chi_{\mathcal{B}} - \delta + o(1))^{\beta(n)},$$

or equivalently

$$|W| \ge \frac{|V|}{(\chi_{\mathcal{B}} - \delta + o(1))^{\beta(n)}}. \tag{5}$$

In the rest of the proof, to obtain a contradiction, we will consider only values of n for which inequality (5) is valid.

For each $\mathfrak{a} \in \mathfrak{A}$, let $W_{\mathfrak{a}} = W \cap (\{\mathfrak{a}\} \times \mathfrak{B})$. Denote by $w_{\mathfrak{a}}$ the number of copies of $\mathcal{B}$ contained in $W_{\mathfrak{a}}$. Then for each set $\mathfrak{b} \subset \mathfrak{B}$ which is a copy of $\mathcal{B}$ denote by $w^{\mathfrak{b}}$ the number of layers $W_{\mathfrak{a}}$ containing $\mathfrak{b}$.

We will obtain a contradiction by counting in two ways the quantity

$$S = \sum_{\mathfrak{a} \in \mathfrak{A}} w_{\mathfrak{a}} = \sum_{\mathfrak{b} \subset \mathfrak{B}} w^{\mathfrak{b}}.$$

One one hand, since by our assumption W contains no copies of $\mathcal{A} \times_p \mathcal{B}$, we in particular have

$$w^{\mathfrak{b}} \le \operatorname{Ind}((\mathfrak{A}, \ell_p); \mathcal{A}) \le \frac{|\mathfrak{A}|}{(\chi_{\mathcal{A}} + o(1))^{\alpha(n)}}.$$

Furthermore, by the definition $\mathfrak{B}$ contains at most $|\mathfrak{B}|(c_{\mathcal{B}} + o(1))^{\beta(n)}$ copies of $\mathcal{B}$. Hence,

$$S = \sum_{\mathfrak{b} \subset \mathfrak{B}} w^{\mathfrak{b}} \le \frac{|\mathfrak{A}||\mathfrak{B}|(c_{\mathcal{B}} + o(1))^{\beta(n)}}{(\chi_{\mathcal{A}} + o(1))^{\alpha(n)}}. \tag{6}$$

On the other hand, to estimate S from below, we may invoke Lemmas 1 and 2. As follows from Lemma 1,

$$w_{\mathfrak{a}} \ge f(|W_{\mathfrak{a}}|; \operatorname{Ind}((\mathfrak{B}, \ell_p); \mathcal{B})) \ge f\left(|W_{\mathfrak{a}}|; \frac{|\mathfrak{B}|}{(\chi_B + o(1))^{\beta(n)}}\right),$$

and therefore

$$S = \sum_{\mathfrak{a} \in \mathfrak{A}} w_{\mathfrak{a}} \ge S' = \sum_{\mathfrak{a} \in \mathfrak{A}} f\left(|W_{\mathfrak{a}}|; \frac{|\mathfrak{B}|}{(\chi_B + o(1))^{\beta(n)}}\right). \tag{7}$$

To estimate S' from below, we may use Lemma 2, which states that in the worst case

$$\left| |W_{\mathfrak{a}}| - \frac{|W|}{|\mathfrak{A}|} \right| \le \frac{|\mathfrak{B}|}{(\chi_{\mathcal{B}} + o(1))^{\beta(n)}}, \quad \text{for all } \mathfrak{a} \in \mathfrak{A}.$$

Taking into account (5), we obtain

$$|W_{\mathfrak{a}}| \ge \frac{|\mathfrak{B}|}{(\chi_{\mathcal{B}} - \delta + o(1))^{\beta(n)}}, \quad \text{for all } \mathfrak{a} \in \mathfrak{A}.$$

Finally, applying Lemma 1 again, we obtain a lower bound for S':

$$S' \ge |\mathfrak{A}| f\left(\frac{|\mathfrak{B}|}{(\chi_{\mathcal{B}} - \delta + o(1))^{\beta(n)}}; \frac{|\mathfrak{B}|}{(\chi_{\mathcal{B}} + o(1))^{\beta(n)}}\right) = |\mathfrak{A}||\mathfrak{B}| \frac{(\chi_{\mathcal{B}} + o(1))^{\beta(n)}}{(\chi_{\mathcal{B}} - \delta + o(1))^{2\beta(n)}}. \tag{8}$$

Taking into account inequality (7), the obtained estimate (8) is at the same time a lower bound for S.

Let us show that inequalities (6) and (8) contradict each other for large n. Indeed, if we assume both to be valid, then after eliminating similar terms we find

$$\frac{(c_{\mathcal{B}} + o(1))^{\beta(n)}}{(\chi_{\mathcal{A}} + o(1))^{n-\beta(n)}} \ge \frac{(\chi_{\mathcal{B}} + o(1))^{\beta(n)}}{(\chi_{\mathcal{B}} - \delta + o(1))^{2\beta(n)}}.$$

Extracting the nth root and passing to the limit, we obtain

$$\frac{c_{\mathcal{B}}^{\beta} \chi_{\mathcal{A}}^{\beta}}{\chi_{\mathcal{A}}} \ge \frac{\chi_{\mathcal{B}}^{\beta}}{(\chi_{\mathcal{B}} - \delta)^{2\beta}} > \frac{1}{\chi_{\mathcal{B}}^{\beta}},$$

or equivalently

$$\chi_{\mathcal{A}} < (\chi_{\mathcal{A}} \chi_{\mathcal{B}} c_{\mathcal{B}})^{\beta}.$$

But this is impossible, since by the choice of β both sides of the inequality are the same. △

Remark 2. Sometimes, and even for rather explicit constructions of $V_{\mathcal{B}}(n)$, it is difficult to compute the exact value of $c_{\mathcal{B}}$. However, to apply the proposition, we need not know the exact value of $c_{\mathcal{B}}$, an upper bound will suffice. As such an upper bound, we may always take $c_{\mathcal{B}} \le F_{\mathcal{B}}^{b-1}$, as follows from the chain of inequalities

$$\frac{C(\mathcal{V}_{\mathcal{B}}(n);\mathcal{B})}{|V_{\mathcal{B}}(n)|} \le \frac{C_{|V_{\mathcal{B}}(n)|}^{|B|}}{|V_{\mathcal{B}}(n)|} \le \frac{|V_{\mathcal{B}}(n)|^b}{|V_{\mathcal{B}}(n)|} \le (F_{\mathcal{B}}^{b-1}+o(1))^n.$$

Remark 3. Let $V_{\mathcal{B}}(n)$ be a "homogeneous" set, i.e., let each element $v \in V_{\mathcal{B}}(n)$ be contained in the same number $C_{\deg}(n)$ of copies of $\mathcal{B}$. Also, let

$$c_{\deg} = \limsup_{n\to\infty} \sqrt[n]{C_{\deg}(n)}.$$

Then $c_{\mathcal{B}} = c_{\deg}$. This easily follows from the fact that

$$\frac{C(\mathcal{V}_{\mathcal{B}}(n);\mathcal{B})}{|V_{\mathcal{B}}(n)|} = \frac{C_{\deg}(n)}{b}.$$

4. CASE OF ARBITRARY p. PROOF OF THEOREM 3

To prove Theorem 3, we will use the auxiliary proposition in some particular case. Let $p \ge 1$. Let $\mathcal{I} = \mathcal{I}(s)$ be a pair of points at distance s from each other. To verify the super-Ramsey property of $\mathcal{I}(s)$, sets of the following special form are usually employed.

Let $0 < y < x \le \dfrac{1}{2}$ be arbitrary parameters. Choose natural-valued functions $x(n)$ and $y(n)$ obeying the following four properties:

1. $\displaystyle\lim_{n\to\infty} \frac{x(n)}{n} = x$;
2. $\displaystyle\lim_{n\to\infty} \frac{y(n)}{n} = y$;
3. $\forall\, n \;\; y(n) \le x(n) \le \dfrac{n}{2}$;
4. $x(n) - y(n)$ is a prime.

