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LARGE SYSTEMS

A Local Large Deviation Principle for Inhomogeneous Birth–Death Processes

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Abstract—The paper considers a continuous-time birth–death process where the jump rate has an asymptotically polynomial dependence on the process position. We obtain a rough exponential asymptotic for the probability of trajectories of a re-scaled process contained within a neighborhood of a given continuous nonnegative function.

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1. INTRODUCTION

In the modern literature on the large deviation principle, one considers various conditions for random processes guaranteeing rough exponential asymptotics for probabilities of rare events; see, for example, [1–6]. In this paper we deal with birth-and-death Markov processes that are inhomogeneous in the state space: the rates of jumps are polynomially dependent on the position of the process. For these processes we obtain an exponential asymptotic for the probabilities of the normalized process to be in a neighborhood of a continuous function. Moreover, we provide this asymptotic for both ergodic processes and transient (even exploding) processes.

The study of birth-and-death processes is of a certain mathematical interest and, moreover, is important for a number of applications. As examples, we can refer to information theory (encoding and storage of information [7, 8]), biology and chemistry (models of growth and extinction in systems with multiple components [9]), and economics (models of competitive production and pricing [10–12]).

Consider a continuous-time Markov process $\xi(t)$, $t \geq 0$, with state space $\mathbb{Z}^+ \cup \{\infty\}$, where $\mathbb{Z}^+ = \{0\} \cup \mathbb{N}$. Let us assume that the process starts at 0.

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The evolution of the process $\xi(\cdot)$ is described as follows. For a given $t \geq 0$, let $\xi(t) = x \in \mathbb{Z}^+$. The state of the random process does not change during random time τ_x having exponential distribution with parameter $h(x) > 0$. At the moment $t + \tau_x$ the process jumps to the states $x \pm 1$ with probabilities

$$\mathbf{P}(\xi(t+\tau_x) = x+1) = \frac{\lambda(x)}{h(x)}, \qquad \mathbf{P}(\xi(t+\tau_x) = x-1) = \frac{\mu(x)}{h(x)}, \tag{1}$$

respectively, where $\lambda(x) + \mu(x) = h(x)$, $\lambda(x) > 0$ when $x \in \mathbb{Z}^+$, and $\mu(x) > 0$ for $x \in \mathbb{N}$.

Assume that for $x = 0$ the rates are $\mu(x) = 0$ and $\lambda(x) = \lambda_0 > 0$ (i.e., the process cannot leave the set $\mathbb{Z}^+$), and the following asymptotics hold true:

$$\lim_{x\to\infty} \frac{\lambda(x)}{P_\ell x^\ell} = \lim_{x\to\infty} \frac{\mu(x)}{Q_m x^m} = 1, \tag{2}$$

where P_ℓ and Q_m are positive constants, $\ell \geq 0$, $m \geq 0$, and $\max(\ell, m) > 0$.

When $\ell \leq 1$, the existence of a Markov process with the above properties is established in the standard way (see, e.g., [13, ch. 17, Sections 4 and 5; 14, ch. 2, Section 5, Theorem 2.5.5; 15, chs. 6 and 7]).

When $\ell > 1$, the process $\xi(\cdot)$, generally speaking, can go to infinity ("explode") during a random time, finite with probability 1. There are two approaches to construct such processes: (1) One can stop the process at a random time point (the time of explosion) (see, e.g., [16, ch. 15, Section 4; 17, ch. 6]); (2) One can extend the phase space $\mathbb{Z}^+$ by adding an absorbing state, denoted by ∞ (see, e.g., [18, ch. 4, Section 48; 13, ch. 17, Section 10]). For our results it makes no difference which version is used.

The above class of random processes is referred to as birth-and-death processes (see, e.g., [13, 18–20]).

There exist conditions on ℓ and m which are sufficient for explosion and non-explosion. For example, when $\ell > 1$ and $m < \ell$, the process $\xi(\cdot)$ explodes, while if $m > \ell$, it does not. For references, see original papers [21, 22] and references therein; see also [23, ch. 23, Section 7; 14, ch. 2, Section 5; 24, ch. 5, Section 3] (in [24] there are also results for general Markov chains, not only birth-and-death processes).

We are interested in a local large deviation principle (LLDP) for the family of scaled processes

$$\xi_T(t) = \frac{\xi(tT)}{T}, \quad 0 \leq t \leq 1,$$

where $T > 0$ is a parameter (see, e.g., [25, 26]). In a sense, the formulation and analysis of the LLDP should precede the study of other forms of the large deviation principle.

The validity of our results does not depend on whether or not the process $\xi(\cdot)$ explodes within a finite time. We focus on the asymptotic of the probability of the event that the trajectories of the process $\xi_T(\cdot)$ stay in a neighborhood of a continuous positive function defined on the interval $[0, 1]$. This means that we are working on the set of trajectories which do not tend to infinity in the time interval $[0, T]$. The considered probabilities are positive even if the process $\xi(\cdot)$ explodes (see equation (3) below).

Let $\mathbb{D}[0, 1]$ denote the space of right-continuous functions with left-limit at each $t \in [0, 1]$. For any $f, g \in \mathbb{D}[0, 1]$, set

$$\rho(f, g) = \sup_{t\in[0,1]} |f(t) - g(t)|.$$

Definition. The family of random processes $\xi_T(\cdot)$ satisfies the LLDP on the set $G \subseteq \mathbb{D}[0, 1]$ with a rate function $I = I(f)\colon \mathbb{D}[0, 1] \to [0, \infty]$ and a normalizing function $\psi(T)$ with $\lim\limits_{T\to\infty} \psi(T) = \infty$

if, for any function $f \in G$, the following equality holds true:

$$\lim_{\varepsilon\to 0}\limsup_{T\to\infty}\frac{1}{\psi(T)}\ln \mathbf{P}(\xi_T(\cdot)\in U_\varepsilon(f)) = \lim_{\varepsilon\to 0}\liminf_{T\to\infty}\frac{1}{\psi(T)}\ln \mathbf{P}(\xi_T(\cdot)\in U_\varepsilon(f)) = -I(f),$$

Here

$$U_\varepsilon(f) = \{g \in \mathbb{D}[0,1] :\ \rho(f,g) < \varepsilon\}.$$

In the framework of this definition, there are various cases to consider. We separate three cases: (1) $\ell > m$; (2) $\ell < m$; and (3) $\ell = m$.

Note that the case $m = 1$ and $\ell = 0$ follows from [11] (where a two-dimensional Markov process is treated). A similar result is obtained in [27] for solutions of stochastic differential Itô's equations. The classical case $\ell = m = 0$ and $\varphi(T) = T$ follows, for example, from [28].

In this paper we use the approach developed in [11]. We would like to note that the large deviation principle for the sequence of processes $\xi_T(\cdot)$ in the space $\mathbb{D}[0,1]$ with Skorokhod metric cannot be obtained even for non-exploding processes: one can show that the corresponding family of measures is not exponentially dense, except for the classical case $\ell = m = 0$.

The paper is organized as follows: in Section 2 we introduce our definitions and notation and state the main result (theorem) and key lemmas. In Section 3 we prove the theorem and key lemmas. In the Appendix, some auxiliary technical assertions are established.

2. MAIN RESULTS AND NOTATION

Let F denote the set of functions $f(t) \in \mathbb{C}[0,1]$ such that $f(0) = 0$ and $f(t) > 0$ for $0 < t \le 1$.

Theorem. *Let conditions* (1) *and* (2) *be fulfilled. Then the family of random processes* $\xi_T(\cdot)$ *satisfies the following LLDP on* F:

(a) *If* $\ell > m$, *then the normalizing function is* $\psi(T) = T^{\ell+1}$, *and the rate function has the form*

$$I(f) = P_\ell \int_0^1 f^\ell(t)\,dt, \quad f \in F;$$

(b) *If* $\ell = m$ *and* $P_\ell \ne Q_m$, *then* $\psi(T) = T^{\ell+1}$ *and*

$$I(f) = \left(\sqrt{P_\ell} - \sqrt{Q_m}\,\right)^2 \int_0^1 f^\ell(t)\,dt, \quad f \in F;$$

(c) *If* $\ell < m$, *then* $\psi(T) = T^{m+1}$ *and*

$$I(f) = Q_m \int_0^1 f^m(t)\,dt, \quad f \in F.$$

The case where $\ell = m$ and $P_\ell = Q_m$ needs a different normalization; we do not discuss it in this paper.

Consider a space- and time-homogeneous Markov process $\zeta(t)$, $t \in [0,T]$, on the phase space $\mathbb{Z}$, where the jump rate is 1 and the jump size is ± 1, occurring with probability $1/2$.

Denote by X_T the set of all right-continuous step functions with a finite number of ± 1-jumps on $[0,T]$.

