ISSN 0030-400X, Optics and Spectroscopy, 2017, Vol. 122, No. 5, pp. 826–834. © Pleiades Publishing, Ltd., 2017.
Original Russian Text © V.D. Vinokurova, N.N. Rosanov, 2017, published in Optika i Spektroskopiya, 2017, Vol. 122, No. 5, pp. 855–863.

PHYSICAL OPTICS

Dynamic Billiards for Particles with Inelastic Reflections

V. D. Vinokurova[a] and N. N. Rosanov[a, b, c, *]

[a] *Vavilov State Optical Institute, St. Petersburg, 199053 Russia*
[b] *ITMO University, St. Petersburg, 197101 Russia*
[c] *Ioffe Physical Technical Institute, Russian Academy of Sciences, St. Petersburg, 194021 Russia*
**e-mail: nnrosanov@mail.ru*

Received November 17, 2016

Abstract—The one-dimensional dynamics of particles that move between a stationary and a harmonically oscillating mirror have been analyzed analytically and numerically taking into account inelastic collisions of particles with mirrors. It has been shown that, in such "billiards," in contrast to the case of elastic collisions, asymptotically stable periodic regimes are established, including the regime of periodic sticking of a particle to the oscillating mirror, as well as regimes of dynamic chaos.

DOI: 10.1134/S0030400X17050216

INTRODUCTION

Control of motion [1, 2] and localization of neutral atoms in magnetic, electric, laser, and laser–magnetic fields [3] have both important basic and applied significance. In particular, atomic mirrors can be created due to the reflection of atoms from an inhomogeneity of the intensity of laser radiation upon its total internal reflection at the dielectric boundary [4–8]. In this case, schemes with two mirrors, between which atoms balance, and with one mirror, in which atoms periodically return to the mirror due to the gravitational field, are both realistic [9].

Because the mass of atoms is considerable and the time of their collision with the mirror is short if the velocity of their incidence on the mirror is not too slow, the motion of atoms can be described classically and a collision can be considered as a short-term impact. It is then allowable to speak of the trajectory of an atom (within the limits imposed by the quantum-mechanical principle of uncertainty). For elastic collisions, these trajectories are described in terms of the well-developed theory of billiards [10–13], which can be either static (with a stationary configuration of mirrors) or dynamic (with moving mirrors). We note that the reflection of classical particles from moving mirrors forms the basis of the Fermi acceleration effect [14], which was proposed to explain significant energies of cosmic particles and which, along with Ulam's investigation [15] of elastic reflections of particles from oscillating mirrors, significantly influenced modern notions of dynamic chaos and the foundations of statistical physics [11, 13].

Further, we restrict ourselves to the model of one-dimensional motion of a particle between a stationary and a harmonically oscillating mirror. It was shown as early as in ground-breaking work [15] that, depending on the parameters, this motion can be either periodic or chaotic. Periodic regimes have a neutral stability; i.e., small deviations of the initial conditions lead to small perturbations, which do not decay with increasing time. Interest is also presented by the regime of consecutive multiple reflections of a particle from one and the same oscillating mirror, which has been found and studied in [16, 17]. Under "ideal" conditions, this regime turns into a sticking regime, in which the particle is localized on the mirror during the half-period of its oscillations.

The dynamic regimes of billiards have been much less studied in the case of inelastic reflections of particles from a mirror, although inelastic reflection occurs for atoms as well, and it can be used for their cooling [9, 18]. At the same time, energy losses upon reflection change qualitatively the dynamics of the particle motion, leading to the possibility of absolutely stable regimes with attractors, for which small perturbations decay with time [19]. Although some portion of the kinetic energy of a particle is lost as a result of its inelastic reflection, this loss can be compensated by the energy inflow from the oscillating mirror. For these reasons, it is of interest to analyze this dynamics in more detail, including the sticking regime, which is the subject of this paper.

Below, we will initially consider the reflection of a particle from a single oscillating mirror, paying the main attention to the regime of (single) sticking. Then, different scenarios of the particle motion between a stationary and a harmonically oscillating mirror will be examined, taking into account inelastic collisions of a particle with the stationary mirror. The last section is devoted to the analysis of periodic sticking. General inferences are presented in Conclusions.