Existence of such functions for each pair of parameters x and y follows from the fact that for some positive function $\varphi(n) = o(n)$ as $n \to \infty$ it is true that for each n in the interval $(n; n+\varphi(n))$ there exists a prime number (see, e.g., [33, 34]).

Set

$$t(n) = s(2x(n) - 2y(n))^{-1/p}$$

and define $V_{\mathcal{I}}(n) \subset \mathbb{R}^n$ as the set of all points with exactly $x(n)$ coordinates equal to $t(n)$, the other $n - x(n)$ coordinates being zero.

By estimating $\mathrm{Ind}((V_{\mathcal{I}}(n), \ell_p); \mathcal{I})$ with the help of the so-called *linear algebraic method* (see [35]), one can show that $\mathcal{I}$ is $(\ell_p; c_1^x, \chi(x,y))$-super-Ramsey, where

$$c_a^b = \frac{a^a}{b^b(a-b)^{(a-b)}}, \qquad \chi(x,y) = \frac{c_1^{\min(x,2x-2y)}}{c_1^{x-y}}.$$

This statement is a reformulation of the Frankl–Wilson theorem from [3] and Ponomarenko's theorem from [36]. The most convenient formulation of Ponomarenko's theorem is given in [37]. On tightness of the Frakl–Wilson and Ponomarenko theorems, see [38–41].

Before applying the auxiliary proposition in the case $\mathcal{B} = \mathcal{I}(s)$, we have to find the value of $c_{\mathcal{I}}$. In our situation this can be done using Remark 3. Indeed, it is easily seen that each vertex of $V_{\mathcal{I}}(n)$ is at distance s from exactly

$$C_{x(n)}^{y(n)} C_{n-x(n)}^{x(n)-y(n)} = (c_x^y c_{1-x}^{x-y} + o(1))^n$$

other vertices, where the last equality is obtained in a standard way from Stirling's formula.

Thus, in our case the proposition looks as follows.

Theorem 7. *Let $\mathcal{A} = (A, d_A)$ be a finite metric space which for some $p \geq 1$ is $(\ell_p; F_{\mathcal{A}}, \chi_{\mathcal{A}})$-super-Ramsey. Let $\mathcal{I}(s)$ be a pair of points at distance s from each other. Then $\mathcal{A} \times_p \mathcal{I}(s)$ is also $(\ell_p; F, \chi)$-super-Ramsey with some values of the parameters F and χ. These values can be chosen, e.g., as follows*:

$$F = F_{\mathcal{A}}^{1-\beta} (c_1^x)^\beta, \qquad \chi = (\chi(x,y))^\beta, \qquad \beta = \frac{\ln \chi_{\mathcal{A}}}{\ln \chi_{\mathcal{A}} + \ln \chi(x,y) + \ln c_x^y + \ln c_{1-x}^{x-y}},$$

where values of the auxiliary parameters $0 < y < x \leq \dfrac{1}{2}$ can be chosen arbitrarily.

To verify the first inequality of Theorem 3, it suffices to apply Theorem 7 to the case

$$\mathcal{A} = \mathcal{I}(s'), \qquad \chi_{\mathcal{A}} = \frac{1+\sqrt{2}}{2}, \qquad x = 0.1400874, \qquad y = \frac{x}{2}.$$

The fact that this $\chi_{\mathcal{A}}$ is admissible follows from the fact that

$$\frac{1+\sqrt{2}}{2} = \chi\left(\frac{2-\sqrt{2}}{2}, \frac{2-\sqrt{2}}{4}\right).$$

(In fact, we have pointed out the global maximum of $\chi(\cdot\,,\cdot)$, but now this is not important for us.) Applying Theorem 7 in the way described above, we obtain

$$\chi_-(\ell_p; \mathcal{I}_p^2(s', s)) \geq 1.04282487\ldots \geq 1.0428\ldots.$$

Since this inequality holds for all s' and s, statement 1 of Theorem 3 is completely proved. In the rest of this section we will often skip parameters indicating sides of ℓ_p-rectangular parallelepipeds, since, as in this example, they will play no role.

The second inequality of Theorem 3 is verified quite similarly. For that, it suffices to apply Theorem 7 to the case

$$\mathcal{A} = \mathcal{I}_p^2, \qquad \chi_{\mathcal{A}} = 1.04282487, \qquad x = 0.06094237, \qquad y = \frac{x}{2},$$

whence we get

$$\chi_-(\ell_p; \mathcal{I}_p^3) \geq 1.01261068\ldots \geq 1.0126\ldots.$$

Verifying inequality 3 is much more difficult. To this end, we consider the following tentative infinite algorithm:

Step 1. Set $\mathcal{A}_1 = \mathcal{I}$, $\chi_{\mathcal{A}_1} = \chi_1 = \dfrac{1+\sqrt{2}}{2}$, and define $c_1 = \ln \chi_1$;

Step 2. Apply Theorem 2 to the case

$$\mathcal{A} = \mathcal{A}_1, \qquad \chi_{\mathcal{A}} = \chi_1, \qquad x_1 = \frac{1}{2}, \qquad y_1 = \frac{1}{4}$$

and store the results in the variables $\mathcal{A}_2$ and χ_2, i.e., $\mathcal{A}_2 = \mathcal{I}_p^2$ and $\chi_2 = \chi_1^{c_2}$, where

$$c_2 = \frac{\ln \chi(x_1, y_1)}{\ln \chi_1 + \ln \chi(x_1, y_1) + \ln c_{x_1}^{y_1} + \ln c_{1-x_1}^{x_1-y_1}};$$

$$\vdots$$

Step $k+1$. Apply Theorem 2 to the case

$$\mathcal{A} = \mathcal{A}_k, \qquad \chi_{\mathcal{A}} = \chi_k, \qquad x_k = \frac{1}{k+1}, \qquad y_k = \frac{1}{2k+2}$$

and store the results in the variables $\mathcal{A}_{k+1}$ and χ_{k+1}, i.e., $\mathcal{A}_{k+1} = \mathcal{I}_p^{k+1}$ and $\chi_{k+1} = \chi_k^{c_{k+1}}$, where

$$c_{k+1} = \frac{\ln \chi(x_k, y_k)}{\ln \chi_k + \ln \chi(x_k, y_k) + \ln c_{x_k}^{y_k} + \ln c_{1-x_k}^{x_k-y_k}}; \qquad \ldots.$$

It is clear that this algorithm justifies the fact that $\chi_-(\ell_p; \mathcal{I}_p^k) \geq \chi_k = e^{c_1 \ldots c_k}$, and if we were able to prove somehow that

$$c_1 \ldots c_k = \left(\frac{1}{2 + o(1)}\right)^k, \tag{9}$$

then inequality 3 would be proved, since we would verify that

$$\chi_-(\ell_p; \mathcal{I}_p^k) \geq \chi_k = e^{\left(\frac{1}{2+o(1)}\right)^k} = 1 + \frac{1}{(2 + o(1))^k}.$$

To justify inequality (9), we invoke the following well-known lemma, which we give without proof.

Lemma 4. *Assume that for any k the condition $x_k > 0$ holds and $\lim\limits_{k\to\infty} x_k = x > 0$. Then $\lim\limits_{k\to\infty} \sqrt[k]{x_1 \ldots x_k} = x$.*

Thus, to complete the proof of inequality 3, it only remains to prove that $\lim\limits_{k\to\infty} c_k = 1/2$. First of all, note that each c_k is not greater than $1/2$. Indeed, it is seen from the definition that for any $k > 1$ we have $c_k \leq F(1/k)$, where

$$F(x) = \frac{\ln \chi(x, x/2)}{\ln \chi(x, x/2) + \ln c_x^{x/2} + \ln c_{1-x}^{x/2}}.$$

By considering the graph of $F(x)$ on the segment $[0, 1/2]$, one can easily check that it does not exceed $1/2$.

