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METHODS OF SIGNAL PROCESSING

Analytical Approach to Multiband Filter Synthesis and Comparison to Other Approaches[1]

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Abstract—A novel analytical approach to the synthesis of multiband electrical filters is presented. This approach allows to obtain the lowest possible order filters for a wide class of specifications including ones with large number of pass- and stopbands and with narrow transition bands. Comparison of the new approach to direct optimization and composite approaches is provided.

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1. INTRODUCTION

Currently, multiband filters are widely used in analog and digital technology. For example, in modern microwave and RF communication systems, filters with several passbands corresponding to various standards of wireless communications are often required. The development of filters with multiband specifications is a difficult engineering task, especially if one wants to simultaneously ensure both high performance of the component and its compact implementation. To the current moment, there are no fully satisfactory and universal solutions to this problem in the literature.

An obvious approach is the composite one: a multiband filter is obtained by interconnecting several single-band filters. Its disadvantage lies in too large orders of the synthesized filters, implying their too large weight and dimensions. This explains the interest in the development of other approaches that are ideologically more sophisticated but allow one to obtain filters of smaller orders under the same specification. At the level of technical implementation, engineers employ methods of microwave filter synthesis based on the use of multimode resonators [1, 2], as well as methods based on the use of frequency transformations (e.g., [3, 4]). The complexity of these approaches grows rapidly with the number of bands and the order of the filter. Also, applicable specifications have significant limitations.

Finally, a common method of filter synthesis is the direct numerical optimization of magnitude response. Here by *optimal* we mean a filter having the smallest order among all physically feasible filters that meet the requirements of a given specification (or mask). The latter consists of the following parameters: (1) boundary frequencies of all pass- and stopbands; (2) maximal acceptable ripple at passbands and (3) minimal acceptable attenuation at stopbands. These data determine the desired shape of the filter magnitude response and only indirectly affect its other characteristics like phase response, impulse response, etc.

Methods of filter synthesis connected with direct numerical optimization usually use Remez-type algorithms (see [5–7].) Due to their principal inherent instability, these methods have a limited

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applicability range: with computations in double precision, the number of filter passbands does not exceed three, as a rule, while the filter order does not exceed twenty.

The authors of the article have developed an analytical approach to the synthesis of optimal multiband filters, which can be applied to specifications with a total number of bands (currently) up to 23 and with orders up to 1000. The aim of the article is to compare the new approach to the synthesis of optimal multiband filters with the direct optimization method based on the Remez-type algorithm, and also to compare optimal filters with nonoptimal composite filters obtained by the corresponding approach. Section 2 introduces the optimization problem underlying the synthesis of the optimal multiband filter. Section 3 contains a brief description of the new analytical approach to this problem. In Section 4 the recipes are given that we used to solve the same problem by direct optimization method. Section 5 describes the composite approach to the synthesis of nonoptimal multiband filters. Section 6 contains the results of numerical experiments made for the comparison of these three approaches.

2. PROBLEM FORMULATION

The search for an optimal filter corresponding to a given specification can be reduced to the solution of a series of least deviation problems of a type given below. To this end, at each step the filter order is fixed and one has to maximize, e.g., the attenuation at stopbands with all other specification parameters fixed. Consider the following two formulations occurring in the literature.

Let E be the set containing m disjoint intervals of the real axis (frequency ranges) divided into two parts, passbands E_+ and stopbands E_-. The ideal transition function F is equal to $+1$ on intervals of E_+ and to -1 on intervals of E_-.

Problem 1. Find a real rational function R_n of degree at most n for which the deviation from the transition function F is minimal in the uniform norm on E:

$$\|R_n - F\|_{C(E)} := \max_{x\in E} |R_n(x) - F(x)| \to \min =: \mu. \tag{1}$$

Problem 2. Find a real rational function $Q_n(x)$ of degree at most n which will minimize the quantity θ under the constraints

$$\min_{w\in E_-} |Q_n(w)| \geq \theta^{-1}, \qquad \max_{w\in E_+} |Q_n(w)| \leq \theta. \tag{2}$$

It is easy to show that these problems are equivalent, while their solutions differ by a linear fractional substitution, i.e., $Q_n = l \circ R_n$, where l is a linear fractional transformation and the quantities inverse to the minimal deviations are related by the Joukowski transformation, $\mu^{-1} = (\theta + \theta^{-1})/2$.

Note that the two given formulations of the rational approximation problem coincide in essence with the third and fourth Zolotarev's problems for the condenser (E_+, E_-) [8]. The problem is multi-extremal: the entire set of rational functions is divided into 2^{m-2} disjoint classes [9, 10], in each of which the solution exists, is unique, and has an equioscillation characterization (or equiripple property): there are $2n+2$ points on E in which the deviation value for the solution of degree n is attained with successive change of sign [11].

Until recently, the exact analytical solution of this problem was known only for the case $m = 2$ (for one passband and one stopband), which was found in 1870s by Chebyshev's pupil E. I. Zolotarev [12]. It was about half a century later that it was used by the German electrical engineer W. Cauer [13] to construct the transfer functions of the optimal high- or low-pass filters, called afterwards the Cauer–Zolotarev (elliptic) filters. To date, these filters are widely used in analog and digital technology, while the method of magnitude response approximation according to Zolotarev–Cauer has become classical.

3. NEW ANALYTICAL APPROACH

Approach to the problems (1) and (2) proposed in this paper is based on explicit analytical formulas obtained in [14] and generalizing Zolotarev's solution to the case of m components of E where m exceeds two. This approach was earlier used by the first author for optimizing the uniform norm of polynomials [15]; its idea is as follows. The solutions of the optimization problem are very specific, since they have a large number of equioscillation points and therefore lie on a certain manifold of small dimension in the space of all rational functions. When solving problems of least deviation, it is reasonable to move from searching the whole space of rational functions of a given degree to the search over this small-dimensional manifold.

Indeed, the solution of the least deviation problem of degree n has $2n+2$ equioscillation points on E [11], and every such point lying in the interior of E will necessarily be a critical point of a solution function which takes there a value in the set Q containing four elements: $\pm 1 \pm \mu$ for the problem (1) and $\pm\theta$ and $\pm\theta^{-1}$ for problem (2). In total, a rational function of degree n has $2n-2$ critical points counted with multiplicities; therefore, a solution of the considered least deviation problem satisfies the following definition with a small parameter g.