Lemma 1. *For any given* T, *the distribution* $\mathbf{P}_T^{(\xi)}(\,\cdot\cap X_T)$ *of the process* $\xi(\cdot)$ *on* X_T *is absolutely continuous with respect to the distribution* $\mathbf{P}_T^{(\zeta)}$ *of the process* $\zeta(\cdot)$ *on* X_T. *The corresponding density* (*the Radon–Nikodym derivative* $d\,\mathbf{P}_T^{(\xi)}(\,\cdot\cap X_T)/d\,\mathbf{P}_T^{(\zeta)}$) *on* X_T *has the form*

$$\mathfrak{p}_T(u)=\begin{cases}2^{N_T(u)}\left(\prod\limits_{i=1}^{N_T(u)}e^{-(h(u(t_{i-1}))-1)\tau_i}\nu(u(t_{i-1}),u(t_i))\right)e^{-(h(u(t_{N_T(u)}))-1))(T-t_{N_T(u)})} & \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{if } N_T(u)\ge 1, \\ e^{-(h(0)-1)T} & \text{if } N_T(u)=0.\end{cases}\tag{3}$$

Here it is assumed that the function $u(\cdot)$ *on* $[0,T]$ *has exactly* $N_T(u)$ *jumps at time points* $t_1,t_2,\ldots,t_{N_T(u)}$, *where* $0=t_0<t_1<\ldots<t_{N_T(u)}\le T$, $\tau_i=t_i-t_{i-1}$, *and*

$$\nu(u(t_{i-1}),u(t_i))=\begin{cases}\lambda(u(t_{i-1})) & \text{if } u(t_i)-u(t_{i-1})=1,\\ \mu(u(t_{i-1})) & \text{if } u(t_i)-u(t_{i-1})=-1.\end{cases}$$

Observe that the probability $\mathbf{P}(\xi(\cdot)\in X_T)$ in Lemma 1 is allowed to be less than 1. (Clearly, this probability is positive.) Note that a similar density was used in [11].

In what follows, we denote by $N_T(\zeta)$ the random number of jumps in the process $\zeta(\cdot)$ on the interval $[0,T]$.

The assertion of Lemma 1 is equivalent to the fact that for any measurable set $G\subseteq X_T$

$$\mathbf{P}(\xi(\cdot)\in G)=e^T\,\mathbf{E}(e^{-A_T(\zeta)}e^{B_T(\zeta)+N_T(\zeta)\ln 2};\zeta(\cdot)\in G).\tag{4}$$

We set

$$\begin{aligned}A_T(\zeta)&=\int\limits_0^T h(\zeta(t))\,dt=\begin{cases}\sum\limits_{i=1}^{N_T(\zeta)}h(\zeta(t_{i-1}))\tau_i+h(\zeta(t_{N_T(\zeta)}))(T-t_{N_T(\zeta)}) & \text{if } N_T(\zeta)\ge 1,\\ h(0)T & \text{if } N_T(\zeta)=0,\end{cases}\\ B_T(\zeta)&=\begin{cases}\sum\limits_{i=1}^{N_T(\zeta)}\ln(\nu(\zeta(t_{i-1}),\zeta(t_i))) & \text{if } N_T(\zeta)\ge 1,\\ 0 & \text{if } N_T(\zeta)=0.\end{cases}\end{aligned}\tag{5}$$

The expressions in (4) specify, in our context, the statement of the Radon–Nikodym theorem (see, e.g., [29, ch. III, Section 10, Theorem 2]). Below we use expressions (4) for analyzing the asymptotic behavior of the logarithm of the probability $\mathbf{P}(\xi_T(\cdot)\in U_\varepsilon(f))$ for $f\in F$.

The theorem shows that for $\ell\ne m$ the main contribution into the asymptotic is brought by $A_T(\zeta)$, whereas in the case $\ell=m$ the asymptotic involves both $A_T(\zeta)$ and $B_T(\zeta)$.

Consider the family of scaled processes

$$\zeta_T(t)=\frac{\zeta(tT)}{T},\quad t\in[0,1].$$

Let k_+ and k_- denote the number of positive and negative jumps in $\zeta_T(\cdot)$ and set $L=k_+-k_-$.

For $\zeta_T(\cdot)\in U_\varepsilon(f)$ we have the inequality

$$f(1)-\varepsilon\le\zeta_T(1)\le f(1)+\varepsilon.\tag{6}$$

The jumps in $\zeta_T(\cdot)$ are $\pm1/T$; therefore, (6) yields the inequalities

$$(f(1)-\varepsilon)T\le L\le(f(1)+\varepsilon)T.\tag{7}$$

With these definitions and observations we can write

$$k_+ + k_- = N_T(\zeta), \quad k_+ = \frac{N_T(\zeta)+L}{2}, \quad k_- = \frac{N_T(\zeta)-L}{2}. \tag{8}$$

For brevity, we write below ξ_T, ζ_T and A_T, B_T instead of $\xi_T(\cdot), \zeta_T(\cdot)$ and $A_T(\zeta), B_T(\zeta)$. Also, we set $v = \max(\ell, m)$.

Lemma 2. *Let $f \in F$. In the case $\ell \neq m$ we have*

$$\lim_{\varepsilon\to 0}\limsup_{T\to\infty}\frac{1}{T^{v+1}}\ln \mathbf{E}\big(e^{B_T+N_T(\zeta)\ln 2}; \zeta_T \in U_\varepsilon(f)\big) \le 0,$$

whereas in the case $\ell = m$,

$$\lim_{\varepsilon\to 0}\limsup_{T\to\infty}\frac{1}{T^{\ell+1}}\ln \mathbf{E}\big(e^{B_T+N_T(\zeta)\ln 2}; \zeta_T \in U_\varepsilon(f)\big) \le 2\sqrt{P_\ell Q_m}\int\limits_0^1 f^\ell(s)\,ds.$$

Lemma 3. *For $f \in F$, in the case $\ell \neq m$ we have*

$$\lim_{\varepsilon\to 0}\liminf_{T\to\infty}\frac{1}{T^{v+1}}\ln \mathbf{E}\big(e^{B_T+N_T(\zeta)\ln 2}; \zeta_T \in U_\varepsilon(f)\big) \ge 0,$$

and in the case $\ell = m$,

$$\lim_{\varepsilon\to 0}\liminf_{T\to\infty}\frac{1}{T^{\ell+1}}\ln \mathbf{E}\big(e^{B_T+N_T(\zeta)\ln 2}; \zeta_T \in U_\varepsilon(f)\big) \ge 2\sqrt{P_\ell Q_m}\int\limits_0^1 f^\ell(s)\,ds.$$

3. PROOFS OF THE THEOREM AND LEMMAS 1–3

Proof of the theorem. We are going to get the LLDP for functions $f \in F$. First let us estimate the quantity A_T. Fix a value $\varepsilon > 0$ until a further notice.

From equation (5) it follows that

$$A_T := \int\limits_0^T h(\zeta(t))\,dt = T\int\limits_0^1 h(T\zeta_T(s))\,ds.$$

If $\zeta_T \in U_\varepsilon(f)$, then

$$(f(s)-\varepsilon) \le \zeta_T(s) \le (f(s)+\varepsilon). \tag{9}$$

Let $\delta \in (0,1)$ be also fixed for the time being, and denote $m_\delta := \min\limits_{t\in[\delta,1]} f(t)$. Since $f \in F$, we have $m_\delta > 0$. Therefore, $k_0 = m_\delta - \varepsilon > 0$ when ε is sufficiently small.

Let us estimate A_T on the set of trajectories ω where inequality (9) is valid. From (9) it follows that $T\zeta_T(s) \ge k_0 T$ for $s \in [\delta, 1]$. Therefore, by virtue of condition (2), for any $\gamma_0 \in (0,1)$ and $s \in [\delta, 1]$, for T large enough we have the inequalities

$$1-\gamma_0 \le \frac{h(T\zeta_T(s))}{P_\ell(T\zeta_T(s))^\ell} \le 1+\gamma_0 \quad \text{in the case } \ell > m, \tag{10}$$

$$1-\gamma_0 \le \frac{h(T\zeta_T(s))}{(P_\ell+Q_m)(T\zeta_T(s))^\ell} \le 1+\gamma_0 \quad \text{in the case } \ell = m, \tag{11}$$

$$1-\gamma_0 \le \frac{h(T\zeta_T(s))}{Q_m(T\zeta_T(s))^m} \le 1+\gamma_0 \quad \text{in the case } \ell < m. \tag{12}$$

Consider the case $\ell > m$. Owing to (9) and (10), for T sufficiently large we get

$$T\int_\delta^1 (1-\gamma_0)P_\ell(T(f(s)-\varepsilon))^\ell\,ds \le A_T \le T\int_0^\delta h(T\zeta_T(s))\,ds + T\int_\delta^1 (1+\gamma_0)P_\ell(T(f(s)+\varepsilon))^\ell\,ds. \tag{13}$$

Set $M := \max(\max_{t\in[0,1]} f(t), 1)$. By using (9), for T large enough we have

$$h(T\zeta_T(s)) \le (1+\gamma_0)P_\ell(T(M+\varepsilon))^\ell.$$

Consequently, from (13) we obtain the inequality

$$T^{\ell+1}P_\ell\int_\delta^1 (1-\gamma_0)(f(s)-\varepsilon)^\ell\,ds \le A_T \le T^{\ell+1}P_\ell\delta(1+\gamma_0)(M+\varepsilon)^\ell + T^{\ell+1}P_\ell\int_\delta^1 (1+\gamma_0)(f(s)+\varepsilon)^\ell\,ds. \tag{14}$$

By using the bound (14) and equation (4), we get the following:

$$\begin{aligned} &e^{-T^{\ell+1}P_\ell\int_\delta^1 (1-\gamma_0)(f(s)-\varepsilon)^\ell\,ds} e^T\,\mathbf{E}\big(e^{B_T+N_T(\zeta)\ln 2};\zeta_T\in U_\varepsilon(f)\big) \\ &\quad\ge \mathbf{P}(\xi_T(\cdot)\in U_\varepsilon(f)) = e^T\,\mathbf{E}\big(e^{-A_T}e^{B_T+N_T(\zeta)\ln 2};\zeta_T\in U_\varepsilon(f)\big) \\ &\quad\ge e^{-T^{\ell+1}P_\ell\delta(1+\gamma_0)(M+\varepsilon)^\ell - T^{\ell+1}P_\ell\int_\delta^1(1+\gamma_0)(f(s)+\varepsilon)^\ell\,ds} e^T\,\mathbf{E}\big(e^{B_T+N_T(\zeta)\ln 2};\zeta_T\in U_\varepsilon(f)\big). \end{aligned} \tag{15}$$