REFLECTION OF A PARTICLE FROM AN OSCILLATING MIRROR

Let us initially consider the reflection of a particle from a single mirror that oscillates with frequency Ω and amplitude μL_0, where L_0 is a quantity with the dimension of length and μ is a (small) dimensionless parameter. Then, the coordinate of the mirror is given by $Z_w(t) = \mu L_0 \cos(\Omega t)$. We introduce dimensionless time $\Omega t \to t$ and dimensionless coordinate of the mirror $z_w(t) = Z_w(t)/(\mu L_0) = \cos t$. For particle velocity V, we also use the dimensionless form $v = V/(\mu L_0)$. Naturally, for the particle to collide with the mirror, the particle velocity should be directed toward the mirror.

Velocity of incidence of a particle v is related to velocity of its reflection v' as

$$v' - \dot{z}_w = -q(v - \dot{z}_w) \qquad (1)$$

(the dot denotes the time derivative). The restoration coefficient is $q = 1$ if the collision is absolutely elastic and is $q = 0$ if the collision is absolutely inelastic (sticking) [20]. We assume that $0 < q < 1$.

As was shown in [17] for the case of elastic reflections, if the velocity of a particle incident on an oscillating mirror is not too high (compared to the maximum velocity of the mirror motion), the particle can collide with the mirror not one but several times before finally moving away from it (we assume that processes of adsorption of particles by the surface do not occur). At a certain selection of conditions, the number of consecutive rereflections can be arbitrarily large (the "sticking regime"). This can be seen from the expression for the time interval between consecutive collisions: $\tau_m = t_{m+1} - t_m$. Indeed, after the mth reflection of the particle at moment of time t_m, the trajectory of the particle is $z_p(t) = \cos t_m + v_{m+1} \times (t - t_m)$. Correspondingly, the equation for determining the time of the next collision has the form

$$\cos t_{m+1} = \cos t_m + v_{m+1} \times (t_{m+1} - t_m). \qquad (2)$$

This equation is universal in the sense that it does not contain any parameters of the scheme. Applying the Taylor expansion in (2) and retaining only the term cubic in τ_m in it, we find

$$\tau_m = \frac{3\cos t_m - \sqrt{9\cos^2 t_m + 24\sin t_m (v_{m+1} + \sin t_m)}}{2\sin t_m}. \qquad (3)$$

As in [17], interval τ_m is small if quantity $v_{m+1} + \sin t_m$ is small. Therefore, an "ideal" sticking regime is achieved when the particle "collides" with the mirror "receding" with unit velocity $v_a = 1$ at time moment $t_a = -\pi/2$, i.e., when the particle velocity is equal to the instantaneous mirror velocity. It is this equality of velocities, as well as the subsequent slowing down the motion of the mirror, that determines the occurrence of the sticking regime. Under ideal conditions, the number of collisions of the particle with the mirror is infinitely large and the particle flies away from the mirror in half the oscillation period (at moment $t = \pi/2$) with the same kinetic energy (the rebound velocity is $v = -v_a = -1$). At small deviations from the ideal conditions at larger values of the velocity and moments of time at $q = 1$, the number of collisions becomes finite. At $q < 1$, i.e., upon damping, time intervals between collisions shorten compared to time intervals in the case of elastic rereflections, because velocity of an inelastically reflected particle v' is larger than velocity $\tilde{v}'$ of an elastically reflected particle:

$$v' - \tilde{v}' = (1 - q)[v - \dot{z}(t)] > 0 \ \text{ at } \ [v - \dot{z}(t)] > 0, \qquad (4)$$

where t is the moment of collision of the particle with the oscillating mirror. As a result, under these conditions, the particle trajectory in the case of inelastic reflections is closer to the trajectory of the mirror and the conditions of realization of the sticking regime are mirror-favorable.