Definition. A rational function $R(x)$ is called g-extremal with respect to the 4-element set of values Q if all its critical points except for $g-1$ ones are simple with their values in Q. The number of exceptional critical points is given by

$$g-1=\sum_{x\colon R(x)\notin \mathrm{Q}} \operatorname{ord} dR(x) + \sum_{x\colon R(x)\in \mathrm{Q}} \left[\frac{1}{2}\operatorname{ord} dR(x)\right], \tag{3}$$

where the sum is over all points in the Riemann sphere; $\operatorname{ord} dR(x)$ is the zero order at a point x of the differential of the holomorphic map $R\colon \mathbb{CP}^1 \to \mathbb{CP}^1$ (for instance, at simple poles of $R(x)$ this quantity is equal to zero), and $[\,\cdot\,]$ is the integer part of a number.

Rational functions with extremality number g possess the following effective, low-parametric for small g, and numerically stable representation [14], which generalizes the construction of Zolotarev's fractions:

$$R(x)=\operatorname{sn}\left(\int_e^x d\zeta + A(e)\,\Big|\,\tau\right), \qquad \mathrm{Q}=\{\pm 1, \pm 1/k(\tau)\}, \tag{4}$$

where $d\zeta$ is a holomorphic differential on a Riemann surface M of genus g whose periods lie in the lattice of periods of elliptic sine, while the phase shift $A(e)$ is also commensurable to this lattice. The surface M is determined by the rational function $R(x)$ as the double covering of the Riemann sphere branched over the points e, where $R(x)$ takes the values from the set Q with odd multiplicity. The emerging Riemann surface is not arbitrary; it is the so-called Calogero–Moser curve: it covers with due branching the torus defined by the set of distinguished values Q. By itself, the algebraic-geometric representation (4) arises in the analysis of a functional equation of the Pell–Abel type which is satisfied by the rational function $R(x)$ (for more details, see [14]).

The use of ansatz (4) for solving the optimization problem 1 or 2 assumes finding all its parameters. First of all, it is necessary to determine the topological type of the real curve M, i.e., its genus and the number of real ovals, and also the location of the latter with respect to components of the set E of filter bands. Assuming the alternation of the pass- and stop-bands, the genus g of the curve M is related to the number m of filter bands by the inequalities $m-1 \le g \le 2m-3$. After the discrete ansatz parameters are determined (currently, it requires a finite search or a qualified work to reduce it), it is possible to write a system of transcendental equations for the moduli of the pair (curve M, differential $d\zeta$), whose solution is the required parameters of ansatz (4), which allow the solution of the optimization problem to be reconstructed.

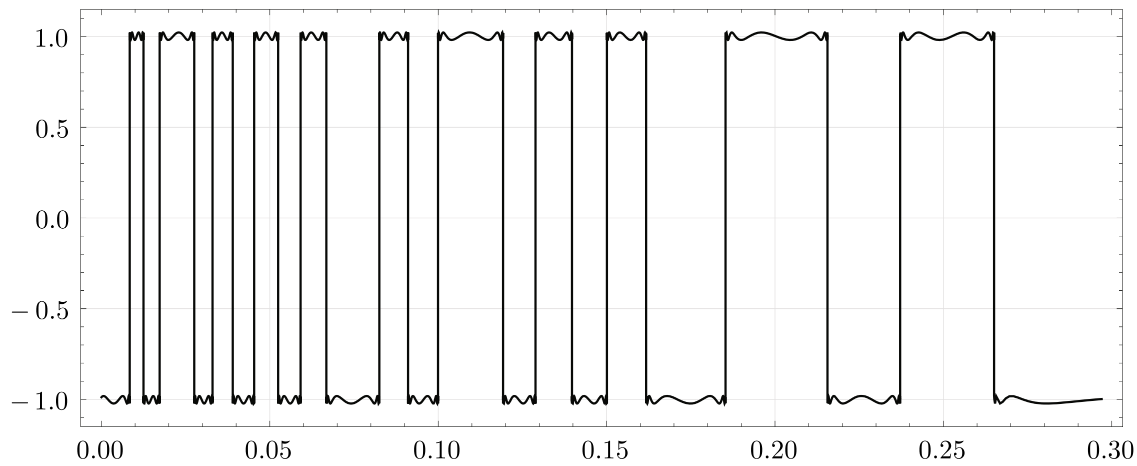

Fig. 1. Graph of the solution of the optimization problem (for a specification with $m = 23$).

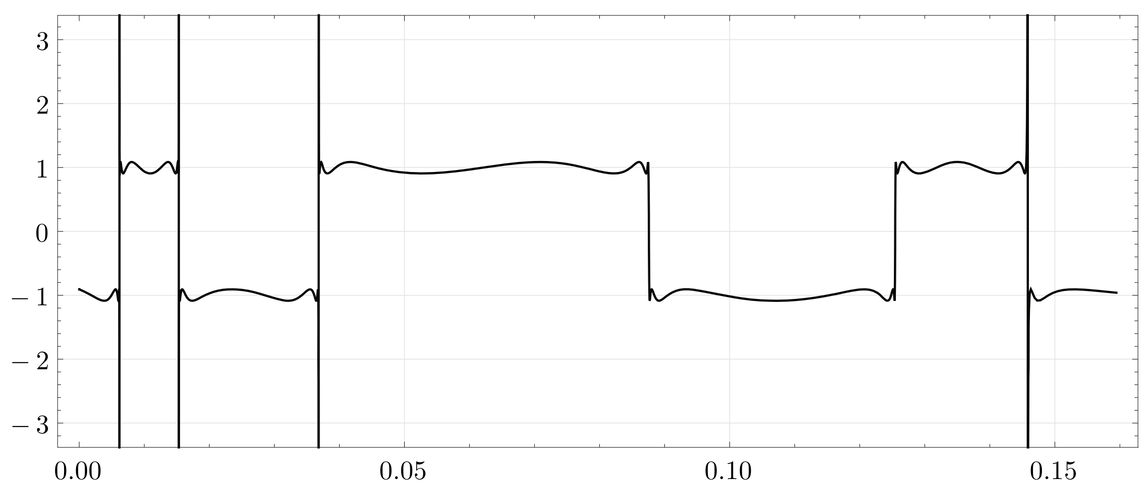

Fig. 2. Graph of the solution of the optimization problem (for a specification with $m = 7$).

A representation of solutions in terms of conformal mappings of rectangular polygons (for instance, computable using theta functions [16]) is given in [14]. A more detailed exposition of the method based on the use of the explicit analytic formula (4) will be given in a separate article.