The fact that each c_k is not greater than $1/2$ implies that $\chi_k \leq e^{2^{-k}}$, or equivalently, $\ln \chi_k \leq 2^{-k}$. Now, using standard methods, one can check that

$$\frac{1}{2} > c_k > \frac{\ln \chi\Big(\dfrac{1}{k}, \dfrac{1}{2k}\Big)}{2^{-k+1} + \ln \chi\Big(\dfrac{1}{k}, \dfrac{1}{2k}\Big) + \ln c_{1/k}^{1/2k} + \ln c_{1-1/k}^{1/2k}} = \frac{\dfrac{1}{2}\dfrac{\ln k}{k} + O\Big(\dfrac{1}{k}\Big)}{2^{-k+1} + \dfrac{\ln k}{k} + O\Big(\dfrac{1}{k}\Big)} \xrightarrow{k\to\infty} \frac{1}{2},$$

which completes the proof of inequality 3.

All other inequalities in Theorem 3 are, in essence, simple consequences of the first three.

Indeed, inequality 4 follows from inequality 2 and the fact that $\mathcal{S}_2 \subset \mathcal{S}_3 \subset \mathcal{I}_p^3(1, 1, 1)$.

Inequality 5 follows from inequality 3 and the fact that $\mathcal{S}_k \subset \mathcal{I}_p^{k+1}(1, \ldots, 1)$.

Inequality 6 follows from inequality 1 and the fact that an ℓ_p-right triangle $\mathcal{T}(a,b,c)$ can be "embedded" in the ℓ_p-rectangle $\mathcal{I}_p^2(a,b)$.

Inequality 7 follows from inequality 2 and the fact that an ℓ_p-acute triangle $\mathcal{T}(a,b,c)$ can be "embedded" in the ℓ_p-rectangular parallelepiped $\mathcal{I}_p^3\left(\sqrt[p]{\dfrac{a^p+b^p-c^p}{2}}, \sqrt[p]{\dfrac{b^p+c^p-a^p}{2}}, \sqrt[p]{\dfrac{c^p+a^p-b^p}{2}}\right)$. △

To conclude this section, we note that though almost all statements of Theorem 3 will be improved for fixed values of p in subsequent sections using other (more tricky) sets $V_{\mathcal{I}}(n)$, inequality 3 cannot be improved in the framework of this method. We deduce this unimprovability from the following lemma.

Lemma 5. *In the notation of the auxiliary proposition, we have* $\chi < \sqrt{\chi_{\mathcal{A}}}$.

Proof. Put

$$\gamma = \frac{\ln \chi_{\mathcal{B}}}{\ln \chi_{\mathcal{A}} + \ln \chi_{\mathcal{B}} + \ln c_{\mathcal{B}}}.$$

Clearly, $\chi = \chi_{\mathcal{B}}^{\beta} = \chi_{\mathcal{A}}^{\gamma}$. It only remains to check that $\gamma < \dfrac{1}{2}$. To this end, we use Lemma 3:

$$\gamma = \frac{\ln \chi_{\mathcal{B}}}{\ln \chi_{\mathcal{A}} + \ln \chi_{\mathcal{B}} + \ln c_{\mathcal{B}}} \le \frac{\ln \chi_{\mathcal{B}}}{\ln \chi_{\mathcal{A}} + 2\ln \chi_{\mathcal{B}}} < \frac{1}{2}. \quad \triangle$$

Now it is clear that in the framework of the method used in this section the estimate for $\chi_-(\ell_p; \mathcal{I}_p^k)$ cannot exceed

$$(\chi_-(\ell_p; \mathcal{I}_p^1))^{2^{-k+1}} = 1 + \frac{1}{(2+o(1))^k},$$

so inequality 3 of Theorem 3 cannot be improved in this way indeed.

5. CASE $p = 2$. PROOF OF THEOREM 4

Using our terminology, we may say that in [5], to justify the $(\ell_2; F_{\mathcal{I}}, \chi_{\mathcal{I}})$-super-Ramsey property for the two-point metric space $\mathcal{I}(s)$, it was proposed to use not the sets $V_{\mathcal{I}}(n)$ that were used in Section 4 and which are usually referred to as $(0,1)$-*graphs* but another (more general) construction, the so-called $(-1,0,1)$-*graphs*. Thanks to that, it was proved that for $\chi_{\mathcal{I}}$ one can take $1.23956674\ldots$ (cf. statement 2 of Theorem 1). Using this result, one can try to improve Theorem 3 for the case of the ℓ_2 metric in two ways.

First, one can literally carry over the method of Section 4 with the only difference that the "initial condition" $\chi_{\mathcal{A}} = \dfrac{1+\sqrt{2}}{2}$ can be replaced with a stronger estimate $\chi_{\mathcal{A}} = 1.239\ldots$.

Second, instead of Theorem 7 one can use the result that will be obtained from the auxiliary proposition if one puts $\mathcal{B} = \mathcal{I}(s)$ in it and takes for $V_{\mathcal{B}}(n)$ the $(-1,0,1)$-graphs proposed in [5]. However, computer experiments have shown that the resulting estimate is even worse in this case, so this approach will be of no use for us.

Thus, to justify the first inequality of Theorem 4, we apply Theorem 7 to the case

$$\mathcal{A} = \mathcal{I}_2^1, \qquad \chi_{\mathcal{A}} = 1.23956674, \qquad x = 0.148421, \qquad y = \frac{x}{2}.$$

This results in

$$\chi_-(\ell_2; \mathcal{I}_2^2) \ge 1.0471486\ldots \ge 1.0471\ldots.$$

The second inequality of Theorem 4 is verified quite similarly. For that, it suffices to apply Theorem 7 to the case

$$\mathcal{A} = \mathcal{I}_2^2, \qquad \chi_{\mathcal{A}} = 1.0471486, \qquad x = 0.0647204, \qquad y = \frac{x}{2},$$

which yields

$$\chi_-(\ell_2;\mathcal{I}_2^3) \geq 1.013684342\ldots \geq 1.0136\ldots.$$

Finally, to justify inequality 3, we apply Theorem 7 to the case

$$\mathcal{A} = \mathcal{I}_2^3, \qquad \chi_{\mathcal{A}} = 1.013684342, \qquad x = 0.0279754, \qquad y = \frac{x}{2}$$

to obtain

$$\chi_-(\ell_2;\mathcal{I}_2^4) \geq 1.00459332\ldots \geq 1.00459\ldots.$$

Inequalities 4 and 6 do not need any justification, since they are particular cases of inequalities 3 and 5 in Theorem 3.

Inequality 5 easily follows from inequality 2 and the fact that $\mathcal{S}_2 \subset \mathcal{S}_3 \subset \mathcal{I}_2^3(1,1,1)$.

To justify inequalities 7 and 8, we would like to use the embedding trick again and embed $\mathcal{O}_k$ in a regular cube $\mathcal{I}^k = \mathcal{I}_2^k(t,\ldots,t)$ for some t. However, this can be done by far not always, and as will be seen below, this question is closely related to existence of the so-called *Hadamard matrices of order* k (see [42, pp. 238–263]). Recall that a Hadamard matrix of order k is a $k \times k$ square matrix with entries 1 and -1 whose rows are pairwise orthogonal.

Lemma 6. *If a Hadamard matrix of order* k *exists, then* $\mathcal{O}_k \subset \mathcal{I}^k$.