In our numerical calculations, moment of sticking of the particle t_{abs} (with respect to harmonic oscillations of the mirror, $-\pi/2 \le t_{\text{abs}} \le \pi/2$) and the corresponding number of its collision m_{abs} ($t_{\text{abs}} = t_{m_{\text{abs}}}$) were determined with a specified accuracy such that the values of dimensionless velocities of the particle and the mirror normalized to μ would differ from each other by no more than 10^{-13}:

$$\left| v_{m_{\text{abs}}} - \dot{z}(t_{m_{\text{abs}}}) \right| \le 10^{-13}\mu. \qquad (5)$$

Upon implementation of this condition, we assume that the particle moves along with the mirror up to moment of slowing down of the latter at $t = \pi/2$, after which the particle moves in the reverse direction with unit velocity $v = -v_a = -1$ (irreversible adsorption does not occur).

The described situation is presented in Fig. 1 for the initial value of the particle velocity $v_0 = 1.01$ and $q = 0.95$. In this case, for the moment of conventionally "absolute" sticking, the time values are $t_{\text{abs}} =$ 0.09515 and, for quantity K, which denotes the number of rereflections of the particle in the near-mirror region, including the moment of the absolute sticking, we have $K = 285$. For $q = 0.85$, we have a similar dynamics with $t_{\text{abs}} = -0.8148$ and $K = 86$. Corre-

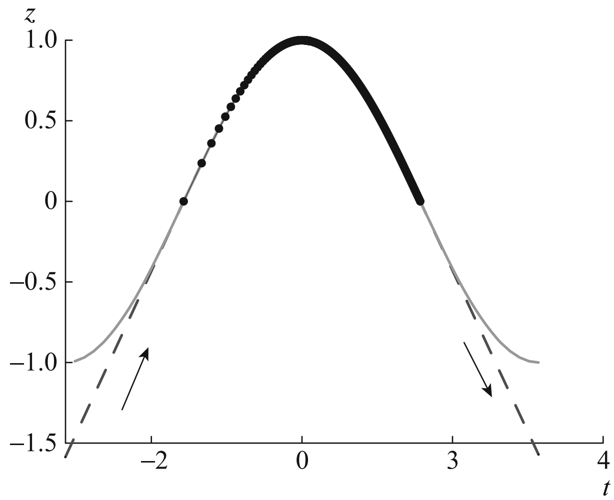

Fig. 1. The solid curve shows the trajectory of a mirror, $z_w(t) = \cos t$, while the dashed curve shows the trajectory of a particle; dots indicate moments of collision of the particle with the mirror. The parameters are as follows: $v_a = 1.01$, $q = 0.95$, $t_{abs} = 0.09515$, and $K = 285$.

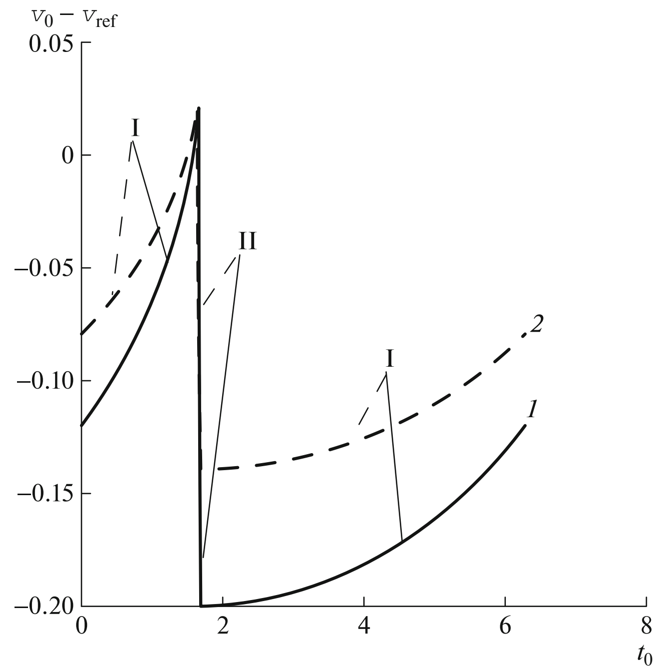

Fig. 2. Dependences of the difference between the moduli of the initial and final velocities of a particle (v_0 and v_{ref}, respectively) reflected at the exit from the zone of mirror oscillations (after possible multiple reflections within a half-period) on the initial phase of the oscillating mirror, t_0, for $q = 1$ (solid curve *1*) and $q = 0.5$ (dashed curve *2*); $v_0 = 0.212$. Roman numeral I indicates segments with a positive derivative, which correspond to the regimes of single reflection from the mirror, while Roman numeral II indicates segments with a negative derivative, which correspond to the regimes of double reflection.

spondingly, an increase in damping upon reflection leads to a decrease in the number of particle rereflections in the near-mirror region of the region up to the moment of absolute sticking, while the absolute sticking itself occurs at a time moment earlier than at weaker damping.