The employed parametrization of the extremal functions allows us to control the behavior of the solution in transition bands of the filter, which corresponds to selection of the class, and solve the problem of the filter of the least degree for a given specification directly, without considering a chain of least deviation problems. Computational tools used to find extremal rational functions by the explicit analytic formula involve the previously developed apparatus [14–18] for effective computations on Riemann surfaces and allow one to compute in a stable way solutions of degrees n up to a thousand and more.

Using the new analytical approach, examples of solutions of optimization problem 1 were computed. A graph of a degree-654 solution function for a specification with the total number of bands $m = 23$ is presented in Fig. 1. Figure 2 shows the graph of a solution (for a specification with $m = 7$) from a class that admits poles in the first, second, third, and sixth transition bands.

4. DIRECT OPTIMIZATION

The solutions of the optimization problem can be computed approximately, e.g., using Remez-type algorithms [9, 19, 20]. A starting point will again be the Akhiezer theorem [11]: an optimal rational function $R(x) = \varphi(x)/\psi(x)$ of degree n must have at least $2n+2$ equioscillation points on E. Assume that we have an approximate fraction given, for instance, by location of zeros and poles or by coefficients of the numerator and denominator in some polynomial basis. Assume also that we have a set of approximate equioscillation points, which we denote by A. The algorithms in question consist in successive refinement of each of these two objects:

1. Having a set of points $A \subset E$, one can modify the fraction so that it will obey the equioscillation condition on the discrete set A;
2. Having the fraction, one can modify the positions of points in order to increase the uniform error norm over A, or make sure that this is impossible, i.e., the uniform error norm does not decrease when being restricted from E to A.

Besides, an initial approximation is required, choosing which is very important due to the local convergence property of Remez-type algorithms. We briefly describe recipes used for all the above-mentioned steps.

Choice of initial approximation. We begin with choosing the set A, namely, we use the $1/(2n+2)$-quantiles of the equilibrium measure for the set E which is computed using a numerical solution of the integral equation with logarithmic kernel [21].

Step 2 (*refinement of* A *using a given fraction* φ/ψ). First, we consider a wider set containing the endpoints of our m intervals and critical points of the error function $\varphi(w)/\psi(w) - F(w)$ lying on E. The points corresponding to deviation less than that obtained previously at Step 1, are discarded. The new set A must obey the sign alternation rule and contain $2n+2$ points. If the number of points is smaller or greater, this is corrected with the help of the previously introduced equilibrium measure; for example, we discard the points corresponding to smaller values of the density of this measure. The search for critical points is carried out by the Brent method or, if the approximation is good enough, in a simpler way, using the Newton method.

Step 1 (*refinement of* φ/ψ *using a given set* A). Unlike polynomial approximations, the equioscillation condition for the set A cannot be represented, to our knowledge, in the form of a linear system; the best formulation that one can obtain is a generalized eigenvalue problem of the form

$$\begin{pmatrix} V_+ & -V_+ \\ V_- & V_- \end{pmatrix} \begin{pmatrix} \varphi \\ \psi \end{pmatrix} = \mu \begin{pmatrix} 0 & \Sigma_+ V_+ \\ 0 & \Sigma_- V_- \end{pmatrix} \begin{pmatrix} \varphi \\ \psi \end{pmatrix} \tag{5}$$

for the first formulation of the extremal problem, and of the form

$$\begin{pmatrix} V_- & 0 \\ 0 & \Sigma_+ V_+ \end{pmatrix} \begin{pmatrix} \varphi \\ \psi \end{pmatrix} = \theta \begin{pmatrix} 0 & \Sigma_- V_- \\ V_+ & 0 \end{pmatrix} \begin{pmatrix} \varphi \\ \psi \end{pmatrix} \tag{6}$$

for the second (see Section 2). The latter problem can be found in [9]. In equations (5) and (6), the following notation is used: $V_\pm$ is the Vandermonde matrix corresponding to the nodes of A which belong to $E_\pm$, $\Sigma_\pm$ are diagonal matrices with entries ± 1 fixed for every pass- or stop-band corresponding to the nodes of A which belong to $E_\pm$, φ and ψ are vectors of numerator and denominator coefficients corresponding to the rational fraction, μ and θ are extremal alternating values. The presence of matrices $\Sigma_\pm$ is due to the possible change of equioscillation signs in adjacent intervals.

As can be seen, the matrix problems (5) and (6) are nonsymmetric, and the corresponding matrix pencils may turn out to be singular; even in the case of two intervals it may happen that standard mantissa computations are not sufficient for getting at least one decimal digit of coefficients of φ and ψ (not to speak of their roots). Besides, the search over all variants of equioscillation signs in adjacent intervals means complexity exponential in m.

A certain advantage over the eigenvalue problems is due to to the reduction of problem (1) formulated for a given set A to a linear programming problem. Here also emerges an exponential-in-m search related to the choice of the sign of denominator of the fraction for each interval. The problem is as follows: minimize the quantity $t > 0$ under the nonlinear constraints

$$|\varphi(x) - \psi(x)F(x)| \leq t|\psi(x)|, \quad x \in A, \quad j = 1, \ldots, m,$$

which turn to linear ones if we fix the sign of $\psi(x)$ on each interval. Good results for this problem are shown by the primal-dual interior point method [22].

Linear parametrization of numerators and denominators of the fraction substantially limits the stability of Remez-type algorithms [19]. The maximal degree of a solution R obtained by such methods working in double precision depends on a configuration of the set E but does not exceed $n = 20$.

5. COMPOSITE APPROACH

The composite approach is not connected with the formulated optimization problem and assumes obtaining a (nonoptimal) multiband filter as a result of combining several single-band filters. Architecturally, this can be implemented, for instance, by parallel connection of bandpass filters each of which maintains one band of a given multiband specification, or by cascade connection of bandpass and band-rejection filters.

Nonoptimal filters obtained by composite approach are hereinafter referred to as composite filters. We used the following recipe for their construction: for each passband, the transfer function of the corresponding passband elliptic filter was computed, and then the resulting functions were added. Parameters of the elliptic filters were optimized by manual search to achieve the smallest possible order of the resulting multiband filter under the condition that its magnitude response fits into the corridor given by the specification.

6. EXAMPLES OF SYNTHESIS

Here we give examples of results of designing digital multiband filters using three approaches: the new analytical approach, direct optimization method, and composite approach. The first two approaches are based on the optimization problem formulated above and give optimal filters (i.e., filters of the minimum possible order for a given specification). The latter approach consists in combining single-band elliptic filters and gives nonoptimal multiband filters.

Optimal digital filters were obtained from optimal analog ones by the standard (so-called bilinear) frequency transformation.