Proof. Let H_k be the Hadamard matrix of order k guaranteed by the condition of the lemma. Put $s = 1/\sqrt{k}$. Let $\boldsymbol{v}_1,\ldots,\boldsymbol{v}_k$ be vectors corresponding to rows of the matrix sH_k, and let $z_1,\ldots,z_k$ be points in $\mathbb{R}^k$ corresponding to these vectors. It is clear that pairwise distances between the elements of $V = \{z_1,\ldots,z_k,-z_1,\ldots,-z_k\}$ are precisely the same as should be in a k-dimensional cross-polytope. Besides, all coordinates of these points are plus or minus s, and hence V is a subset of vertices of the k-dimensional cube

$$C_k = \{(x_1,\ldots,x_k) : \forall\, i \;\; x_i^2 = s^2\},$$

which for $t = 2s$ coincides with $\mathcal{I}^k = \mathcal{I}_2^k(t,\ldots,t)$ as a metric space. △

Since a Hadamard matrix of order 4 exists, inequality 7 follows from inequality 3. Indeed, by what was proved above, $\mathcal{O}_3 \subset \mathcal{O}_4 \subset \mathcal{I}^4$.

A quite plausible Hadamard conjecture says that a Hadamard matrix of order $k > 2$ exists if and only if k is a multiple of 4. At present, this conjecture remains open, though many weaker results have been proved. For us, the following result will be sufficient, which was first proved in [43].

Theorem 8. *If* $p > 2$ *is a prime, then a Hadamard matrix of order* $2(p+1)$ *exists.*

Since prime numbers are known to occur in the natural sequence "sufficiently frequently" (see [33, 34]), the same concerns numbers of the form $2(p+1)$. More formally, if for any natural k we denote by $m(k)$ the smallest integer of the form $2(p+1)$ which is greater than k, then as $k \to \infty$ we have

$$m(k) = k(1 + o(1)). \tag{10}$$

Now Theorem 8 and Lemma 6 imply that $\mathcal{O}_k \subset \mathcal{O}_{m(k)} \subset \mathcal{I}^{m(k)}$. Hence, from (10) and statement 4 of Theorem 4 it follows that

$$\chi_-(\ell_2;\mathcal{O}_k) \geq \chi_-(\ell_2;\mathcal{I}^{m(k)}) \geq 1 + \frac{1}{(2+o(1))^{m(k)}} = 1 + \frac{1}{(2+o(1))^k} \quad \text{as } k \to \infty,$$

which completes the proof of inequality 8.

Inequalities 9 and 10 are verified much easier. Indeed, inequality 8 follows from inequality 1 and the fact that a right triangle $\mathcal{T}(a,b,c)$ can be "embedded" in the rectangle $\mathcal{I}_2^2(a,b)$, and inequality 10 follows from inequality 2 and the fact that an acute triangle $\mathcal{T}(a,b,c)$ can be "embedded" in the rectangular parallelepiped $\mathcal{I}_2^3\left(\sqrt{\dfrac{a^2+b^2-c^2}{2}}, \sqrt{\dfrac{b^2+c^2-a^2}{2}}, \sqrt{\dfrac{c^2+a^2-b^2}{2}}\right)$.

Now Theorem 4 is proved completely. △

6. CASE $p = 1$. PROOF OF THEOREM 5

The idea of our proof of Theorem 5 is very close to arguments in Section 4: we justify the ℓ_1-super-Ramsey property for $\mathcal{I} = \mathcal{I}(s)$ using an explicit sequence of sets $V_{\mathcal{I}}(n)$, then reformulate the auxiliary proposition for our particular case, and finally prove Theorem 5.

The construction which we will consider is usually called a $(-1,0,1)$-*graph in* ℓ_1. Precisely this construction was used in [7] to establish the lower bound in statement 3 of Theorem 1.

Let k_{-1}, k_1, and q be arbitrary positive numbers satisfying the following inequalities:

$$0 < k_{-1} \le k_1, \qquad \frac{k_{-1}+k_1}{2} \le q \le k_{-1}+k_1 \le \frac{1}{2}. \tag{11}$$

Set $k_0 = 1 - k_{-1} - k_1$ and choose some natural-valued functions $k_{-1}(n)$, $k_0(n)$, $k_1(n)$, and $q(n)$ satisfying the following conditions:

1. $k_i(n) = k_i n + o(n) \quad$ for $i \in \{-1,0,1\}$;
2. $k_{-1}(n) + k_0(n) + k_1(n) = n$;
3. $q(n) = qn + o(n)$ is a prime;
4. $\dfrac{k_{-1}(n)+k_1(n)}{2} \le q(n) \le k_{-1}(n)+k_1(n) \le \dfrac{n}{2}$.

Existence of such functions follows from the fact already mentioned above that for some positive function $\varphi(n) = o(n)$ as $n \to \infty$ it is true that for each n in the interval $(n, n+\varphi(n))$ there exists a prime number (see, e.g., [33,34]).

Put $t(n) = \dfrac{s}{2q(n)}$ and define $V_{\mathcal{I}}(n)$ as the set of all points in $\mathbb{R}^n$ which for each $i \in \{-1,0,1\}$ have exactly $k_i(n)$ coordinates equal to $it(n)$. By estimating $\mathrm{Ind}((V_{\mathcal{I}}(n),\ell_1);\mathcal{I})$ using the linear algebraic method of [7], one can show that $\mathcal{I}$ is $(\ell_1; F(k_{-1},k_1), \chi(k_{-1},k_1,q))$-super-Ramsey, where

$$c_a^b = \frac{a^a}{b^b(a-b)^{(a-b)}}, \qquad \alpha(q) = c_1^q \cdot 2^q, \qquad F(k_{-1},k_1) = c_1^{k_1} c_{1-k_1}^{k_{-1}}, \qquad \chi(k_{-1},k_1,q) = \frac{F(k_{-1},k_1)}{\alpha(q)}.$$

At the moment we cannot apply the auxiliary proposition in our case, since $c_{\mathcal{I}}$ has not yet been calculated. As in Section 4, a precise value of $c_{\mathcal{I}}$ can be found using Remark 3, which states that $c_{\mathcal{I}} = c_{\deg}$. However, this time calculating the constant $c_{\deg}$ is rather difficult and spreads over several lemmas.

Lemma 7. *Let a set* $\mathfrak{M}_1$ *consist of all ordered nontuples*

$$\boldsymbol{m} = (m_{-1,-1}, m_{-1,0}, m_{-1,1}, m_{0,-1}, m_{0,0}, m_{0,1}, m_{1,-1}, m_{1,0}, m_{1,1})$$

satisfying the inequalities

$$\begin{gathered} 0 \le m_{-1,-1}, \qquad 0 \le m_{-1,1}, \qquad 0 \le m_{1,-1}, \qquad m_{-1,-1} \le k_{-1}+k_1-q, \\ m_{-1,-1}+m_{-1,1} \le k_{-1}, \qquad m_{-1,-1}-m_{1,-1} \ge k_{-1}-q, \\ m_{-1,-1}+m_{1,-1} \le k_{-1}, \qquad m_{-1,-1}-m_{-1,1} \ge k_{-1}-q \end{gathered}$$

and equalities

$$\begin{aligned} m_{-1,0} &= k_{-1} - m_{-1,-1} - m_{-1,1}, & m_{1,0} &= k_1 - m_{1,-1} - m_{1,1},\\ m_{0,-1} &= k_{-1} - m_{-1,-1} - m_{1,-1}, & m_{0,1} &= k_1 - m_{-1,1} - m_{1,1},\\ m_{0,0} &= k_0 - m_{0,1} - m_{0,-1}, & m_{1,1} &= k_{-1} + k_1 - q - m_{-1,-1}. \end{aligned}$$