Further, by virtue of (15),

$$\begin{aligned} &-P_\ell\int_\delta^1 (1-\gamma_0)(f(s)-\varepsilon)^\ell\,ds + \limsup_{T\to\infty}\frac{1}{T^{\ell+1}}\ln\mathbf{E}\big(e^{B_T+N_T(\zeta)\ln 2};\zeta_T\in U_\varepsilon(f)\big) \\ &\quad\ge \limsup_{T\to\infty}\frac{1}{T^{\ell+1}}\ln\mathbf{P}(\xi_T\in U_\varepsilon(f)) \ge \liminf_{T\to\infty}\frac{1}{T^{\ell+1}}\ln\mathbf{P}(\xi_T\in U_\varepsilon(f)) \\ &\quad\ge -P_\ell\delta(1+\gamma_0)(M+\varepsilon)^\ell - P_\ell\int_\delta^1(1+\gamma_0)(f(s)+\varepsilon)^\ell\,ds \\ &\qquad + \liminf_{T\to\infty}\frac{1}{T^{\ell+1}}\ln\mathbf{E}\big(e^{B_T+N_T(\zeta)\ln 2};\zeta_T\in U_\varepsilon(f)\big). \end{aligned} \tag{16}$$

Next, from(16) it follows that

$$\begin{aligned} &-P_\ell\int_\delta^1 (1-\gamma_0)f^\ell(s)\,ds + \lim_{\varepsilon\to 0}\limsup_{T\to\infty}\frac{1}{T^{\ell+1}}\ln\mathbf{E}\big(e^{B_T+N_T(\zeta)\ln 2};\zeta_T\in U_\varepsilon(f)\big) \\ &\quad\ge \lim_{\varepsilon\to 0}\limsup_{T\to\infty}\frac{1}{T^{\ell+1}}\ln\mathbf{P}(\xi_T(\cdot)\in U_\varepsilon(f)) \ge \lim_{\varepsilon\to 0}\liminf_{T\to\infty}\frac{1}{T^{\ell+1}}\ln\mathbf{P}(\xi_T(\cdot)\in U_\varepsilon(f)) \\ &\quad\ge -P_\ell\delta(1+\gamma_0)M^\ell - P_\ell\int_0^1(1+\gamma_0)f^\ell(s)\,ds \\ &\qquad + \lim_{\varepsilon\to 0}\liminf_{T\to\infty}\frac{1}{T^{\ell+1}}\ln\mathbf{E}\big(e^{B_T+N_T(\zeta)\ln 2};\zeta_T\in U_\varepsilon(f)\big). \end{aligned} \tag{17}$$

Note that inequality (17) is valid for all $\gamma_0 > 0$ and $\delta > 0$. Letting $\gamma_0 \to 0$ and $\delta \to 0$, we get

$$\begin{aligned} &- P_\ell \int\limits_0^1 f^\ell(s)\,ds + \lim_{\varepsilon\to 0}\limsup_{T\to\infty} \frac{1}{T^{\ell+1}} \ln \mathbf{E}\big(e^{B_T+N_T(\zeta)\ln 2}; \zeta_T \in U_\varepsilon(f)\big) \\ &\qquad \ge \lim_{\varepsilon\to 0}\limsup_{T\to\infty} \frac{1}{T^{\ell+1}} \ln \mathbf{P}(\xi_T(\cdot) \in U_\varepsilon(f)) \ge \lim_{\varepsilon\to 0}\liminf_{T\to\infty} \frac{1}{T^{\ell+1}} \ln \mathbf{P}(\xi_T(\cdot) \in U_\varepsilon(f)) \\ &\qquad \ge -P_\ell \int\limits_0^1 f^\ell(s)\,ds + \lim_{\varepsilon\to 0}\liminf_{T\to\infty} \frac{1}{T^{\ell+1}} \ln \mathbf{E}\big(e^{B_T+N_T(\zeta)\ln 2}; \zeta_T \in U_\varepsilon(f)\big). \end{aligned} \tag{18}$$

In a similar way, by using (11) and (12), we obtain inequalities for the case $\ell = m$:

$$\begin{aligned} &- (P_\ell + Q_m) \int\limits_0^1 f^\ell(s)\,ds + \lim_{\varepsilon\to 0}\limsup_{T\to\infty} \frac{1}{T^{\ell+1}} \ln \mathbf{E}\big(e^{B_T+N_T(\zeta)\ln 2}; \zeta_T \in U_\varepsilon(f)\big) \\ &\qquad \ge \lim_{\varepsilon\to 0}\limsup_{T\to\infty} \frac{1}{T^{\ell+1}} \ln \mathbf{P}(\xi_T \in U_\varepsilon(f)) \ge \lim_{\varepsilon\to 0}\liminf_{T\to\infty} \frac{1}{T^{\ell+1}} \ln \mathbf{P}(\xi_T \in U_\varepsilon(f)) \\ &\qquad \ge -(P_\ell + Q_m) \int\limits_0^1 f^\ell(s)\,ds + \lim_{\varepsilon\to 0}\liminf_{T\to\infty} \frac{1}{T^{\ell+1}} \ln \mathbf{E}\big(e^{B_T+N_T(\zeta)\ln 2}; \zeta_T \in U_\varepsilon(f)\big), \end{aligned} \tag{19}$$

and for the case $\ell < m$:

$$\begin{aligned} &- Q_m \int\limits_0^1 f^m(s)\,ds + \lim_{\varepsilon\to 0}\limsup_{T\to\infty} \frac{1}{T^{m+1}} \ln \mathbf{E}\big(e^{B_T+N_T(\zeta)\ln 2}; \zeta_T \in U_\varepsilon(f)\big) \\ &\qquad \ge \lim_{\varepsilon\to 0}\limsup_{T\to\infty} \frac{1}{T^{m+1}} \ln \mathbf{P}(\xi_T(\cdot) \in U_\varepsilon(f)) \ge \lim_{\varepsilon\to 0}\liminf_{T\to\infty} \frac{1}{T^{m+1}} \ln \mathbf{P}(\xi_T(\cdot) \in U_\varepsilon(f)) \\ &\qquad \ge -Q_m \int\limits_0^1 f^m(s)\,ds + \lim_{\varepsilon\to 0}\liminf_{T\to\infty} \frac{1}{T^{m+1}} \ln \mathbf{E}\big(e^{B_T+N_T(\zeta)\ln 2}; \zeta_T \in U_\varepsilon(f)\big). \end{aligned} \tag{20}$$

Observe that in the course of deducing the estimates (18)–(20) the limit $T \to \infty$ precedes the limit $\varepsilon \to 0$.

Applying Lemmas 2 and 3 to (18)–(20) completes the proof of the LLDP for the functions in F. △

Remark 1. The above argument allows us to extend the assertion of the theorem to the set of functions $f \in \mathbb{C}[0,1]$ with $f(0) = 0$, $f(t) \ge 0$ for $0 < t \le 1$, and $f(t) = 0$ at finitely many points in $[0,1]$.

Remark 2. For the Yule process (a process of pure birth with $\ell > 0$, $P_\ell > 0$, and $\mu(x) \equiv 0$; see, e.g., [13]), the rate functional has the form

$$I(f) = P_\ell \int\limits_0^1 f^\ell(t)\,dt, \quad f \in F_M,$$

where F_M is the set of nondecreasing continuous functions $f(t)$ on $[0,1]$ with $f(0) = 0$.

Proof of Lemma 1. Let $N_T(\xi)$ be the number of jumps in the process $\xi(\cdot)$ in the time interval $[0,T]$. In the course of the proof we work on the event that the trajectory of $\xi(\cdot)$ belongs to X_T, i.e., that $N_T < \infty$. This event has a positive probability.

As was mentioned above, the statement of the lemma means that for any measurable set $G \subseteq X_T$ equality (4) is valid. Denote by $X_T^{(n)}$ the set of functions $u(\cdot) \in X_T$ with $N_T(u) = n$, $n = 0, 1, \ldots$. Consider the one-to-one mapping

$$u \in X_T^{(n)} \mapsto (t_1, \ldots, t_n; \Delta_1, \ldots, \Delta_n) \in \mathfrak{X}_T^{(n)} = [0, T]^n_< \times \{+1, -1\}^n, \quad n = 1, 2, \ldots. \tag{21}$$

Here $t_1, \ldots, t_n$ is a sequence of jump times for the function $u(\cdot)$ in $[0, T]$, and Δ_i is the size of the jump $u(t_i) - u(t_{i-1})$ (with $\Delta_1 = u(t_1)$). Next, $[0, T]^n_<$ stands for the n-dimensional simplex $\{(t_1, \ldots, t_n) :\ 0 < t_1 < \ldots < t_n \le T\}$.