The direct optimization did not yield satisfactory results for any of the considered examples due to the complexity of the specifications. Its results are given for the first and second examples. The composite approach, as can be seen from the examples, yields filters with significantly higher orders in comparison with the optimal filters.

6.1. Single-Band Filter

With the help of the new analytical approach, we constructed a single-band optimal filter of order 18 with strongly asymmetric widths of transition bands equal to 0.016 and $2 \cdot 10^{-5}$. The graph of its magnitude response is shown in Fig. 3.

The standard frequency transformation used in the synthesis of the passband elliptic filter from the low pass prototype cannot provide such a difference in the widths of the transition bands; the elliptic filter computed according to the same specification has a larger order, namely 28. The graph of its magnitude response is presented in Fig. 4.

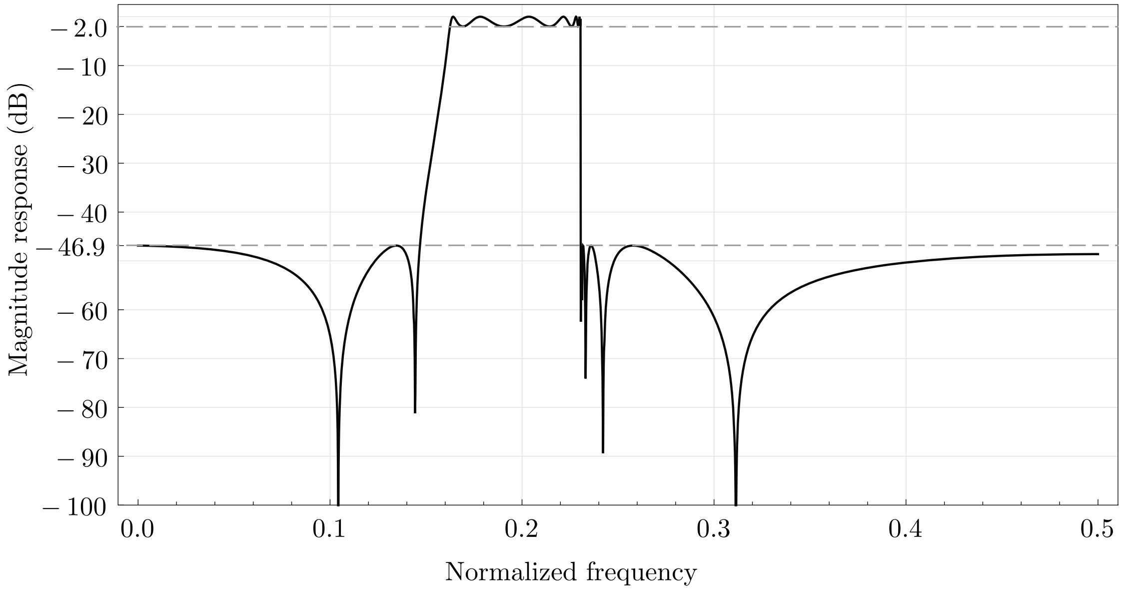

Fig. 3. Magnitude response of a single-band optimal filter.

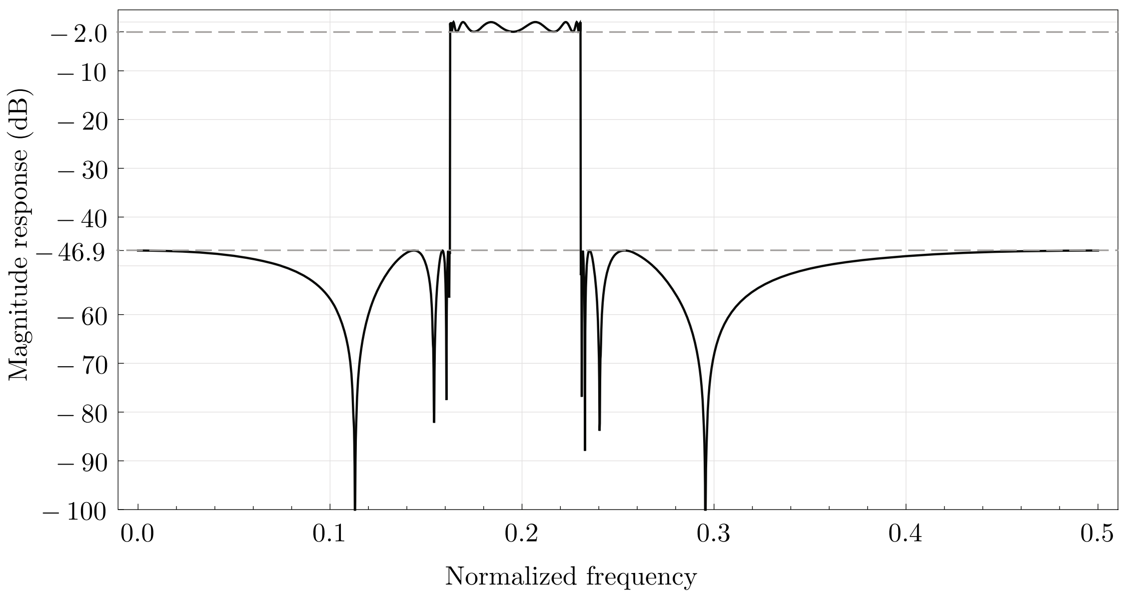

Fig. 4. Magnitude response of a single-band elliptic filter.

Results of the direct numerical optimization for that specification are given in Table 1. In computations using the Remez algorithm, boundaries of the filter bands were given by specification, the passband ripple was fixed equal to $-2\,$dB, and the attenuation in the stopbands (the second column in the table) was determined from the order of the optimal filter obtained. The $-46.9\,$dB attenuation in the stopbands turned out to be unattainable for the direct optimization method due to the fact that even the computation of the filter of order 10 required quite a lot of time.

6.2. Dual-Band Filter

As an example of an optimal dual-band filter, a filter of order 16 was constructed using the analytical approach with a minimum attenuation at the stopbands equal to $-40\,$dB and a passband

Table 1. Results of the direct numerical optimization for a single-band filter.

3	−3.381	102
4	−4.227	154
5	−7.589	211
6	−10.612	308
8	−19.355	457
9	−22.378	652
10	–	> 1000
18	−46.9	?

Table 2. Results of the direct optimization for a dual-band filter.

3	−5.342	151
4	−9.178	273
5	−12.649	351
6	−16.013	408
7	−19.215	594
8	–	> 1000
16	−40	?

ripple equal to −2.6 dB. Widths of the transition bands were 0.012, 0.012, 0.003, and 0.008. The magnitude response graph is shown in Fig. 5.