Depending on time of the first collision or the initial phase of the mirror oscillations t_0 the particle can either decrease or increase its kinetic energy as a result of collisions. As was shown in [17], the dependence of the difference between the moduli of the final velocity of the particle after its reflection from the oscillating mirror and the initial particle velocity on the phase of mirror oscillations, t_0, at the moment of time at which the (dimensionless) coordinate of the particle is $z_{p0} = -1$ may have jumps, and, while the function itself is continuous, its derivative has discontinuities. They are caused by a change in the number of successive reflections of the particle from the mirror upon a change in the phase t_0. Figure 2 shows the results of calculations for different values of damping. At $q = 1$ (curve *1*), segments with a positive derivative (indicated by the Roman numeral I) correspond to regimes of a single reflection. Segments with a negative derivative (indicated by the Roman numeral II) correspond to regimes of a double reflection. At $q < 1$ (curve *2*), the corresponding dependence has a jump and a discontinuity in the derivative, but the absolute value of the jump decreases compared to the case of elastic reflections. Similar discontinuities also appear upon variations in the initial velocity of the particle.

MOTION OF A PARTICLE BETWEEN A STATIONARY AND OSCILLATING MIRROR

For this motion, parameter μ cannot be eliminated by renormalization. In this case, apart from the dimensionless time $\Omega t \to t$, we will introduce another dimensionless coordinate of the particle: $z(t) = Z(t)/L_0$, where L_0 is the time-averaged distance between the mirrors. The dimensionless coordinate of the oscillating mirror is then $z_w(t) = Z_w(t)/L_0 = 1 + \mu\cos t$. For particle velocity V, we will also use the dimensionless form: $v = V/L_0$. We assume that the reflection from the stationary mirror ($z = 0$) is elastic, while, from the oscillating mirror, the particle is reflected inelastically.

Regimes of a motionless particle located beyond the zone of mirror oscillations prove to be unstable, since the acquirement of even a small initial velocity by the particle leads to its reflections from the mirrors with an increase in the kinetic energy. For stationary regimes without rereflections (with the unvaried