The probabilities $\mathbf{P}(\xi(\cdot) \in G)$ and $\mathbf{P}(\zeta(\cdot) \in G)$ are determined by

(a) the respective densities $\mathfrak{f}_\xi$ and $\mathfrak{f}_\zeta$ relative to the summation measure $\sum\limits_{n\ge1} \prod\limits_{j=1}^{n} dt_j$ on $\mathfrak{X}_T := \bigcup\limits_{n\ge1} \mathfrak{X}_T^{(n)}$ (here $t_0 = 0$ for $j = 1$), and

(b) the probabilities $\mathbf{P}(\xi(t) = 0,\ 0 \le t \le T) = e^{-\lambda(0)T}$, $\mathbf{P}(\zeta(t) = 0,\ 0 \le t \le T) = e^{-T}$.

The densities $\mathfrak{f}_\xi$ and $\mathfrak{f}_\zeta$ are of the form

$$\mathfrak{f}_\xi(t_1, \ldots, t_n; \Delta_1, \ldots, \Delta_n) = \left(\prod_{i=1}^{n} \nu(x_{i-1}, x_i) e^{-h(x_{i-1})\tau_i}\right) e^{-h(x_n)(T-t_n)}, \tag{22}$$

$$\mathfrak{f}_\zeta(t_1, \ldots, t_n; \Delta_1, \ldots, \Delta_n) = 2^{-n} \left(\prod_{i=1}^{n} e^{-\tau_i}\right) e^{-(T-t_n)}, \tag{23}$$

where $x_0 = 0$, $x_i = \sum\limits_{j=1}^{i} \Delta_j$, $i = 1, \ldots, n$.

Each factor $\nu(x_{i-1}, x_i) e^{-h(x_{i-1})\tau_i}$ in (22) gives the probability density $h(x_{i-1}) e^{-h(x_{i-1})\tau_i}$ for the time that the process ξ spent at state x_{i-1} multiplied by the probability $\nu(x_{i-1}, x_i)/h(x_{i-1})$ of a jump from x_{i-1} to x_i. The factor $e^{-h(x_n)(T-t_n)}$ is the probability to stay at x_n until time T. A similar meaning is attributed to the factors $\frac{1}{2} e^{-\tau_i}$ and $e^{-(T-t_n)}$. The products of terms in (22) and (23) reflect the Markovian character of both processes.

The Radon–Nikodym derivative $d\mathbf{P}_T^{(\xi)}(\,\cdot \cap X_T)/d\mathbf{P}_T^{(\zeta)}$ in (3) is the ratio $\mathfrak{f}_\xi/\mathfrak{f}_\zeta$, because the mapping $X_T^{(n)} \to \mathfrak{X}_T^{(n)}$ is one-to-one. The Radon–Nikodym theorem can be applied here, since both densities $\mathfrak{f}_\xi$ and $\mathfrak{f}_\zeta$ are positive on $\mathfrak{X}_T^{(n)}$, and the measure $\sum\limits_{n\ge1} \prod\limits_{j=1}^{n} dt_j$ on $\mathfrak{X}_T$ is finite (for the formulation and proof of the Radon–Nikodym theorem, see, e.g., [29, ch. III, Section 10, Theorem 2; 30, Theorem 6.10]). $\triangle$

Proof of Lemma 2. First, we upper bound the expected value $\mathbf{E}\big(e^{B_T + N_T(\zeta)\ln 2}; \zeta_T \in U_\varepsilon(f)\big)$. Given $a > 1$, represent this value as

$$\begin{aligned} \mathbf{E}\big(e^{B_T + N_T(\zeta)\ln 2}; \zeta_T \in U_\varepsilon(f)\big) &= E_1 + E_2, \\ E_1 := \mathbf{E}\big(e^{B_T + N_T(\zeta)\ln 2}&; \zeta_T \in U_\varepsilon(f); N_T(\zeta) \le T^a\big), \\ E_2 := \mathbf{E}\big(e^{B_T + N_T(\zeta)\ln 2}&; \zeta_T \in U_\varepsilon(f); N_T(\zeta) > T^a\big). \end{aligned} \tag{24}$$

Let us bound E_1 from above. If $\zeta_T \in U_\varepsilon(f)$ and $N_T(\zeta) \le T^a$, then by virtue of (2) it follows that for any $\gamma_1 > 0$ and T large enough,

$$\begin{aligned} B_T &= \sum_{i=1}^{N_T(\zeta)} \ln(\nu(\zeta(t_{i-1}), \zeta(t_i))) \\ &\le T^a\big(\ln(P_\ell T^\ell (M+\varepsilon)^\ell (1+\gamma_1)) + \ln(Q_m T^m (M+\varepsilon)^m (1+\gamma_1))\big), \end{aligned}$$

Here, as above, $M = \max\big(\max\limits_{t\in[0,1]} f(t), 1\big)$.

Set $k_1 = P_\ell Q_m (M+\varepsilon)^{\ell+m}(1+\gamma_1)^2$. Then the following inequality is fulfilled:

$$E_1 \le \exp\{(T^a+1)\ln(k_1 T^{\ell+m})\}. \tag{25}$$

Next, we establish an upper bound for E_2. Denote $M_\delta := \max\limits_{s\in[0,\delta]} f(s)$. Given $u(\cdot)\in X_T$, set

$$\widetilde{\nu}(u(t_{i-1}),u(t_i)) = \begin{cases} P_\ell(u(t_{i-1}))^\ell & \text{if } u(t_i)-u(t_{i-1})=1,\ t_i\ge \delta T,\\ Q_m(u(t_{i-1}))^m & \text{if } u(t_i)-u(t_{i-1})=-1,\ t_i\ge \delta T,\\ P_\ell(T(M_\delta+\varepsilon))^\ell & \text{if } u(t_i)-u(t_{i-1})=1,\ t_i< \delta T,\\ Q_m(T(M_\delta+\varepsilon))^m & \text{if } u(t_i)-u(t_{i-1})=-1,\ t_i< \delta T.\end{cases}$$

As above, the t_i are the times of jumps in $u(\cdot)$.

If $\zeta_T\in U_\varepsilon(f)$, then by (2) and because of the form of the function $\widetilde{\nu}(\zeta(t_{i-1}),\zeta(t_i))$, for T sufficiently large and $t_{i-1}<\delta T$, we have the inequality

$$\widetilde{\nu}(\zeta(t_{i-1}),\zeta(t_i)) \ge \nu(\zeta(t_{i-1}),\zeta(t_i)). \tag{26}$$

Next, if $\zeta_T\in U_\varepsilon(f)$ and ε is sufficiently small, then for $s>\delta$ we have $\zeta_T(s) > \min\limits_{s\in[\delta,1]} f(s)-\varepsilon > 0$. Thus, for $t_{i-1}\ge\delta T$, condition (2) implies that for any $\gamma_2\in(0,1)$ and T large enough, we have

$$(1-\gamma_2) \le \frac{\nu(\zeta(t_{i-1}),\zeta(t_i))}{\widetilde{\nu}(\zeta(t_{i-1}),\zeta(t_i))} \le (1+\gamma_2). \tag{27}$$

Owing to inequalities (26) and (27), for any $\gamma_2>0$ and T sufficiently large,

$$\begin{aligned}&\prod_{i=1}^{N_T(\zeta)} \nu(\zeta(t_{i-1}),\zeta(t_i))\mathbf{1}(\zeta_T\in U_\varepsilon(f), N_T(\zeta)>T^a)\\ &\qquad\le (1+\gamma_2)^{N_T(\zeta)} \prod_{i=1}^{N_T(\zeta)} \widetilde{\nu}(\zeta(t_{i-1}),\zeta(t_i))\mathbf{1}(\zeta_T\in U_\varepsilon(f), N_T(\zeta)>T^a).\end{aligned}$$

Next, set

$$\widetilde{f}_\delta(s) = \begin{cases} M_\delta & \text{if } t\in[0,\delta),\\ f(s) & \text{if } t\in[\delta,1].\end{cases}$$

From the form of $\widetilde{\nu}(\zeta(t_{i-1}),\zeta(t_i))$ it follows that for $\zeta_T\in U_\varepsilon(f)$ one of the inequalities

$$\widetilde{\nu}(\zeta(t_{i-1}),\zeta(t_i)) \le P_\ell(T(\widetilde{f}_\delta(t_{i-1}/T)+\varepsilon))^\ell \tag{28}$$

or

$$\widetilde{\nu}(\zeta(t_{i-1}),\zeta(t_i)) \le Q_m(T(\widetilde{f}_\delta(t_{i-1}/T)+\varepsilon))^m \tag{29}$$

holds true, depending on the sign of $\zeta(t_i)-\zeta(t_{i-1})$.