The composite filter constructed according to the same specification is of order 23. The graph of its magnitude response is presented in Fig. 6.

Results of the direct optimization for that specification are given in Table 2. In computations, boundaries of the filter bands were given by specification, the passband ripple was fixed equal to −2 dB, and the attenuation in the stopbands (the second column in the table) was determined from the order of the optimal filter obtained. In this case the Remez-type algorithm turned out to be inapplicable for the search of optimal filters starting already from order eight, which means that the optimal filter of order 16 obtained by the analytical approach cannot be found by the direct optimization method.

6.3. Four-Band Filter

With the help of the analytical approach, we synthesized a four-band optimal filter of order 36 with a minimum attenuation at the stopbands equal to −42.8 dB, ripple at the passbands equal to −2.0 dB, and widths of the transition bands from 0.002 to 0.005. The magnitude response graph of the obtained filter is shown in Fig. 7.

The composite filter constructed according to the same specification is of order 55. The graph of its magnitude response is presented in Fig. 8.

6.4. Five-Band Filter

For the five-band specification, an optimal filter of order 76 was computed with a minimum magnitude response attenuation at the stopbands equal to −50 dB and passband ripple equal to −2.0 dB. The widths of the transition bands lie in the interval from 0.002 to 0.005. The magnitude response graph is shown in Fig. 9.

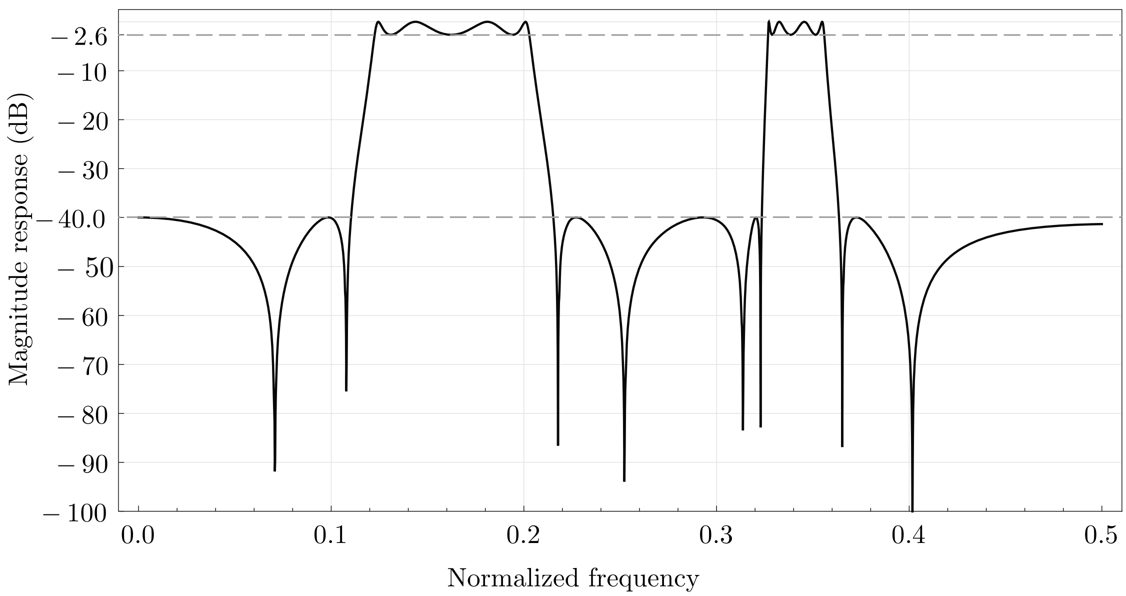

Fig. 5. Magnitude response of a dual-band optimal filter.

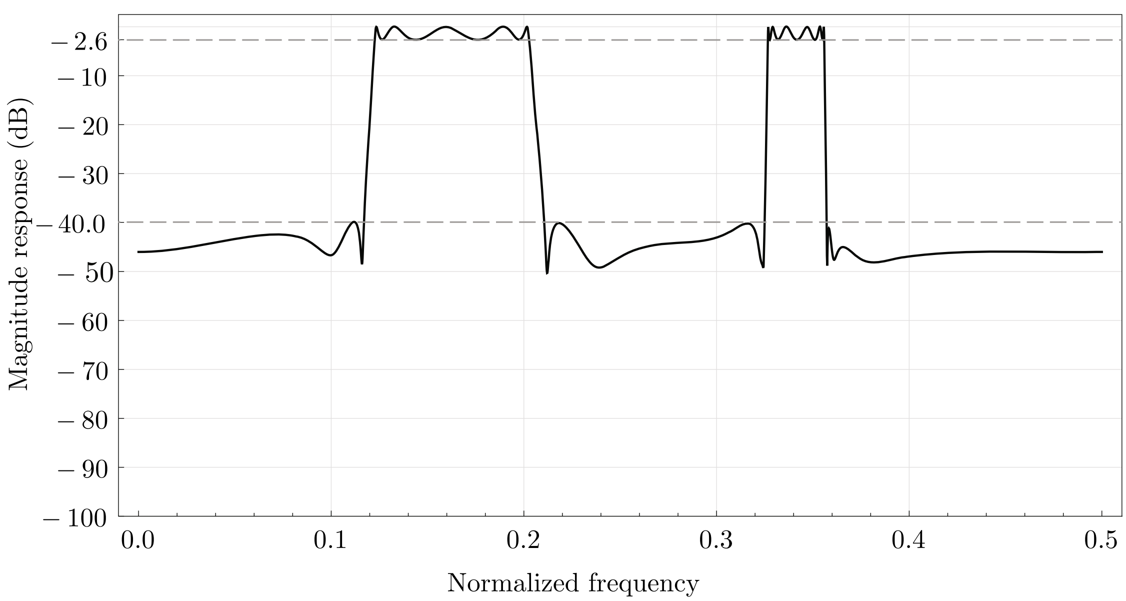

Fig. 6. Magnitude response of a dual-band composite filter.

The composite filter constructed according to the same specification is of order 121. The graph of its magnitude response is presented in Fig. 10.

6.5. Notch Filter with Two Rejection Bands

The magnitude response graph of an optimal notch filter of order 16 performing accurate cutting of two given frequencies is presented in Fig. 11. A fragment of the magnitude response containing the rejected frequencies is presented on a larger scale in Fig. 12.

The composite filter maintaining the same magnitude response approximation quality is of order 62.