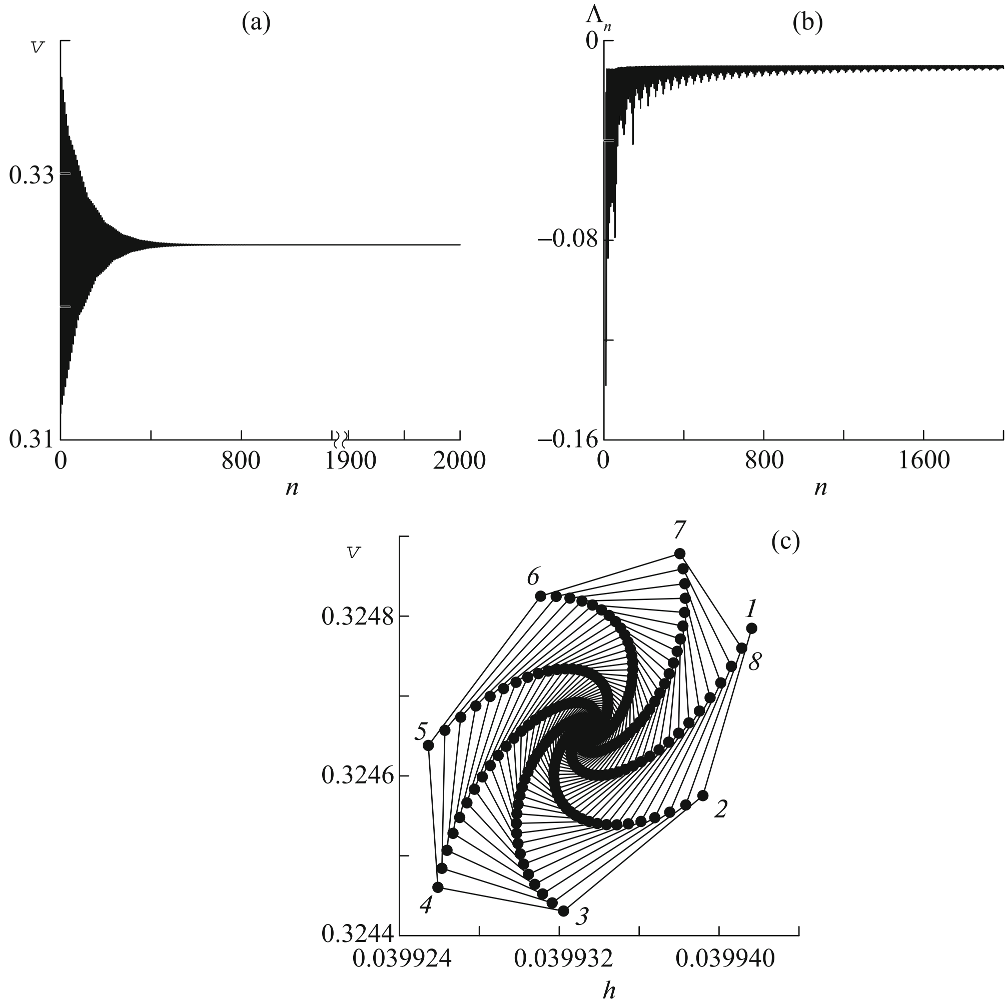

Fig. 3. Dynamics of setting up of a steady-state regime of motion of a particle between oscillating inelastic and stationary elastic mirrors for $\mu = 0.02$, $v_0 = 0.312$, and $q = 0.99$: (a) dependence of the particle velocity on the number of collision with the stationary mirror; (b) setting up of the Lyapunov exponent in relation to the number of reflections, n; and (c) phase portrait.

kinetic energy of the particle), assuming $v' = -v$ in (1), we find

$$v = -\frac{1+q}{1-q}\dot{z}_w. \tag{6}$$

Because $v > 0$, then $\dot{z}_w < 0$. For stationary regimes with collisions of the particle with the mirror in N periods of its oscillations at $t = t_n$, we have

$$\begin{gathered} z_n = (1 + \mu\cos t_n), \quad \dot{z}_n = -\mu\sin t_n, \\ v_n = \frac{1+q}{1-q}\mu\sin t_n, \end{gathered} \tag{7}$$

while, at $t_{n+1} = t_n + 2\pi N$,

$$v_n = \frac{z_n}{\pi N} = \mu\frac{1+q}{1-q}\sin t_n = \frac{1}{\pi N}[1 + \mu\cos t_n]. \tag{8}$$

From this, we obtain the following equation for $\varphi = t_n$:

$$b\sin\varphi = 1 + \mu\cos\varphi, \quad b = \mu\pi N\frac{1+q}{1-q}. \tag{9}$$

The solution of this equation is

$$\begin{gathered} \varphi = \beta + \arcsin\left(\frac{\sin\beta}{\mu}\right), \\ \text{where} \quad \beta = \arcsin\left(\frac{(1-q)}{\sqrt{(\pi N(1+q))^2 + (1-q)^2}}\right), \end{gathered} \tag{10}$$

and the inequality

$$\left|\frac{(1-q)}{\sqrt{(\pi N(1+q))^2 + (1-q)^2}}\right| \le \mu \tag{11}$$

should be implemented.