If $\zeta_T\in U_\varepsilon(f)$, then, by virtue of (8), the process $\zeta_T(\cdot)$ has $\dfrac{N_T(\zeta)+L}{2}$ positive and $\dfrac{N_T(\zeta)-L}{2}$ negative jumps. Hence, from (28) and (29) we obtain

$$\begin{aligned}&\prod_{i=1}^{N_T(\zeta)} \nu(\zeta(t_{i-1}),\zeta(t_i))\mathbf{1}(\zeta_T\in U_\varepsilon(f), N_T(\zeta)>T^a)\\ &\qquad\le \begin{cases} (1+\gamma_2)^{N_T(\zeta)} T^{vL/2} P_\ell^{\frac{N_T(\zeta)+L}{2}} Q_m^{\frac{N_T(\zeta)-L}{2}} \prod\limits_{i=1}^{N_T(\zeta)} T^{\frac{\ell+m}{2}}(M+\varepsilon)^v & \text{if } \ell\ne m,\\ (1+\gamma_2)^{N_T(\zeta)} P_\ell^{\frac{N_T(\zeta)+L}{2}} Q_m^{\frac{N_T(\zeta)-L}{2}} \prod\limits_{i=1}^{N_T(\zeta)} T^\ell(\widetilde{f}_\delta(t_{i-1}/T)+\varepsilon)^\ell & \text{if } \ell=m.\end{cases}\end{aligned} \tag{30}$$

Set

$$k_2(T) := \min\big(1, (P_\ell/Q_m)^{(f(1)-\varepsilon)T/2}\big), \qquad k_3(T) := \max\big(1, (P_\ell/Q_m)^{(f(1)+\varepsilon)T/2}\big).$$

Then from (7) it follows that

$$k_2(T) \le \left(\frac{P_\ell}{Q_m}\right)^{L/2} \le k_3(T). \tag{31}$$

In addition, set $k_4 = \left(\dfrac{\widetilde{f}_\delta(0)+\varepsilon}{\widetilde{f}_\delta(t_{N_T(\zeta)}/T)+\varepsilon}\right)^{\ell}$. By inequalities (30) and (31), for T sufficiently large we have

$$E_2 \le \begin{cases} k_3(T)T^{\frac{v(M+\varepsilon)T}{2}}\,\mathbf{E}\prod\limits_{i=1}^{N_T(\zeta)} 2P_\ell^{\frac12}Q_m^{\frac12}(1+\gamma_2)T^{(\ell+m)/2}(M+\varepsilon)^v & \text{if } \ell\ne m,\\ k_3(T)k_4\,\mathbf{E}\prod\limits_{i=1}^{N_T(\zeta)} 2P_\ell^{\frac12}Q_m^{\frac12}(1+\gamma_2)T^{\ell}(\widetilde{f}_\delta(t_i/T)+\varepsilon)^\ell & \text{if } \ell=m.\end{cases}$$

Following Remark 3 (see the Appendix), we get an exponential bound for E_2:

$$E_2 \le \begin{cases} k_3(T)e^{-T}T^{\frac{v(M+\varepsilon)T}{2}}\exp\Big\{2P_\ell^{\frac12}Q_m^{\frac12}(1+\gamma_2)T^{(\ell+m)/2+1}(M+\varepsilon)^v\Big\} & \text{if } \ell\ne m,\\ k_3(T)k_4e^{-T}\exp\Big\{2P_\ell^{\frac12}Q_m^{\frac12}(1+\gamma_2)T^{\ell+1}\int\limits_0^1(\widetilde{f}_\delta(s)+\varepsilon)^\ell\,ds\Big\} & \text{if } \ell=m.\end{cases}$$

Then, for T sufficiently large, selecting $a < \dfrac{\ell+m}{2}+1$, we obtain from (25) that

$$\mathbf{E}\big(e^{B_T+N_T(\zeta)\ln 2};\zeta_T\in U_\varepsilon(f)\big) = E_1+E_2 \le 2E_2.$$

Finally, by taking into account that the value $\ln\big(k_3(T)T^{\frac{v(M+\varepsilon)T}{2}}\big)$ is of order $T\ln T$, while $\ln(k_3(T)k_4)$ is of order T, we conclude that for any $\gamma_2\in(0,1)$ and $\delta\in(0,1)$ the following bounds hold true:

$$\lim_{\varepsilon\to0}\limsup_{T\to\infty}\frac{1}{T^{v+1}}\ln\mathbf{E}(e^{B_T+N_T(\zeta)\ln 2};\zeta_T\in U_\varepsilon(f)) \le \begin{cases}0 & \text{if } \ell\ne m,\\ 2\sqrt{P_\ell Q_m}(1+\gamma_2)\int\limits_0^1\widetilde{f}_\delta^{\,\ell}(s)\,ds & \text{if } \ell=m.\end{cases}$$

Taking the limit as $\gamma_2\to0$ and $\delta\to0$ completes the proof. △

Proof of Lemma 3. Now let us lower bound E_2 from (24). As above, we fix a sufficiently small ε until the end of the argument. Throughout what follows, $[\,\cdot\,]$ stands for the integer part.

Introduce the event $D := \{\omega : \max\limits_{1\le k\le N_T(\zeta)+1}\tau_k \le T^{1-\beta}\}$, where $1<\beta<a$ and $\tau_{N_T(\zeta)+1} := T - t_{N_T(\zeta)}$. Also consider the event $C_\varepsilon := \{\omega : \inf\limits_{t\in[t_{[\varepsilon T/4]},T]}\zeta(t) > \varepsilon/16\}$, where $t_{[\varepsilon T/4]}$ is the time of the $[\varepsilon T/4]$th jump in $\zeta(t)$.

Obviously,

$$\begin{aligned} E_2 &= \mathbf{E}\,2^{N_T(\zeta)}\prod_{i=1}^{N_T(\zeta)}\nu(\zeta(t_{i-1}),\zeta(t_i))\mathbf{1}(\zeta_T\in U_\varepsilon(f), N_T(\zeta)>T^a)\\ &\ge \mathbf{E}\,2^{N_T(\zeta)}\prod_{i=1}^{N_T(\zeta)}\nu(\zeta(t_{i-1}),\zeta(t_i))\mathbf{1}(D, C_\varepsilon, \zeta_T\in U_\varepsilon^+(f), N_T(\zeta)>T^a),\end{aligned}$$

where $U_\varepsilon^+(f) := \{g : \min\limits_{t\in[0,1]} g(t)\ge 0\}\cap U_\varepsilon(f)$.

Let $\delta = \min\{s : \min\limits_{t\in[s,1]} f(t) \ge 2\varepsilon\}$. Denote $r(\delta) := \min\{i : t_i \ge T\delta\}$.

Assume that $\zeta_T \in U_\varepsilon^+(f)$ and $r(\delta)+1 \le i \le N_T(\zeta)$. Then by condition (2), for any $\gamma_3 \in (0,1)$ and T large enough, we have either

$$\nu(\zeta(t_{i-1}),\zeta(t_i)) \ge (1-\gamma_3)P_\ell(T(f(t_{i-1}/T)-\varepsilon))^\ell \tag{32}$$

or

$$\nu(\zeta(t_{i-1}),\zeta(t_i)) \ge (1-\gamma_3)Q_m(T(f(t_{i-1}/T)-\varepsilon))^m, \tag{33}$$

depending on the sign of $\zeta(t_i)-\zeta(t_{i-1})$.

If the event C_ε has occurred and $[\varepsilon T/4] \le i \le r(\delta)$, then, owing to condition (2), for any $\gamma_3 \in (0,1)$ and a sufficiently large T the following inequality holds true:

$$\nu(\zeta(t_{i-1}),\zeta(t_i)) \ge (1-\gamma_3)(T\varepsilon/16)^w, \tag{34}$$

where $w := \min(\ell, m)$.

For $\zeta_T \in U_\varepsilon^+(f)$ and $1 \le i \le [\varepsilon T/4]$ we have

$$\nu(\zeta(t_{i-1}),\zeta(t_i)) \ge k_5 := \min\Big(\inf_{x\in\mathbb{Z}^+}\lambda(x), \inf_{x\in\mathbb{N}}\mu(x)\Big). \tag{35}$$

Let us introduce the function

$$\widehat{f}_\varepsilon(s) = \begin{cases} \dfrac{\varepsilon}{16\max(1,P_\ell,Q_m)} & \text{if } s\in[0,\delta),\\ f(s)-\varepsilon & \text{if } s\in[\delta,1]. \end{cases}$$

Using (8) and (32)–(35), we get the bound

$$E_2 \ge k_6(T)\,\mathbf{E}\,P_\ell^{\frac{N_T(\zeta)+L}{2}} Q_m^{\frac{N_T(\zeta)-L}{2}}(1-\gamma_3)^{N_T(\zeta)}2^{N_T(\zeta)} \times \prod_{i=[\varepsilon T/4]+1}^{N_T(\zeta)} (T\widehat{f}_\varepsilon(t_{i-1}/T))^w \mathbf{1}(D, C_\varepsilon, \zeta_T\in U_\varepsilon^+(f), N_T(\zeta) > T^a),$$

where $k_6(T) := \Big(\dfrac{k_5}{\max(P_\ell,Q_m)}\Big)^{[\varepsilon T/4]}$.

From inequalities (7) and (31) we obtain

$$E_2 \ge k_7(T)\,\mathbf{E}\,P_\ell^{\frac{N_T(\zeta)}{2}} Q_m^{\frac{N_T(\zeta)}{2}}(1-\gamma_3)^{N_T(\zeta)}2^{N_T(\zeta)} \times \prod_{i=1}^{N_T(\zeta)} (T\widehat{f}_\varepsilon(t_i/T))^w \mathbf{1}(D, C_\varepsilon, \zeta_T\in U_\varepsilon^+(f), N_T(\zeta) > T^a),$$

where $k_7(T) := \dfrac{k_6(T)k_2(T)}{(MT)^{w[\varepsilon T/4]}}$.