Put

$$G(\boldsymbol{m}) = c_{k_{-1}}^{m_{-1,1}} c_{k_{-1}-m_{-1,1}}^{m_{-1,-1}} c_{k_0}^{m_{0,1}} c_{k_0-m_{0,1}}^{m_{0,-1}} c_{k_1}^{m_{1,1}} c_{k_1-m_{1,1}}^{m_{1,-1}}, \qquad c_1 = \sup_{\boldsymbol{m}\in\mathfrak{M}_1} G(\boldsymbol{m}).$$

Then

$$c_{\mathrm{deg}} = c_1.$$

Proof. Let $\boldsymbol{x} = (x_1, \ldots, x_n) \in V_{\mathcal{I}}(n)$ be an arbitrary fixed element of $V_{\mathcal{I}}(n)$. Put

$$C_{\mathrm{deg}}(n) = |\{\boldsymbol{y} = (y_1, \ldots, y_n) \in V_{\mathcal{I}}(n) :\ \ell_1(\boldsymbol{x}, \boldsymbol{y}) = s\}|.$$

By the symmetry of the construction it is clear that $C_{\mathrm{deg}}(n)$ depends in no way on which element $\boldsymbol{x} \in V_{\mathcal{I}}(n)$ is fixed. It is clear that if we prove that $C_{\mathrm{deg}}(n) = (c_1 + o(1))^n$, the lemma will be proved due to the definition of c_{deg} given in Remark 3.

To calculate $C_{\mathrm{deg}}(n)$, assume that an element $\boldsymbol{y}$ is also fixed and reformulate the property $\ell_1(\boldsymbol{x}, \boldsymbol{y}) = s$ in slightly other terms.

For $i, j \in \{-1, 0, 1\}$, define

$$m_{i,j}(n) = |\{k :\ x_k = it(n),\ y_k = jt(n)\}|.$$

Clearly, not all of these quantities are "independent" of each other, since, for instance,

$$\begin{aligned} m_{-1,0}(n) &= k_{-1}(n) - m_{-1,-1}(n) - m_{-1,1}(n),\\ m_{1,0}(n) &= k_1(n) - m_{1,-1}(n) - m_{1,1}(n),\\ m_{0,-1}(n) &= k_{-1}(n) - m_{-1,-1}(n) - m_{1,-1}(n),\\ m_{0,1}(n) &= k_1(n) - m_{-1,1}(n) - m_{1,1}(n),\\ m_{0,0}(n) &= k_0(n) - m_{0,1}(n) - m_{0,-1}(n). \end{aligned} \tag{12}$$

Let us try to express the distance $\ell_1(\boldsymbol{x}, \boldsymbol{y})$ through the introduced functions:

$$\begin{aligned} \ell_1(\boldsymbol{x}, \boldsymbol{y}) &= t(n)(m_{-1,0}(n) + 2m_{-1,1}(n) + m_{0,-1}(n) + m_{0,1}(n) + 2m_{1,-1}(n) + m_{1,0}(n))\\ &= \frac{s}{2q(n)} 2(k_{-1}(n) + k_1(n) - m_{-1,-1}(n) - m_{1,1}(n)); \end{aligned}$$

hence, $\ell_1(\boldsymbol{x}, \boldsymbol{y}) = s$ if and only if

$$m_{1,1}(n) = k_{-1}(n) + k_1(n) - q(n) - m_{-1,-1}(n). \tag{13}$$

The remaining three quantities $m_{-1,-1}(n)$, $m_{-1,1}(n)$, and $m_{1,-1}(n)$ are in a sense "independent": neither of them can be rigidly expressed through the other two. However, now we will write out inequality-type constraints satisfied by them. This set of constraints is equivalent to the fact that all $m_{i,j}(n) \geq 0$:

$$\begin{gathered} 0 \leq m_{-1,-1}(n), \qquad 0 \leq m_{-1,1}(n), \qquad 0 \leq m_{1,-1}(n),\\ m_{-1,-1}(n) \leq k_{-1}(n) + k_1(n) - q(n), \qquad m_{-1,-1}(n) + m_{-1,1}(n) \leq k_{-1}(n),\\ m_{-1,-1}(n) - m_{1,-1}(n) \geq k_{-1}(n) - q(n), \qquad m_{-1,-1}(n) + m_{1,-1}(n) \leq k_{-1}(n),\\ m_{-1,-1}(n) - m_{-1,1}(n) \geq k_{-1}(n) - q(n). \end{gathered} \tag{14}$$

Note that we have written out only eight inequalities, since the ninth, expressing the nonnegativity of $m_{0,0}(n)$, happens to be satisfied automatically in our situation.

Thus, we have proved that if $\ell_1(\boldsymbol{x},\boldsymbol{y}) = s$, then the ordered nontuple $\boldsymbol{m}(n)$ corresponding to $\boldsymbol{y}$ and consisting of the elements $m_{i,j}(n)$ belongs to the set $\mathfrak{M}_1(n)$ consisting of all ordered nontuples satisfying the set of equalities (12), equality (13), and inequalities (14).

Furthermore, is clear that for each $\boldsymbol{m}(n) \in \mathfrak{M}_1(n)$ there exit exactly

$$C_{k_{-1}(n)}^{m_{-1,1}(n)} C_{k_{-1}(n)-m_{-1,1}(n)}^{m_{-1,-1}(n)} C_{k_0(n)}^{m_{0,1}(n)} C_{k_0(n)-m_{0,1}(n)}^{m_{0,-1}(n)} C_{k_1(n)}^{m_{1,1}(n)} C_{k_1(n)-m_{1,1}(n)}^{m_{1,-1}(n)}$$

different $\boldsymbol{y} \in V_{\mathcal{I}}(n)$ located at distance s from $\boldsymbol{x}$ and corresponding to $\boldsymbol{m}(n)$.

Hence,

$$C_{\deg}(n) = \sum_{\boldsymbol{m}(n)\in\mathfrak{M}_1(n)} C_{k_{-1}(n)}^{m_{-1,1}(n)} C_{k_{-1}(n)-m_{-1,1}(n)}^{m_{-1,-1}(n)} C_{k_0(n)}^{m_{0,1}(n)} C_{k_0(n)-m_{0,1}(n)}^{m_{0,-1}(n)} C_{k_1(n)}^{m_{1,1}(n)} C_{k_1(n)-m_{1,1}(n)}^{m_{1,-1}(n)}. \tag{15}$$

Now standard application of Stirling's formula to (15) completes the proof of the fact that

$$C_{\deg}(n) = (c_1 + o(1))^n.$$

Thus, the lemma is proved. △

Using Lemma 7, for any fixed k_{-1}, k_1, and q we can approximately evaluate $c_{\deg}$ using computer. However, we cannot do this with a required precision in a reasonable time, so our nearest goal is to simplify Lemma 7.

Lemma 8. *Let $\mathfrak{M}_2$ be the set of all ordered nontuples*

$$\boldsymbol{m} = (m_{-1,-1}, m_{-1,0}, m_{-1,1}, m_{0,-1}, m_{0,0}, m_{0,1}, m_{1,-1}, m_{1,0}, m_{1,1})$$

satisfying the inequalities

$$0 \le m_{1,-1} \le \min\Big(k_{-1}, \frac{q}{2}\Big),$$
$$\max(0, k_{-1} + m_{1,-1} - q) \le m_{-1,-1} \le \min(k_{-1} + k_1 - q, k_{-1} - m_{1,-1})$$

and equalities

$$m_{-1,1} = \frac{(k_{-1} - m_{-1,-1})(m_{-1,-1} + q - k_{-1})}{1 - k_{-1} - k_1 + m_{1,-1}},$$
$$\begin{aligned} m_{-1,0} &= k_{-1} - m_{-1,-1} - m_{-1,1}, & m_{1,0} &= k_1 - m_{1,-1} - m_{1,1}, \\ m_{0,-1} &= k_{-1} - m_{-1,-1} - m_{1,-1}, & m_{0,1} &= k_1 - m_{-1,1} - m_{1,1}, \\ m_{0,0} &= k_0 - m_{0,1} - m_{0,-1}, & m_{1,1} &= k_{-1} + k_1 - q - m_{-1,-1}. \end{aligned}$$

Put

$$c_2 = \sup_{\boldsymbol{m}\in\mathfrak{M}_2} G(\boldsymbol{m}).$$

Then

$$c_{\deg} = c_2.$$

Proof. In the notation of Lemma 7, it only suffices to prove that $c_1 = c_2$.