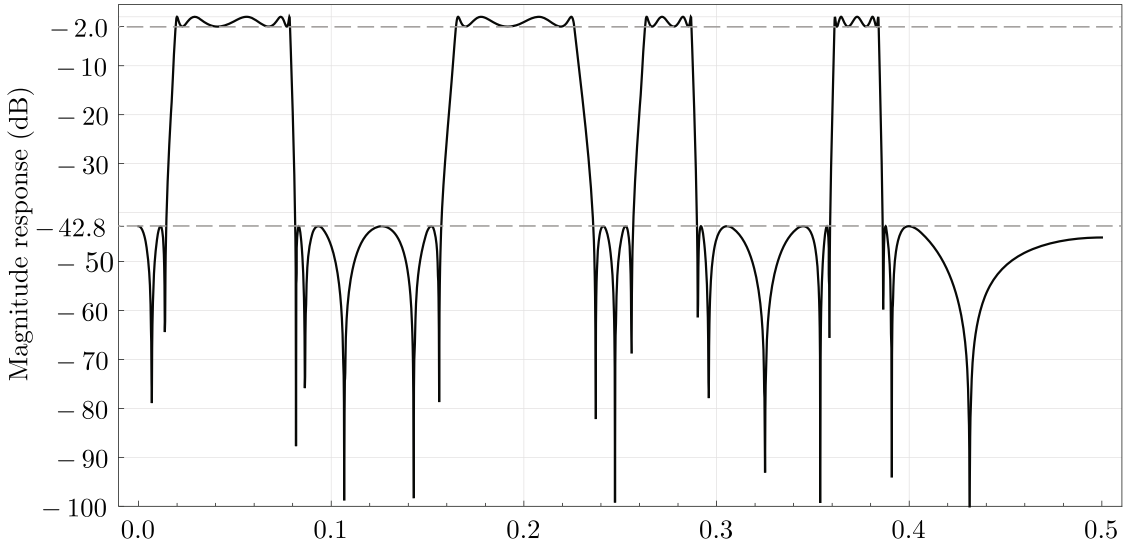

Fig. 7. Magnitude response of a four-band optimal filter.

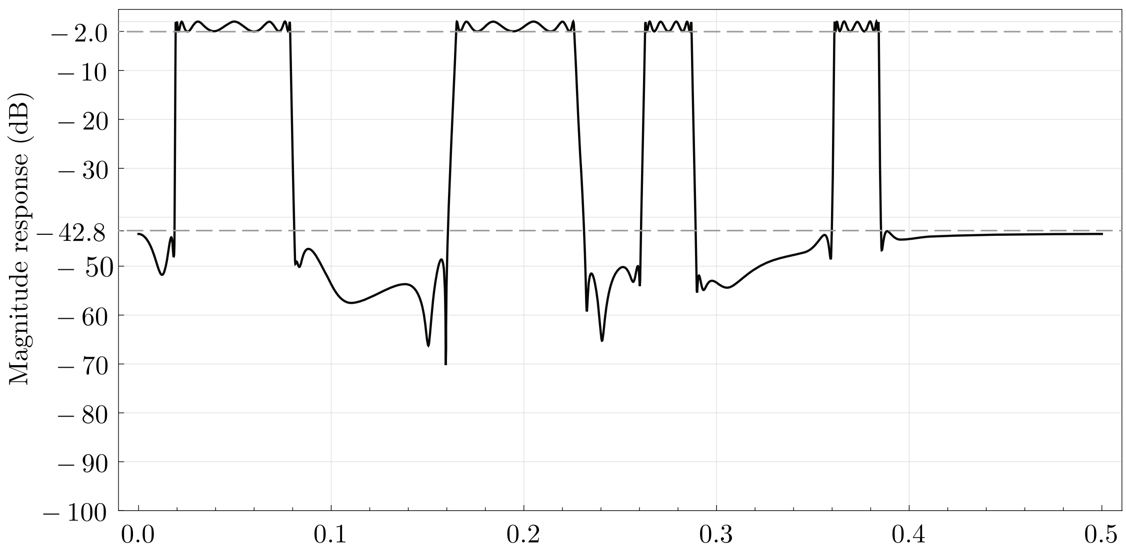

Fig. 8. Magnitude response of a four-band composite filter.

6.6. Filter with Two Passbands Critically Close to Each Other

With the help of the analytical approach, we constructed an optimal filter of order 24 with two passbands critically close to each other. The corresponding magnitude response graph is given in Fig. 13. Such filters can be used if it is required to extract a certain frequency range from the spectrum while rejecting one or several frequencies contained within that range.

The composite filter maintaining the same magnitude response approximation quality is of order 59.

7. CONCLUSION

The article gives a comparison of three approaches to the synthesis of multiband filters: the new analytical approach, direct numerical optimization based on Remez method, and semi-analytical

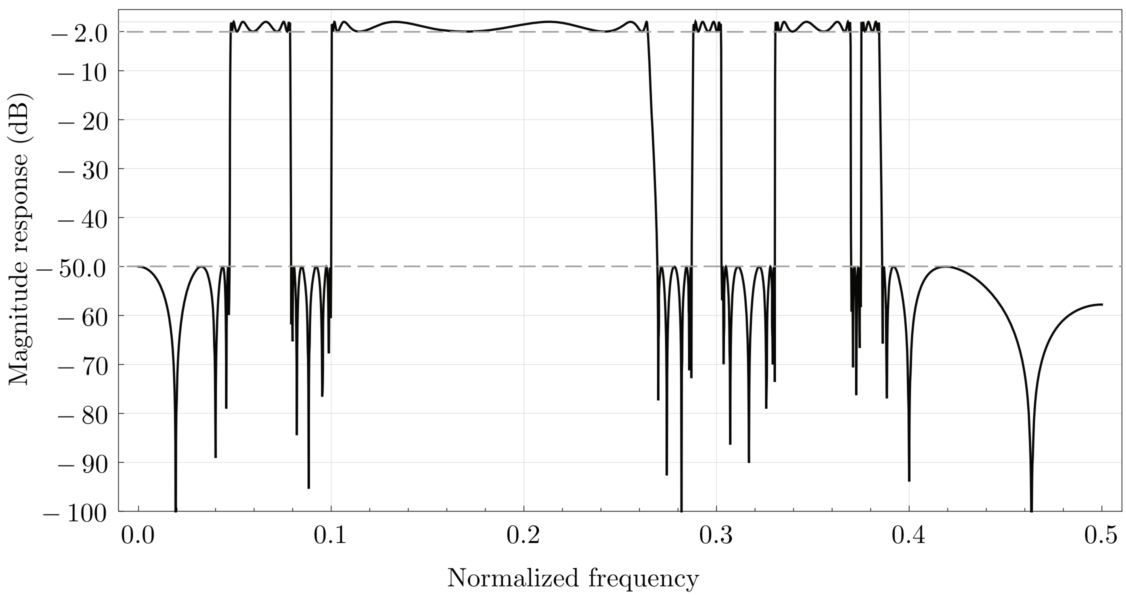

Fig. 9. Magnitude response of a four-band optimal filter.