From Lemma 6 (see the Appendix) it follows that for any $\gamma_4 \in (0,1)$ and T sufficiently large the following holds true:

$$E_2 \ge k_7(T)\sum_{n=[T^a]+1}^{\infty} 2^n(1-\gamma_4)^n P_\ell^{\frac{n}{2}} Q_m^{\frac{n}{2}}\,\mathbf{E}\prod_{i=1}^{n} T^w(\widehat{f}_\varepsilon(t_i/T))^w\mathbf{1}(D, N_T(\zeta)=n). \tag{36}$$

Here γ_4 is expressed via γ_3 and θ, whereas $\theta\in(0,1)$ is introduced in Lemmas 5 and 6. To estimate the product from (36), we use Lemma 4. Taking into account that $n > T^a$, we get that for T large enough,

$$\begin{aligned}
&\mathbf{E}\prod_{i=1}^{n} T^w(\widehat{f}_\varepsilon(t_i/T))^w \mathbf{1}(D, N_T(\zeta)=n)\\
&\quad= \mathbf{E}\prod_{i=1}^{n} T^w(\widehat{f}_\varepsilon(t_i/T))^w \mathbf{1}(N_T(\zeta)=n) - \mathbf{E}\prod_{i=1}^{n} T^w(\widehat{f}_\varepsilon(t_i/T))^w \mathbf{1}(\overline{D}, N_T(\zeta)=n)\\
&\quad\ge \frac{\left(T^w\int\limits_0^T(\widehat{f}_\varepsilon(t_i/T))^w\,dt\right)^n}{n!}e^{-T} - 2T^\beta \frac{\left(T^w\int\limits_0^T(\widehat{f}_\varepsilon(t_i/T))^w\,dt - T^{w+1}\alpha_{\frac{1}{T^\beta}}\right)^n}{n!}e^{-T}, \qquad (37)
\end{aligned}$$

Here $\alpha_{\frac{1}{T^\beta}} = \dfrac{1}{2T^\beta}\inf\limits_{s\in[0,1]}(\widehat{f}_\varepsilon(s))^w = \dfrac{1}{2T^\beta}\Big(\dfrac{\varepsilon}{16\max(1,P_\ell,Q_m)}\Big)^w$ (cf. equation (40) in the Appendix).

Now let us now estimate the last term on the right-hand side of (37). Denote $k_8 := \sup\limits_{s\in[0,1]}(\widehat{f}_\varepsilon(s))^w$. Since $a > \beta$, for a sufficiently large T the following inequalities hold true:

$$\begin{aligned}
&2T^\beta\left(T^w\int\limits_0^T(\widehat{f}_\varepsilon(t/T))^w\,dt - T^{w+1}\alpha_{\frac{1}{T^\beta}}\right)^n\\
&\qquad\le 2T^\beta\left(T^w\int\limits_0^T(\widehat{f}_\varepsilon(t/T))^w\,dt\right)^n\left(1-\frac{\varepsilon^w}{2k_8T^\beta(16\max(1,P_\ell,Q_m))^w}\right)^n\\
&\qquad\le 2T^\beta\left(T^w\int\limits_0^T(\widehat{f}_\varepsilon(t/T))^w\,dt\right)^n\left(1-\frac{\varepsilon^w}{2k_8T^\beta(16\max(1,P_\ell,Q_m))^w}\right)^{T^a}\\
&\qquad\le \left(T^w\int\limits_0^T(\widehat{f}_\varepsilon(t/T))^w\,dt\right)^n \exp\left(\beta\ln(2T) - \frac{\varepsilon^w}{2k_8(16\max(1,P_\ell,Q_m))^w}T^{a-\beta}\right)\\
&\qquad\le \frac{1}{2}\left(T^w\int\limits_0^T(\widehat{f}_\varepsilon(t/T))^w\,dt\right)^n.
\end{aligned}$$

Consequently, from (37) it follows that

$$\mathbf{E}\prod_{i=1}^{n} T^w(\widehat{f}_\varepsilon(t_i/T))^w \mathbf{1}(D, N_T(\zeta)=n) \ge \frac{1}{2}\frac{\left(T^w\int\limits_0^T(\widehat{f}_\varepsilon(t/T))^w\,dt\right)^n}{n!}e^{-T}.$$

By virtue of (36), for T sufficiently large,

$$E_2 \ge \frac{k_7(T)}{2}\sum_{n=[T^a]+1}^{\infty} 2^n(1-\gamma_4)^n(P_\ell Q_m)^{n/2}\frac{\left(T^w\int\limits_0^T(\widehat{f}_\varepsilon(t/T))^w\,dt\right)^n}{n!}e^{-T}.$$

From this it follows that, selecting $a < w+1$, for T large enough we obtain the inequalities

$$\begin{aligned}
E_2 &\ge \frac{k_7(T)e^{-T}}{2}\exp\left(2(1-\gamma_4)\sqrt{P_\ell Q_m}\,T^{w+1}\int\limits_0^1(\widehat{f}_\varepsilon(s))^w\,ds\right)\\
&\quad - \frac{k_7(T)e^{-T}}{2}\exp(a\ln(T) + (w+2)T^a\ln(T))\\
&\ge \frac{k_7(T)e^{-T}}{4}\exp\left(2(1-\gamma_4)\sqrt{P_\ell Q_m}\,T^{w+1}\int\limits_0^1(\widehat{f}_\varepsilon(s))^w\,ds\right). \qquad (38)
\end{aligned}$$

By virtue of (38) and the fact that $\ln k_7(T)$ is a quantity of order $T\ln T$, we now conclude that

$$\liminf_{T\to\infty}\frac{1}{T^{v+1}}\ln\mathbf{E}\big(e^{B_T+N_T(\zeta)\ln 2};\zeta_T\in U_\varepsilon(f)\big)\ge\begin{cases}0 & \text{if } \ell\ne m,\\ 2(1-\gamma_4)\sqrt{P_\ell Q_m}\int\limits_0^1\widehat{f}_\varepsilon(s)^\ell\,ds & \text{if } \ell=m.\end{cases}$$

Furthermore, taking into account the definition of function $\widehat{f}(s)$, we obtain

$$\lim_{\varepsilon\to 0}\liminf_{T\to\infty}\frac{1}{T^{v+1}}\ln\mathbf{E}\big(e^{B_T+N_T(\zeta)\ln 2};\zeta_T\in U_\varepsilon(f)\big)\ge\begin{cases}0 & \text{if } \ell\ne m,\\ 2(1-\gamma_4)\sqrt{P_\ell Q_m}\int\limits_\delta^1 f^\ell(s)\,ds & \text{if } \ell=m.\end{cases}$$

Taking the limit as $\delta\to 0$ and $\gamma_4\to 0$ completes the proof of the lemma. △

APPENDIX

Here we prove the auxiliary assertions used in the arguments above.

Let $X_T^{(n)}$ stand for the event that the process ζ has exactly n jumps on the interval $[0,T]$.

Lemma 4. *Let $g(t)$ be a nonnegative bounded Borel function and $n\ge 1$. Then*

$$\mathbf{E}\prod_{i=1}^n g(t_i)\mathbf{1}(X_T^{(n)})=\frac{\left(\int\limits_0^T g(s)\,ds\right)^n}{n!}e^{-T},\tag{39}$$

$$\mathbf{E}\prod_{i=1}^n g(t_i)\mathbf{1}(X_T^{(n)})\mathbf{1}\Big(\max_{1\le k\le n+1}\tau_k>T\Delta\Big)\le\frac{2}{\Delta}\frac{\left(\int\limits_0^T g(s)\,ds-T\alpha_\Delta\right)^n}{n!}e^{-T}.\tag{40}$$

Here $\Delta>0$ is a constant and $\alpha_\Delta:=\dfrac{\Delta}{2}\inf\limits_{t\in[0,T]}g(t)$. Further, $t_1,\dots,t_n$ are jump times on $[0,T]$ in the process $\zeta(\cdot)$, and $\tau_{n+1}:=T-t_n$.

Proof. First, we prove (39). To this end, write

$$\mathbf{E}\bigg(\prod_{i=1}^n g(t_i)\;\Big|\;X_T^{(n)}\bigg)=\mathbf{E}\bigg(\prod_{i=1}^n g(t_i)\;\Big|\;\eta(T)=n\bigg),$$

where η is a Poisson process with mean $\mathbf{E}\,\eta(t)=t$.

From [15, Theorem 2.3, p. 126] it follows that

$$\mathbf{E}\bigg(\prod_{i=1}^n g(t_i)\;\Big|\;\eta(T)=n\bigg)=\frac{n!}{T^n}\int\limits_0^T\bigg(\int\limits_{s_1}^T\dots\bigg(\int\limits_{s_{n-1}}^T\prod_{i=1}^n g(s_i)\,ds_n\bigg)\dots ds_2\bigg)ds_1=\frac{1}{T^n}\bigg(\int\limits_0^T g(s)\,ds\bigg)^n.$$

Therefore,

$$\mathbf{E}\prod_{i=1}^n g(t_i)\mathbf{1}(X_T^{(n)})=\frac{1}{T^n}\bigg(\int\limits_0^T g(s)\,ds\bigg)^n\mathbf{P}(\eta(T)=n)=\frac{\left(\int\limits_0^T g(s)\,ds\right)^n}{n!}e^{-T}.$$

Next, we turn to the proof of (40):

$$\mathbf{E}\prod_{i=1}^n g(t_i)\mathbf{1}(X_T^{(n)})\mathbf{1}\Big(\max_{1\le k\le n+1}\tau_k>T\Delta\Big)$$
$$\le\sum_{r=1}^{[2/\Delta]}\mathbf{E}\prod_{i=1}^n g(t_i)\mathbf{1}(\eta(T)=n)\mathbf{1}\bigg(\eta\Big(\frac{rT\Delta}{2}\Big)-\eta\Big(\frac{(r-1)T\Delta}{2}\Big)=0\bigg):=\sum_{r=1}^{[2/\Delta]}D_r;$$

here we used the fact that if $\max\limits_{1\le k\le n+1}\tau_k > T\Delta$, then there exists an r with $1 \le r \le \left[\frac{2}{\Delta}\right]$ and with no jumps on the interval $\left[\frac{(r-1)T\Delta}{2}, \frac{rT\Delta}{2}\right]$.