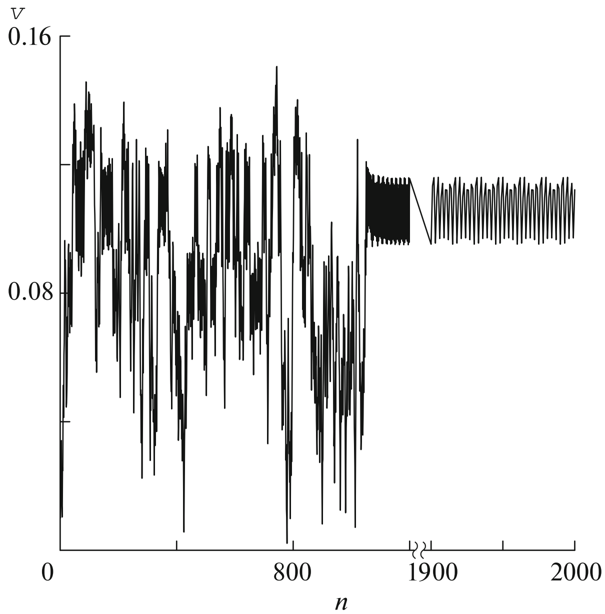

Fig. 4. Dependence of the particle velocity on the number of collision with a stationary mirror at $q = 0.99$, $\mu = 0.01$, and $v_0 = 1.01\mu$.

Numerical calculations confirm the validity of formula (10) and restriction (11).

Figures 3–7 present calculation results for different values of the parameters of the problem. Thus, Figs. 3–5 show typical dependences for the restoration coefficient $q = 0.99$. In this case, the initial phase of the process, i.e., the phase of the mirror oscillations at the moment of the first reflection of the particle from the stationary mirror, is $t_0 = 0$. Figure 3a presents the dependence of the particle velocity on the number of collision at the moment of reflection of the particle from the stationary mirror for $\mu = 0.02$ and $v_0 = 0.312$. In this case, a regime with a single (in modulus) value of the particle velocity is established. Figure 3b shows the results of setting up of the Lyapunov exponent, which characterizes the rate of recession of close trajectories in the phase space [2] and is defined as

$$\Lambda = \lim_{n\to\infty} \Lambda_n,$$
$$\Lambda_n = \lim_{dv\to 0} \sqrt{(v_n - v'_n)^2 + (h_n - h'_n)^2} \Big/ (n dv). \quad (12)$$

Here, n is the number of reflections from the stationary mirror. Primes mark the velocities and ordinates of the particle at the moment of the nth collision with the oscillating mirror after the nth reflection from the stationary mirror for the (displaced) trajectory in the phase space with coordinates (h, v), the initial value of the velocity for which differs by dv from the initial value of the velocity for the main trajectory (under study) for the same initial phase of oscillations of the mirror ($v'_0 = v_0 + dv$). The figure demonstrates the setting up of a negative value of the Lyapunov exponent. Figure 3c presents the phase portrait of the corresponding process of setting up of the stationary regime. The coordinates of points on it correspond to position h_n of the particle in the near-mirror region (region of mirror oscillations, $-\mu \le h \le \mu$) and velocity v_n of the particle at the moments of its collisions with the oscillating mirror. The figures on it enumerate several first collisions. It is significant that this stable regime is an attractor; i.e., small initial deviations from corresponding asymptotic values at long times exponentially decrease.

Under the conditions of Fig. 4 (smaller amplitude of mirror oscillations and initial particle velocity), after an irregular transient process, a periodic regime

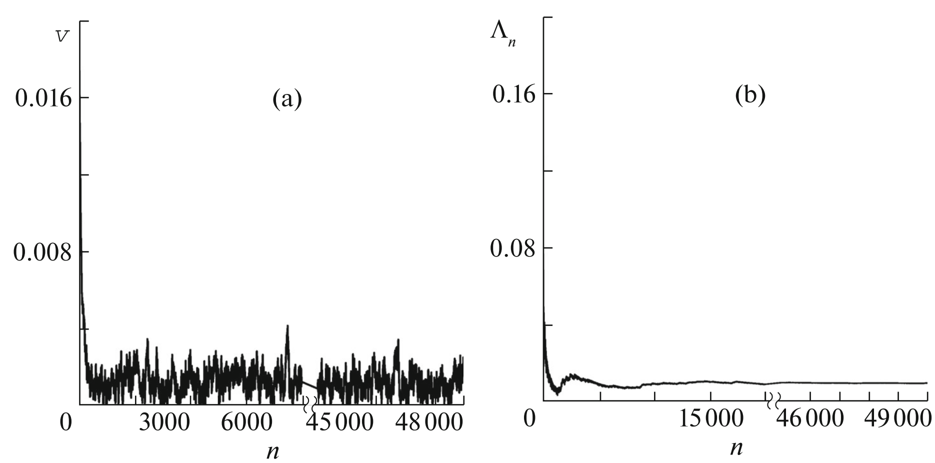

Fig. 5. Dependences of (a) particle velocity $v = v_n$ and (b) the intermediate value of the Lyapunov exponent, Λ_n, on the collision number, n, with the stationary mirror at $\mu = 0.0001$, $v_0 = 0.02$, and $q = 0.99$.