One can check that the system of inequalities on $m_{-1,-1}$, $m_{-1,1}$, and $m_{1,-1}$ given in Lemma 7 is equivalent to the following:

$$0 \le m_{1,-1} \le \min\Big(k_{-1}, \frac{q}{2}\Big), \tag{16}$$

$$\max(0, k_{-1} + m_{1,-1} - q) \le m_{-1,-1} \le \min(k_{-1} + k_1 - q, k_{-1} - m_{1,-1}), \tag{17}$$
$$0 \le m_{-1,1} \le \min(k_{-1} - m_{-1,-1}, m_{-1,-1} + q - k_{-1}). \tag{18}$$

Since $\mathfrak{M}_1$ is a compact set and the function $G(\cdot)$ is continuous on it, the maximum of $G(\cdot)$, equal to c_1, is attained.

Note that

$$\frac{dG(\boldsymbol{m})}{dm_{-1,1}} = G(\boldsymbol{m}) \ln\left(\frac{(k_{-1} - m_{-1,-1} - m_{-1,1})(q + m_{-1,-1} - k_{-1} - m_{-1,1})}{m_{-1,1}(1 - k_{-1} - k_1 - q + m_{1,-1} + m_{-1,1})}\right). \tag{19}$$

Here we have found the total derivative of $G(\cdot)$ but not the partial one. These derivatives do not coincide, since the equalities given in Lemma 7 imply that some arguments of $G(\cdot)$ depend on $m_{-1,1}$.

It follows from (19) that in $\mathfrak{M}_1$ we have

$$\begin{aligned}\frac{dG(\boldsymbol{m})}{dm_{-1,1}} \ge 0 &\iff \frac{(k_{-1} - m_{-1,-1} - m_{-1,1})(q + m_{-1,-1} - k_{-1} - m_{-1,1})}{m_{-1,1}(1 - k_{-1} - k_1 - q + m_{1,-1} + m_{-1,1})} \ge 1 \\ &\iff (k_{-1} - m_{-1,-1} - m_{-1,1})(q + m_{-1,-1} - k_{-1} - m_{-1,1}) - \\ &\qquad - m_{-1,1}(1 - k_{-1} - k_1 - q + m_{1,-1} + m_{-1,1}) \ge 0 \\ &\iff m_{-1,1} \le \xi_0 = \frac{(k_{-1} - m_{-1,-1})(m_{-1,-1} + q - k_{-1})}{1 - k_{-1} - k_1 + m_{1,-1}}.\end{aligned}$$

Thus, we have proved that for fixed values of $m_{1,-1}$ and $m_{-1,-1}$ the function $G(\boldsymbol{m})$ attains its maximum either at the point $m_{-1,1} = \xi_0$ or on the boundary of the domain $\mathfrak{M}_1$ if ξ_0 does not satisfy (18).

Now let us show that $m_{-1,1} = \xi_0$ always satisfies inequality (18). Indeed,

$$\begin{aligned}\xi_0 \le k_{-1} - m_{-1,-1} &\Longleftarrow m_{-1,-1} + q - k_{-1} \le 1 - k_{-1} - k_1 + m_{1,-1} \\ &\Longleftarrow 0 \le 1 - k_1 - q + m_{1,-1} - m_{-1,-1} \\ &\Longleftarrow 0 \le 1 - k_1 - (k_{-1} + k_1) + 0 - k_{-1} = 1 - 2(k_{-1} + k_1), \\ \xi_0 \le m_{-1,-1} + q - k_{-1} &\Longleftarrow k_{-1} - m_{-1,-1} \le 1 - k_{-1} - k_1 + m_{1,-1} \\ &\Longleftarrow 0 \le 1 - 2k_{-1} - k_1 + m_{1,-1} + m_{-1,-1} \Longleftarrow 0 \le 1 - 2k_{-1} - k_1,\end{aligned}$$

and therefore $\xi_0 \le \min(k_{-1} - m_{-1,-1}, m_{-1,-1} + q - k_{-1})$. At the same time, nonnegativity of ξ_0 is obvious.

Thus, we have proved that for fixed values of $m_{1,-1}$ and $m_{-1,-1}$ the function $G(\boldsymbol{m})$ attains its maximum at $m_{-1,1} = \xi_0$. Since global maxima of $G(\cdot)$ are at the same time its maxima in $m_{-1,1}$ for some fixed values of $m_{1,-1}$ and $m_{-1,-1}$, the equality $m_{-1,1} = \xi_0$ holds for them as well.

Hence, it suffices to look for an extremum of $G(\cdot)$ not over the whole domain $\mathfrak{M}_1$ but over its subdomain $\mathfrak{M}_2$, which completes the proof of the lemma. △

It is easier to approximately evaluate $c_{\deg}$ using Lemma 8 rather than Lemma 7, since the domain $\mathfrak{M}_2$ is two-dimensional, whereas $\mathfrak{M}_1$ is three-dimensional. In the general case, further simplification is not possible.

However, preliminary computer calculations have shown that to obtain the best estimates for chromatic numbers using the auxiliary proposition, one should set $k_{-1} = k_1 = q = k \le 1/4$ (it is easily seen that inequality (11) allows for this choice of parameters). In this particular case we can explicitly indicate the point $\boldsymbol{m}(k) \in \mathfrak{M}_2$ where $G(\cdot)$ attains its maximum.

Lemma 9. *Let* $0 < k \le \dfrac{1}{4}$ $k_{-1} = k_1 = q = k$. *Define*

$$\boldsymbol{m}(k) = (m_{-1,-1}(k), m_{-1,0}(k), m_{-1,1}(k), m_{0,-1}(k), m_{0,0}(k), m_{0,1}(k), m_{1,-1}(k), m_{1,0}(k), m_{1,1}(k))$$

as follows:

$$m_{-1,-1}(k) = m_{1,1}(k) = \frac{k}{2}, \qquad m_{1,-1}(k) = m_{-1,1}(k) = -\frac{1}{2} + k + \frac{\sqrt{1-4k+5k^2}}{2},$$

$$m_{1,0}(k) = m_{-1,0}(k) = m_{0,-1}(k) = m_{0,1}(k) = \frac{1}{2} - \frac{k}{2} - \frac{\sqrt{1-4k+5k^2}}{2},$$

$$m_{0,0}(k) = -k + \sqrt{1-4k+5k^2}.$$

Then

$$c_{\mathrm{deg}} = G(\boldsymbol{m}(k)).$$

Proof. In the notation of Lemma 8, it only suffices to prove that $c_2 = G(\boldsymbol{m}(k))$.

Since $G(\cdot)$ is continuous on the compact $\mathfrak{M}_2$, it attains its maximum value c_2 at some point. We only need to check that this point is $\boldsymbol{m}(k)$.

Inequalities (16) and (17), which must be satisfied by all nontuples $\boldsymbol{m} \in \mathfrak{M}_2$, considerably simplify in our case:

$$0 \le m_{1,-1} \le \frac{k}{2}, \tag{20}$$

$$m_{1,-1} \le m_{-1,-1} \le k - m_{1,-1}. \tag{21}$$

Let us try to use the same trick by which we have found in Lemma 8 the value of $m_{-1,1}$ at the global maximum of $G(\cdot)$. However, this time we will find the value of $m_{-1,-1}$.