Consider

$$D_1 = \mathbf{E}\prod_{i=1}^n g(t_i)\mathbf{1}(\eta(T)=n)\mathbf{1}\left(\eta\left(\frac{T\Delta}{2}\right)=0\right)$$
$$= \mathbf{E}\prod_{i=1}^n g(t_i)\mathbf{1}\left(\eta(T)-\eta\left(\frac{T\Delta}{2}\right)=n\right)\mathbf{1}\left(\eta\left(\frac{T\Delta}{2}\right)=0\right).$$

By using the independence of increments in and the homogeneity of the Poisson process and formula (39), we obtain

$$D_1 = \frac{\left(\int\limits_{\frac{T\Delta}{2}}^{T} g(s)\,ds\right)^n}{n!}e^{-T(1-\Delta/2)}\,\mathbf{P}\left(\eta\left(\frac{T\Delta}{2}\right)=0\right) = \frac{\left(\int\limits_{\frac{T\Delta}{2}}^{T} g(s)\,ds\right)^n}{n!}e^{-T}.$$

Similarly, for any r, $1 \le r \le \left[\frac{2}{\Delta}\right]$, one obtains

$$D_r = \frac{\left(\int\limits_{[0,T]/B_{r,\Delta}} g(s)\,ds\right)^n}{n!}e^{-T},$$

where $B_{r,\Delta} = \left[\frac{(r-1)T\Delta}{2}, \frac{rT\Delta}{2}\right]$.

In view of the relations $\min\limits_{1\le r\le\left[\frac{2}{\Delta}\right]}\int\limits_{\left[\frac{(r-1)T\Delta}{2},\frac{rT\Delta}{2}\right]} g(s)\,ds \ge T\frac{\Delta}{2}\inf\limits_{s\in[0,T]} g(s) = T\alpha_\Delta$, we get

$$\mathbf{E}\prod_{i=1}^n g(t_i)\mathbf{1}(X_T^{(n)})\mathbf{1}\left(\max_{1\le k\le n+1}\tau_k > T\Delta\right) \le \frac{2}{\Delta}\frac{\left(\int\limits_0^T g(s)\,ds - T\alpha_\Delta\right)^n}{n!}e^{-T}. \quad \triangle$$

Remark 3. Lemma 4 implies that

$$\mathbf{E}\prod_{i=1}^{\eta(T)} g(t_i)\mathbf{1}(\eta(T)\ge 1) = e^{-T}\left(\exp\left\{\int\limits_0^T g(s)\,ds\right\}-1\right).$$

Lemma 5. *Consider a sequence $b_1, b_2, \ldots, b_n$ where each b_i equals either -1 or 1. Denote by c_d the number of sequences with the following property*:

$$\left|\sum_{k=1}^r b_k\right| \le d, \quad \forall r,\ 1\le r\le n.$$

Take $d = [T\Delta]$ and $n = O(T^\beta)$ where $T\to\infty$ while $\Delta > 0$ and $\beta > 1$. Then for any $\theta\in(0,1)$ and a sufficiently large T we have the bound

$$c_d \ge (1-\theta)^{n+1}2^n.$$

Proof. It is clear that if a sequence $b_{2(p-1)d+1}, \ldots, b_{2pd}$ with $1 \le p \le \frac{n}{2d}$ has equally many 1's and -1's, and in the sequence $b_{2d[\frac{n}{2d}]+1}, \ldots, b_n$ the difference between the numbers of 1's and -1's is at most 1 in the absolute value, then the required property is fulfilled. The number of such sequences is not less than $(C_{2d}^d)^{[\frac{n}{2d}]}$.

Using Stirling's formula gives

$$(C_{2d}^d)^{[\frac{n}{2d}]} \sim \left(\frac{\sqrt{2}(2d)^{2d}}{\sqrt{\pi d}d^{2d}}\right)^{[\frac{n}{2d}]} = \left(\frac{\sqrt{2}2^{2d}}{\sqrt{\pi d}}\right)^{[\frac{n}{2d}]} \geq 2^{n-2d}(\pi d)^{-\frac{n}{4d}}.$$

Thus, owing to the fact that $-2d\ln 2 - \dfrac{n\ln \pi d}{4d} = o(n)$, we obtain that, for any $\theta \in (0,1)$ and T sufficiently large,

$$c_d \geq (1-\theta)2^n \exp\left\{-2d\ln 2 - \frac{n\ln\pi d}{4d}\right\} \geq (1-\theta)^{n+1}2^n. \quad \triangle$$

Lemma 6. *Take $\beta > 1$ and $n \geq T^\beta$, and let $g(\cdot)$ be a nonnegative bounded Borel function. For any $\theta > 0$ and all T sufficiently large, the following estimate holds true*:

$$\begin{aligned}\mathbf{E}\, g(t_1,\dots,t_n)\mathbf{1}\Big(\max_{1\le k\le n+1}\tau_k \le T^{1-\beta}\Big)\mathbf{1}(X_T^{(n)})\mathbf{1}(\zeta_T \in U_\varepsilon^+(f))\mathbf{1}(C_\varepsilon)\\ \geq (1-\theta)^{2n}\,\mathbf{E}\, g(t_1,\dots,t_n)\mathbf{1}\Big(\max_{1\le k\le n+1}\tau_k \le T^{1-\beta}\Big)\mathbf{1}(X_T^{(n)}),\end{aligned}$$

where $U_\varepsilon^+(f) := \{g : \min\limits_{t\in[0,1]} g(t) \geq 0\} \cap U_\varepsilon(f)$, $C_\varepsilon := \{\omega : \inf\limits_{t\in[t_{[\varepsilon T/4]},T]} \zeta(t) > \varepsilon/16\}$, and $t_{[\varepsilon T/4]}$ is the point of the $[\varepsilon T/4]$th jump in process $\zeta(t)$.

Proof. Since $f(t)$ is uniformly continuous on $[0,1]$, for $\delta > 0$ sufficiently small we have the inequality

$$\sup_{s,t:\,|s-t|\le\delta} |f(s)-f(t)| < \frac{\varepsilon}{4}.$$

Fix δ with $1/\delta \in \mathbb{N}$, and let $1 \leq r \leq 1/\delta$.

Let $B_{m_r,\delta r}$ be the event that the process ζ has exactly m_r jumps on the interval $[T\delta(r-1), T\delta r]$. Then we can write

$$\begin{aligned}&\mathbf{E}\, g(t_1,\dots,t_n)\mathbf{1}\Big(\max_{1\le k\le n+1}\tau_k \le T^{1-\beta}\Big)\mathbf{1}(X_T^{(n)})\mathbf{1}(\zeta_T \in U_\varepsilon^+(f))\mathbf{1}(C_\varepsilon)\\ &= \sum_{m_1,\dots,m_{1/\delta}} \mathbf{E}\, g(t_1,\dots,t_n)\mathbf{1}\Big(\max_{1\le k\le n+1}\tau_k \le T^{1-\beta}\Big)\mathbf{1}(X_T^{(n)})\mathbf{1}(\zeta_T \in U_\varepsilon^+(f))\mathbf{1}(C_\varepsilon)\prod_{r=1}^{1/\delta}\mathbf{1}(B_{m_r,\delta r}),\end{aligned}$$

The summation here is over all collections with $\min\limits_r m_r \geq \delta T^\beta$, $\sum\limits_r m_r = n$.

Take a collection $m_1,\dots,m_r$ satisfying the above condition. Consider a piece of a trajectory of ζ on the interval $[0,\delta T]$. Denote by $t_{1,1},\dots,t_{m_1,1}$ the jump points of ζ lying in this interval. Assume that the jumps satisfy the following conditions:

1. The jumps at times $t_{1,1},\dots,t_{d_{\varepsilon,1},1}$ are positive, where $d_{\varepsilon,1} := [T\varepsilon/4]$;
2. The jumps at times $t_{d_{\varepsilon,1}+1,1},\dots,t_{m_1,1}$ are such that for any integer $k \in [d_{\varepsilon,1}+1, m_1]$ we have the inequality
$$\left|\sum_{\ell=d_{\varepsilon,1}+1}^{k} \zeta(t_{\ell,1})\right| \leq \left[\frac{T\varepsilon}{8}\right].$$

Then, for T large enough, the trajectory $\zeta_T(t)$ has the following properties:

1. The trajectory is nonnegative and lies in an ε-neighborhood of f for $t \in [0,\delta]$;
2. $\zeta_T(t) \geq \varepsilon/16$ for $t > t_{d_{\varepsilon,1},1}/T$;
3. $|\zeta_T(\delta) - f(\delta)| \leq 3\varepsilon/8$.

Now consider a piece of a trajectory of $\zeta(\cdot)$ defined on the interval $[\delta T, 2\delta T]$ and having the property $|\zeta(\delta T) - Tf(\delta)| \le 3\varepsilon T/8$. Denote by $t_{1,2}, \ldots, t_{m_2,2}$ the jump points of $\zeta(\cdot)$ in this interval. Let these jumps satisfy the following conditions:

1. At times $t_{1,2}, \ldots, t_{|d_{\varepsilon,2}|,2}$ the jumps are positive or negative in accordance with the sign of $d_{\varepsilon,2} := [T(\max(\varepsilon/4, f(2\delta)) - \zeta_T(\delta))]$;
2. At times $t_{|d_{\varepsilon,2}|+1,2}, \ldots, t_{m_2,2}$, the jumps are such that for any integer $k \in [|d_{\varepsilon,2}| + 1, m_2]$ the following inequality holds true:

$$\left| \sum_{\ell=|d_{\varepsilon,2}|+1}^{k} \zeta(t_{\ell,2}) \right| \le \left[\frac{T\varepsilon}{8} \right].$$

Then, again for T large enough, the trajectory $\zeta_T(t)$ has the following properties:

1. The trajectory is nonnegative and lies in a ε-neighborhood of f for $t \in [\delta, 2\delta]$;
2. $\zeta_T(t) \ge \varepsilon/16$ for $t \in [\delta, 2\delta]$;
3. $|\zeta_T(2\delta) - f(2\delta)| \le 3\varepsilon/8$.