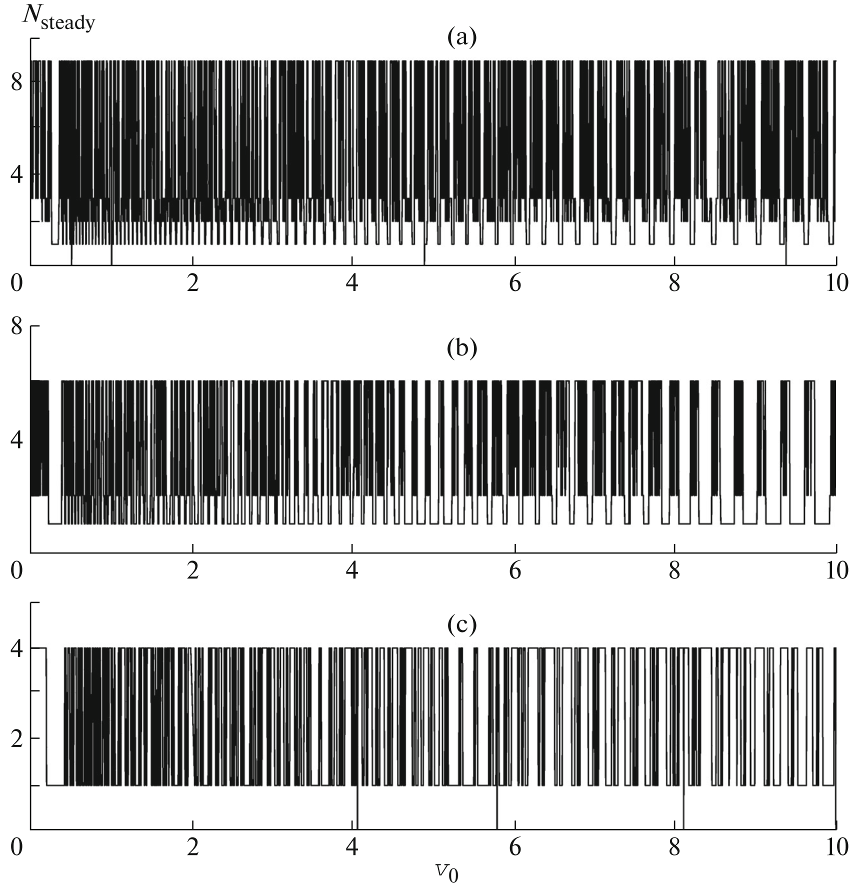

Fig. 6. Dependences of the smallest number of periods, N_{steady}, in steady-state processes over which the dynamic situation repeating takes place in the case of damping with $q = 0.97$ on the initial velocity, v_0, for μ = (a) 0.01, (b) 0.02, and (c) 0.03.

is established in which 15 reflections from the stationary mirror occur during 45 oscillations of the mirror.

Finally, under the conditions of Fig. 5, a regime of dynamic chaos is observed. This is confirmed by a positive value of the Lyapunov exponent (Fig. 5b), which corresponds to an exponential recession of trajectories with close initial conditions.

In Fig. 6, for steady-state processes, the smallest number of periods, N_{steady}, during which the dynamic situation is repeated, is shown as a function of initial velocity v_0. Results are presented for different values of μ with the initial phase of mirror oscillations being $t_0 = 0$. As is seen, if the value of the velocity is small, $v_0 < 0.5$, there is a rather wide "window" of values of v_0 for which the smallest number of periods is unity, $N_{steady} = 1$. As μ increases, this window widens. At larger values of the velocity ($v_0 > 4$), relatively narrow windows with this value $N_{steady} = 1$ appear, which widen with increasing v_0. For different μ, the stepwise dependence $N_{steady}(v_0)$, apart from unity values $N_{steady} = 1$, contains different values $N_{steady}(v_0)$, which are inherent in the given value of μ. For example, at $\mu = 0.01$, there are additional steps with $N_{steady} = 9$ and $N_{steady} = 3$; at $\mu = 0.02$, there are steps with $N_{steady} = 6$ and $N_{steady} = 2$; and at $\mu = 0.03$, there are steps with $N_{steady} = 4$.