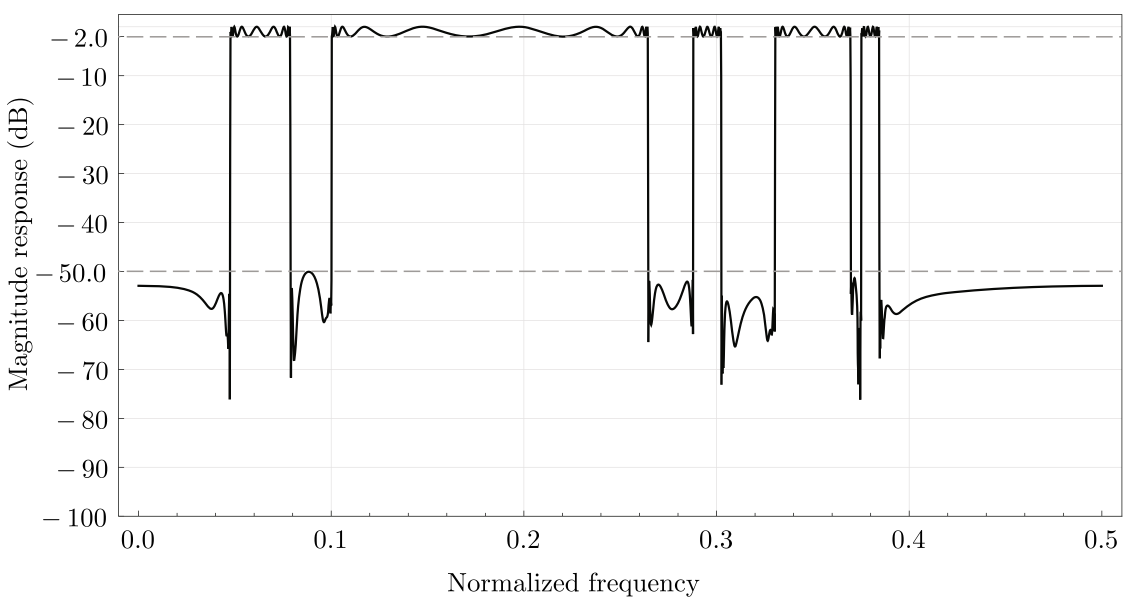

Fig. 10. Magnitude response of a four-band composite filter.

composite approach. Today, the direct numerical optimization has probably the widest algorithmic support: there exist (and are being improved) mature packages for engineering computations. Unfortunately, the unavoidable instability of Remez-type algorithms does not allow to solve too complicated problems: in double precision arithmetic (15 decimal digits), the filter order does not attain 20, and good approximation properties are unreachable in the case of complicated specifications, for instance, with the large number of pass- and stop-bands, narrow transition bands, or critically close passbands. The composite approach consists in breaking up a complex problem into a series of simple ones and successively solving them by using Zolotarev's fraction for constructing the passband filter magnitude response. Its advantage is that in this way it is always possible to obtain an (ersatz) solution for a given specification. As a rule, it is far from being optimal: the

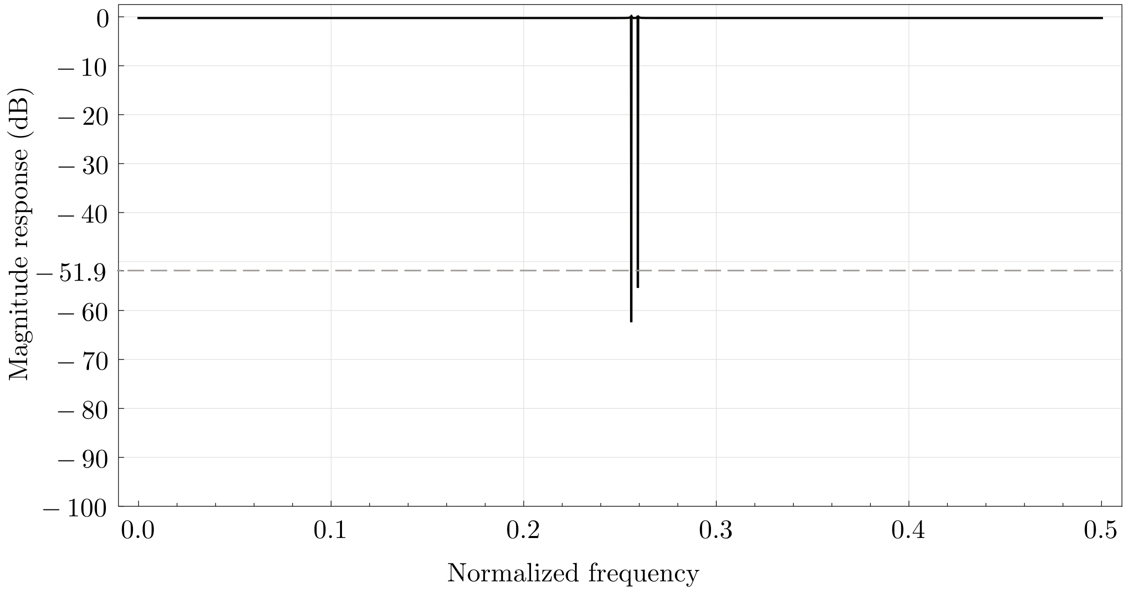

Fig. 11. Magnitude response of the optimal notch filter with two rejection bands.