One can check that in $\mathfrak{M}_2$ we have

$$\frac{dG(\boldsymbol{m})}{dm_{-1,-1}} = G(\boldsymbol{m}) \ln\left(\frac{(1-2k+m_{1,-1}-m_{-1,-1})(k-m_{-1,-1}-m_{1,-1})(k-m_{-1,-1})^2}{(1-3k+m_{-1,-1}+m_{1,-1})(m_{-1,-1}-m_{1,-1})m_{-1,-1}^2}\right),$$

and therefore

$$\begin{aligned}\frac{dG(\boldsymbol{m})}{dm_{-1,-1}} \ge 0 \iff & \frac{(1-2k+m_{1,-1}-m_{-1,-1})(k-m_{-1,-1}-m_{1,-1})(k-m_{-1,-1})^2}{(1-3k+m_{-1,-1}+m_{1,-1})(m_{-1,-1}-m_{1,-1})m_{-1,-1}^2} \ge 1 \\ \iff & -(2m_{-1,-1}-k)\big((1-k)m_{-1,-1}^2 + (k^2-k)m_{-1,-1} + \\ & + k(k-m_{1,-1})(1-2k+m_{1,-1})\big) \ge 0.\end{aligned} \tag{22}$$

The discriminant of the obtained quadratic expression in $m_{-1,-1}$ is

$$D = k(1-k)(7k^2 - 3k - 12km_{1,-1} + 4m_{1,-1} + 4m_{1,-1}^2).$$

One can check, taking into account inequality (20), that the discriminant D monotonically increases with $m_{1,-1}$. But even at the point $m_{1,-1} = \dfrac{k}{2}$ the discriminant is negative, since it equals

$$-k^2(1-k)(1-2k) < 0,$$

and therefore the quadratic expression in $m_{-1,-1}$ obtained in (22) is always positive. Thus, we arrive at the following equivalence:

$$\frac{dG(\boldsymbol{m})}{dm_{-1,-1}} \ge 0 \iff -(2m_{-1,-1}-k) \ge 0.$$

Hence, at the global maximum point of $G(\cdot)$ we indeed must have

$$m_{-1,-1} = m_{-1,-1}(k) = \frac{k}{2}, \tag{23}$$

since $m_{-1,-1}(k)$ satisfies (21) for any $m_{1,-1}$ satisfying (20).

Now let us find what is the value of $m_{1,-1}$ at the global maximum point. To this end, we proceed as follows.

Taking into account (23), one can check that

$$\frac{dG(\boldsymbol{m})}{dm_{1,-1}} = G(\boldsymbol{m}) \ln\left(\frac{(1-2k+m_{1,-1})(k-2m_{1,-1})^2}{m_{1,-1}(2-5k+2m_{1,-1})^2}\right),$$

and therefore

$$\begin{aligned}\frac{dG(\boldsymbol{m})}{dm_{1,-1}} \ge 0 &\iff \frac{(1-2k+m_{1,-1})(k-2m_{1,-1})^2}{m_{1,-1}(2-5k+2m_{1,-1})^2} \ge 1 \\ &\iff (1-2k)(k^2+8km_{1,-1}-4m_{1,-1}-4m_{1,-1}^2) \ge 0.\end{aligned}$$

It is easily checked that this quadratic function in $m_{1,-1}$ has the following roots:

$$\xi_\pm = -\frac{1}{2} + k \pm \frac{\sqrt{1-4k+5k^2}}{2}.$$

Clearly, ξ_- is negative and therefore does not satisfy inequality (20) for any k. At the same time, one can check that ξ_+ satisfies (20) for any $0 < k \le 1/4$. Hence, at the global maximum point of $G(\cdot)$ over $\mathfrak{M}_2$ we must have the equality

$$m_{-1,1} = \xi_+ = -\frac{1}{2} + k + \frac{\sqrt{1-4k+5k^2}}{2},$$

which is precisely equivalent to

$$m_{-1,1} = m_{-1,1}(k). \tag{24}$$

Since all the other $m_{i,j}$ are uniquely expressed through $m_{-1,-1}$ and $m_{-1,1}$ by the identities given in the assertion of Lemma 8, one can now easily deduce from (23) and (24) that at the global maximum point of $G(\cdot)$ over $\mathfrak{M}_2$ we have

$$\forall\, i,j \in \{-1,0,1\} \quad m_{i,j} = m_{i,j}(k).$$

Lemma 9 is completely proved. $\triangle$

Thus, we finally understand how the auxiliary proposition looks like in the case of our construction.

Theorem 9. *Let $\mathcal{A} = (A, d_A)$ be an $(\ell_1; F_\mathcal{A}, \chi_\mathcal{A})$-super-Ramsey finite metric space. Let $\mathcal{I}(s)$ be a pair of points at distance s from each other. Then $\mathcal{A} \times_1 \mathcal{I}(s)$ is also $(\ell_1; F, \chi)$-super-Ramsey with some values of the parameters F and χ. These values can be chosen, e.g., as follows*:

$$F = F_\mathcal{A}^{1-\beta}(F(k,k))^\beta, \qquad \chi = (\chi(k,k,k))^\beta, \qquad \beta = \frac{\ln \chi_\mathcal{A}}{\ln \chi_\mathcal{A} + \ln \chi(k,k,k) + \ln G(\boldsymbol{m}(k))},$$

where a value of the auxiliary parameter $0 < k \le \dfrac{1}{4}$ can be chosen arbitrarily.

To justify the first inequality of Theorem 5, it suffices to apply Theorem 9 to the case

$$\mathcal{A} = \mathcal{I}_1^1, \qquad \chi_\mathcal{A} = \frac{1+\sqrt{3}}{2}, \qquad k = 0.1122.$$

Admissibility of this $\chi_\mathcal{A}$ follows from the fact that

$$\frac{1+\sqrt{3}}{2} = \chi\left(\frac{3-\sqrt{3}}{6}, \frac{3-\sqrt{3}}{6}, \frac{3-\sqrt{3}}{6}\right).$$

Note that we have pointed out the global maximum of $\chi(\cdot\,,\cdot\,,\cdot)$, but this fact is of no importance for us now.

Applying Theorem 9 in the above-described way, we obtain

$$\chi_-(\ell_1;\mathcal{I}_1^2) \geq 1.0738908466\ldots \geq 1.07389\ldots.$$

The second inequality of Theorem 5 is justified quite similarly. To this end, it suffices to apply Theorem 9 to the case

$$\mathcal{A}=\mathcal{I}_1^2, \qquad \chi_{\mathcal{A}}=1.0738908466, \qquad k=0.05194,$$

which yields

$$\chi_-(\ell_1;\mathcal{I}_1^3) \geq 1.0218824299\ldots \geq 1.02188\ldots.$$

All the other inequalities in Theorem 5 are, in essence, consequences of already proved facts.

Inequalities 3 and 5 trivially follow from the corresponding inequalities in Theorem 3.

Inequality 4 follows from inequality 2 and the fact that $\mathcal{S}_2\subset\mathcal{S}_3\subset\mathcal{I}_1^3(1,1,1)$.

Inequality 6 follows from inequality 1 and the fact that an ℓ_1-right triangle $\mathcal{T}(a,b,c)$ can be "embedded" in the ℓ_1-rectangle $\mathcal{I}_1^2(a,b)$.

Finally, inequality 7 follows from inequality 2 and the fact that an ℓ_1-acute triangle $\mathcal{T}(a,b,c)$ can be "embedded" in the ℓ_1-rectangular parallelepiped $\mathcal{I}_1^3\Big(\frac{a+b-c}{2},\frac{b+c-a}{2},\frac{c+a-b}{2}\Big)$. $\triangle$

7. CASE $p=\infty$. PROOF OF THEOREM 6

To prove the first statement of Theorem 6, we will use arguments similar to those used in the proof of the auxiliary proposition but better adapted to our particular case.