Further pieces of the trajectory are dealt with by induction.

Let us count the trajectories whose jumps satisfy the above properties. Since $\max\limits_{1\le k\le n+1} \tau_k \le T^{1-\beta}$, we have that for any r with $1 \le r \le 1/\delta$ the interval $[T\delta(r-1), T\delta r]$ contains at least $[\delta T^{\beta}]$ jumps of the process $\zeta(\cdot)$, where $\beta > 1$. Using Lemma 5 yields that, when T is sufficiently large, on $[T\delta(r-1), T\delta r]$ the number of pieces of the trajectory with the above-described properties will be not less than

$$(1-\theta)^{m_r+1-|d_{\varepsilon,r}|} 2^{m_r - |d_{\varepsilon,r}|} > (1-\theta)^{2m_r} 2^{m_r}.$$

Consequently, the number of trajectories that fulfill the above properties for all r is not less than

$$\prod_r (1-\theta)^{2m_r} 2^{m_r} = (1-\theta)^{2n} 2^n. \tag{41}$$

Next, the jump directions in ζ are mutually independent and depend on neither the number of jumps within the interval nor the jump times. Hence, we can use equality (41) to obtain

$$\begin{aligned}
&\sum_{m_1,\ldots,m_{1/\delta}} \mathbf{E}\, g(t_1,\ldots,t_n) \mathbf{1}\Big(\max_{1\le k\le n+1} \tau_k \le T^{1-\beta} \Big) \mathbf{1}(X_T^{(n)}) \mathbf{1}(\zeta_T \in U_\varepsilon^+(f)) \mathbf{1}(C_\varepsilon) \prod_{r=1}^{1/\delta} \mathbf{1}(B_{m_r,\delta r}) \\
&\quad \ge \sum_{m_1,\ldots,m_{1/\delta}} \frac{(1-\theta)^{2n} 2^n}{2^n} \mathbf{E}\, g(t_1,\ldots,t_n) \mathbf{1}\Big(\max_{1\le k\le n+1} \tau_k \le T^{1-\beta} \Big) \prod_{r=1}^{1/\delta} \mathbf{1}(B_{m_r,\delta r}) \\
&\quad = (1-\theta)^{2n}\, \mathbf{E}\, g(t_1,\ldots,t_n) \mathbf{1}\Big(\max_{1\le k\le n+1} \tau_k \le T^{1-\beta} \Big) \mathbf{1}(X_T^{(n)}). \quad \triangle
\end{aligned}$$

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REFERENCES

1. Dembo, A. and Zeitouni, O., *Large Deviations Techniques and Applications*, New York: Springer, 1998, 2nd ed.

2. Deuschel, J.-D. and Stroock, D.W., *Large Deviations*, Boston: Academic, 1989.
3. den Hollander, F., *Large Deviations*, Providence, RI: Amer. Math. Soc., 2000.
4. Olivieri, E. and Vares, M.E., *Large Deviations and Metastability*, Cambridge, UK: Cambridge Univ. Press, 2005.
5. Puhalskii, A., *Large Deviations and Idempotent Probability*, Boca Raton, FL: Chapman & Hall/CRC, 2001.
6. Varadhan, S.R.S., *Large Deviations and Applications*, Philadelphia: SIAM, 1984.
7. Suhov, Y. and Stuhl, I., On Principles of Large Deviation and Selected Data Compression, `arXiv:1604.06971 [cs.IT]`, 2016.
8. Suhov, Y.M. and Stuhl, I., Selected Data Compression: A Refinement of Shannon's Principle, *Analytical and Computational Methods in Probability Theory (Proc. 1st Int. Conf. ACMPT'2017, Moscow, Russia, Oct. 23–27, 2017)*, Rykov, V., Singpurwalla, N.D., and Zubkov, A.M., Eds., Lect. Notes Comp. Sci., vol. 10684, New York: Springer, 2018.
9. Mazel, A., Suhov, Yu., Stuhl, I., and Zohren, S., Dominance of Most Tolerant Species in Multi-type Lattice Widom–Rowlinson Models, *J. Stat. Mech.*, 2014, no. 8, p. P08010.
10. Kelbert, M., Stuhl, I., and Suhov, Y., Weighted Entropy and Optimal Portfolios for Risk-Averse Kelly Investments, *Aequationes Math.*, 2018, vol. 92, no. 1, pp. 165–200.
11. Mogulskii, A., Pechersky, E., and Yambartsev, A., Large Deviations for Excursions of Non-homogeneous Markov Processes, *Electron. Commun. Probab.*, 2014, vol. 19, Paper no. 37 (8 pp.).
12. Vvedenskaya, N., Suhov, Y., and Belitsky, V., A Non-linear Model of Trading Mechanism on a Financial Market, *Markov Process. Related Fields*, 2013, vol. 19, no. 1, pp. 83–98.
13. Feller, W., *An Introduction to Probability Theory and Its Applications*, New York: Wiley, 1966. Translated under the title *Vvedenie v teoriyu veroyatnostei i ee prilozheniya*, Moscow: Mir, 1967, 2 vols.
14. Kelbert, M. and Suhov, Y., *Probability and Statistics by Example*, vol. 2: *Markov Chains: A Primer in Random Processes and Their Applications*, Cambridge, UK: Cambridge Univ. Press, 2008. Translated under the title *Veroyatnost' i statistika v primerakh i zadachakh*, vol. 2: *Markovskie tsepi kak otpravnaya tochka teorii sluchainykh protsessov*, Moscow: MCCME, 2010.
15. Karlin, S. and Taylor, H.M., *A First Course in Stochastic Processes*, New York: Academic, 1975, 2nd ed.
16. Korolyuk, V.S., Portenko, N.I., Skorokhod, A.V., and Turbin, A.F., *Spravochnik po teorii veroyatnostei i matematicheskoi statistike* (Handbook in Probability Theory and Mathematical Statistics), Moscow: Nauka, 1985, 2nd ed.
17. Dynkin, E.B., *Markovskie protsessy*, Moscow: Fizmatgiz, 1963. Translated under the title *Markov Processes*, Berlin: Springer; New York: Academic, 1965.
18. Itô, K., *Veroyatnostnye protsessy* (Stochastic Processes), vol. II, Moscow: Inostr. Lit., 1963.
19. Itô, K., *Stochastic Processes: Lectures Given at Aarhus University*, Berlin: Springer, 2004.
20. Itô, K., *Essentials of Stochastic Processes*, Providence, RI: Amer. Math. Soc., 2006.
21. Karlin, S. and McGregor, J., The Classification of Birth and Death Processes, *Trans. Amer. Math. Soc.*, 1957, vol. 86, no. 2, pp. 366–400.
22. Ledermann, W. and Reuter, G.E.H., Spectral Theory for the Differential Equations of Simple Birth and Death Processes, *Philos. Trans. Roy. Soc. London, Ser. A*, 1954, vol. 246, pp. 321–369.
23. Norris, J.R., *Markov Chains*, Cambridge, UK: Cambridge Univ. Press, 1998.
24. Stroock, D.W., *An Introduction to Markov Processes*, Berlin: Springer, 2014, 2nd ed.
25. Borovkov, A.A. and Mogul'skii, A.A., On Large Deviation Principles in Metric Spaces, *Sibirsk. Mat. Zh.*, 2010, vol. 51, no. 6, pp. 1251–1269 [*Siberian Math. J.* (Engl. Transl.), 2010, vol. 51, no. 6, pp. 989–1003].

26. Borovkov, A.A. and Mogulskii, A.A., Large Deviation Principles for Random Walk Trajectories. I, *Teor. Veroyatn. Primen.*, 2011, vol. 56, no. 4, pp. 627–655 [*Theory Probab. Appl.* (Engl. Transl.), 2011, vol. 56, no. 4, pp. 538–561].

27. Logachov, A.V., The Local Principle of Large Deviations for Solutions of Itô Stochastic Equations with Quick Drift, *Ukr. Mat. Visn.*, 2015, vol. 12, no. 4, pp. 457–471 [*J. Math. Sci. (N.Y.)* (Engl. Transl.), 2016, vol. 218, no. 1, pp. 28–38].

28. Borovkov, A.A. and Mogul'skiĭ, A.A., Iniqualities and Principle of Large Deviations for the Trajectories of Processes with Independent Increments, *Sibirsk. Mat. Zh.*, 2013, vol. 54, no. 2, pp. 286–297 [*Siberian Math. J.* (Engl. Transl.), 2013, vol. 54, no. 2, pp. 217–226].

29. Dunford, N. and Schwartz, J.T., *Linear Operators*, Part 1: *General Theory*, New York: Interscience, 1958. Translated under the title *Lineinye operatory*, vol. 1: *Obshchaya teoriya*, Moscow: Inostr. Lit., 1962.

30. Rudin, W., *Real and Complex Analysis*, New York: McGraw-Hill, 1987, 3rd ed.