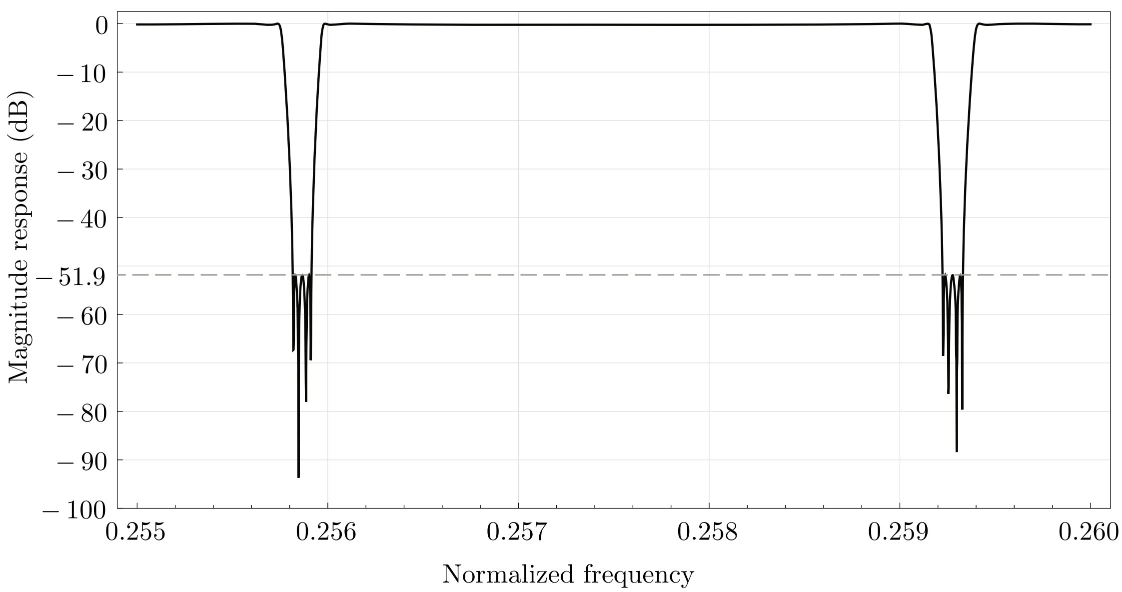

Fig. 12. Fragment of the magnitude response containing the rejected frequencies.

order of composite filter can be several times as large as the order of an optimal filter with the same specification. The situation gets worse with increasing complexity of the filter specification. In our opinion, the most promising—as well as the least studied from the algorithmic aspect—is the analytical approach based on a complex mathematical apparatus. The authors intend to continue their research in this direction.

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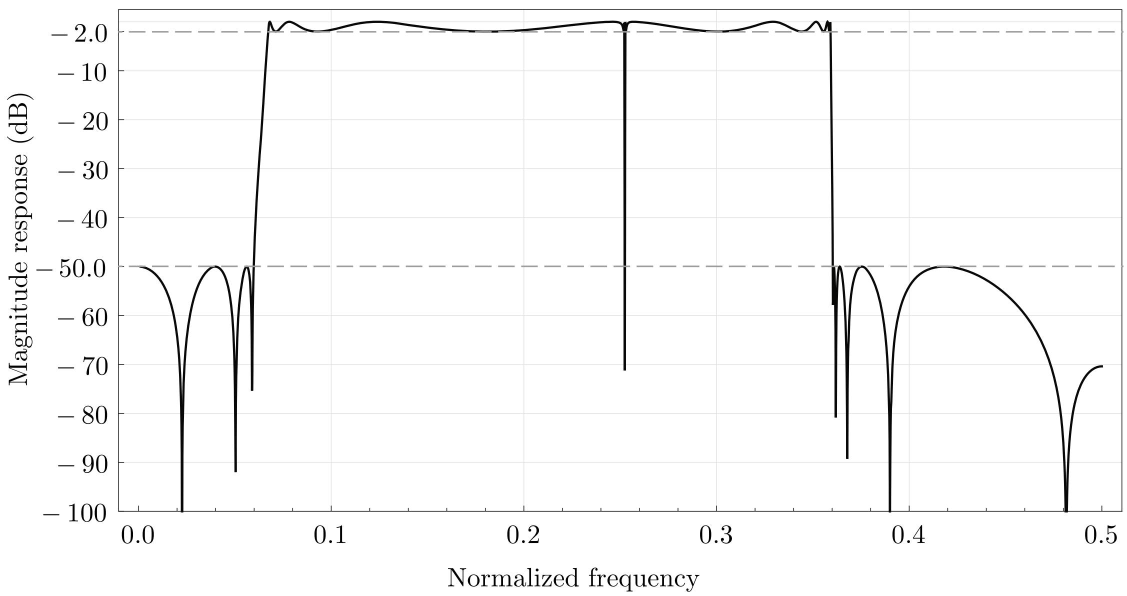

Fig. 13. Magnitude response of the optimal filter with two passbands critically close to each other.

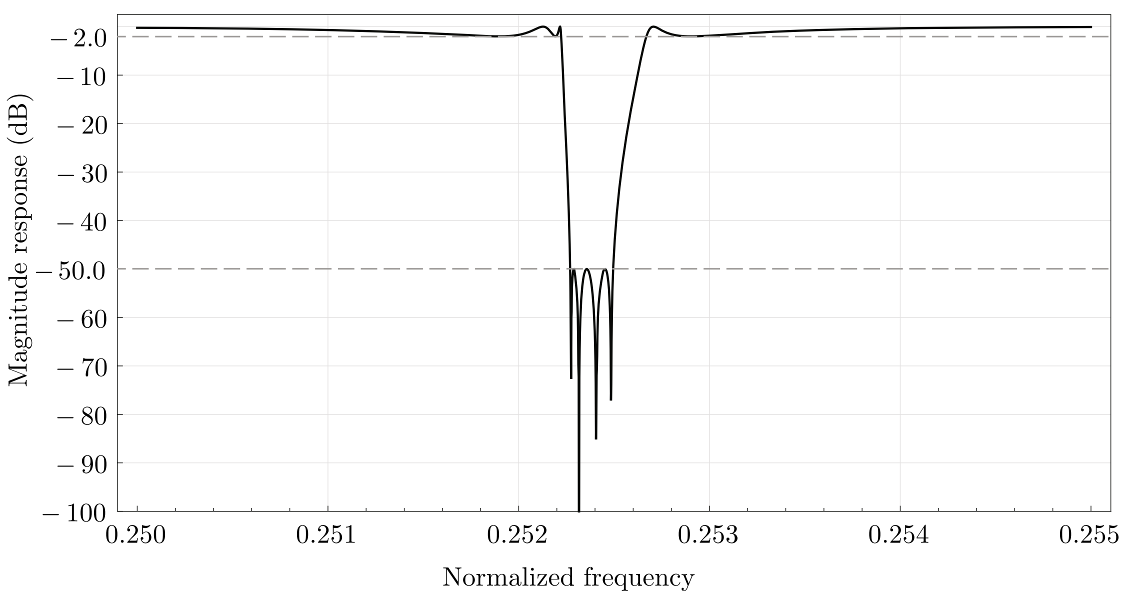

Fig. 14. Fragment of the magnitude response containing the rejected frequency.

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