Let among positive numbers $a_1\leq\ldots\leq a_k$ there be exactly s distinct numbers, which we denote by $b_1,\ldots,b_s$ arranging them in ascending order. Put $m_i=|\{j: a_j=b_i\}|$. Clearly, $m_1+\ldots+m_s=k$.

For an arbitrary $c>0$ define a sequence of sets $V(n;c)$ as follows:

$$V(n;c)=\{(x_1,\ldots,x_n):\ \forall\, i\ \ x_i\in\{0,c\}\}.$$

Clearly, $|V(n;c)|=2^n$.

Let $f_1(n),\ldots,f_s(n)$ be arbitrary natural-valued functions satisfying the following two conditions:

$$1.\ \forall\, n\in\mathbb{N}\ \ \sum_{i=1}^{s} f_i(n)=n; \qquad\qquad 2.\ \forall\, i\ \ f_i(n)=\frac{n}{s}+o(n)\quad \text{as } n\to\infty.$$

Set

$$V_s(n)=V(f_1(n);b_1)\times\ldots\times V(f_s(n);b_s).$$

Clearly, $V_s(n)\subset\mathbb{R}^n$ and $|V_s(n)|=2^n$. Let $\mathcal{V}_s(n)=(V_s(n),\ell_\infty)$.

A key point of our proof of Theorem 6 is the following theorem.

Theorem 10. *As $n\to\infty$ we have*

$$\operatorname{Ind}\big(\mathcal{V}_s(n);\mathcal{I}_\infty^k(a_1,\ldots,a_k)\big)\leq\big(2^{\frac{s-1}{s}}+o(1)\big)^n.$$

Proof. We prove the theorem by induction on s. The induction base, the case $s=1$, is almost obvious. Indeed, if $s=1$, then both in the space $\mathcal{I}_\infty^k(a_1,\ldots,a_k)$ and in the space $\mathcal{V}_s(n)$ the distance between any two distinct points equals b_1, and therefore

$$\operatorname{Ind}\big(\mathcal{V}_s(n);\mathcal{I}_\infty^k(a_1,\ldots,a_k)\big)=|\mathcal{I}_\infty^k(a_1,\ldots,a_k)|-1=2^k-1,$$

which, of course, can be represented as $(2^0+o(1))^n$.

Now let prove the induction step. For that, represent $V_s(n)$ in the form $A \times B$ with

$$A = V(f_1(n); b_1) \times \ldots \times V(f_{s-1}(n); b_{s-1}), \qquad B = V(f_s(n); b_s).$$

For any $b \in B$, the set $A \times \{b\}$ will be referred to as a *layer*. It is clear that in this way we partition $V_s(n)$ into $r = 2^{f_s(n)}$ layers.

Let $W \subset V_s(n)$ be any one of small independent sets, i.e.,

$$|W| = \operatorname{Ind}\bigl(\mathcal{V}_s(n); \mathcal{I}_\infty^k(a_1, \ldots, a_k)\bigr),$$

and let W_i be the intersection of W with the ith layer, $1 \le i \le r$. Without loss of generality we may assume that for some t the sets W_i contain a copy of the metric space $\mathcal{I}_\infty^{k-m_s}(a_1, \ldots, a_{k-m_s})$ if and only if $i \le t$. Clearly,

$$\operatorname{Ind}\bigl(\mathcal{V}_s(n); \mathcal{I}_\infty^k(a_1, \ldots, a_k)\bigr) = |W| = \sum_{i=1}^{t} |W_i| + \sum_{i=t+1}^{r} |W_i|, \tag{25}$$

and to complete the proof of the theorem it only remains to estimate these two sums from above.

We start with estimating the first sum. It is clear that each $|W_i|$ can be estimated from above trivially:

$$|W_i| \le |A| = 2^{n - f_s(n)} = \bigl(2^{\frac{s-1}{s}} + o(1)\bigr)^n. \tag{26}$$

Furthermore, it is clear that t cannot be "too large." Namely, if we had $t \ge 2^{m_s}$, by taking one copy of the space $\mathcal{I}_\infty^{k-m_s}(a_1, \ldots, a_{k-m_s})$ from each of the first 2^{m_s} layers we would find in W a copy of $\mathcal{I}_\infty^k(a_1, \ldots, a_k)$, which is impossible. Hence, $t \le 2^{m_s}$. This inequality and (26) immediately imply that

$$\sum_{i=1}^{t} |W_i| \le 2^{m_s} \bigl(2^{\frac{s-1}{s}} + o(1)\bigr)^n = \bigl(2^{\frac{s-1}{s}} + o(1)\bigr)^n. \tag{27}$$

Now we pass to estimating the second sum on the right-hand side of (25). First, it is clear that for $i > t$ we have

$$|W_i| \le \operatorname{Ind}\bigl((A, \ell_\infty); \mathcal{I}_\infty^{k-m_s}(a_1, \ldots, a_{k-m_s})\bigr). \tag{28}$$

Since among the $a_1, \ldots, a_{k-m_s}$ there are only $s-1$ distinct numbers, we may apply the induction hypothesis, which states that

$$\operatorname{Ind}\bigl((A, \ell_\infty); \mathcal{I}_\infty^{k-m_s}(a_1, \ldots, a_{k-m_s})\bigr) \le \bigl(2^{\frac{s-2}{s-1}} + o(1)\bigr)^{\frac{s-1}{s} n + o(n)} = \bigl(2^{\frac{s-2}{s}} + o(1)\bigr)^n. \tag{29}$$

Furthermore, it is clear that in this sum there are

$$r - t \le r = 2^{f_s(n)}$$

terms. This inequality, (28), and (29) imply that

$$\sum_{i=t+1}^{r} |W_i| \le 2^{f_s(n)} \bigl(2^{\frac{s-2}{s}} + o(1)\bigr)^n = \bigl(2^{\frac{s-1}{s}} + o(1)\bigr)^n. \tag{30}$$

Relations (25), (27), and (30) complete the proof of the induction step and thereby of the whole theorem. △

One can easily see that Theorem 10 in essence says that our metric space $\mathcal{I}_\infty^k(a_1,\dots,a_k)$ is $(\ell_\infty;2,2^{1/s})$-super-Ramsey. As we have already noted, this implies

$$\chi_-(\ell_\infty;\mathcal{I}_\infty^k(a_1,\dots,a_k))\ge 2^{1/s},$$

which completes the proof of the first statement of Theorem 6.

Let us see what we have proved for the case $1=a_1=\ldots=a_k$. In this case $s=1$, and thus we have proved that

$$\chi_-(\ell_\infty;\mathcal{I}_\infty^k(1,\dots,1))\ge 2.$$

Since $\mathcal{S}_1\subset\mathcal{S}_k\subset\mathcal{I}_\infty^{k+1}(1,\dots,1)$, we have the following chain of inequalities:

$$2\le\chi_-(\ell_\infty;\mathcal{I}_\infty^{k+1}(1,\dots,1))\le\chi_-(\ell_\infty;\mathcal{S}_k)\le\chi_-(\ell_\infty;\mathcal{S}_1)=2,$$

where the last exact equality follows from statement 4 of Theorem 1. Hence, Theorem 6 is completely proved. $\triangle$

Remark 4. Note that we have also proved that $\chi_-(\ell_\infty;\mathcal{I}_\infty^k(1,\dots,1))=2$. However, this equality adds nothing new to the assertion of Theorem 6, since it is easy to check that $\mathcal{I}_\infty^k(1,\dots,1)=\mathcal{S}_{2^k-1}$.

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