THE FEASIBILITY OF THE MULTIPLE REPETITION OF THE STICKING REGIME

As was shown in [17], upon elastic reflection, $q = 1$, the multiple repetition of the sticking regime in the two-mirror scheme is problematic. This is related to the fact that, upon elastic reflections in the region passed by the oscillating mirror, the particle is ultimately reflected from the oscillating mirror at a relative moment of time that is different from $\pi/2$. In contrast, upon inelastic reflection in the case of "absolute" sticking, the particle is detached from the mirror with a high accuracy at the moment of time $\pi/2$, and this gives grounds for repeating the situation of the absolute sticking.

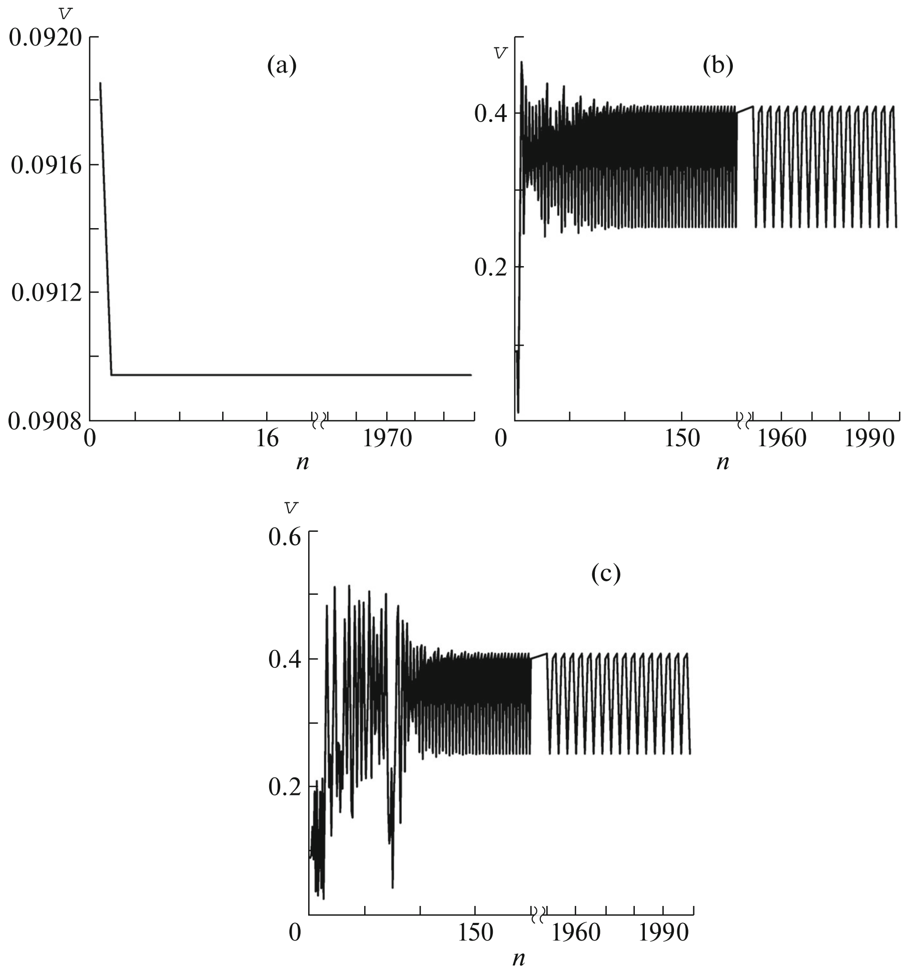

Fig. 7. Dependences of the velocity of a particle completely reflected from an oscillating mirror and the number of rereflections in the near-mirror region on the number of collision with a stationary mirror at $q = 0.95$: (a) $\mu = 2/(7\pi)$, $v_0 = \mu(1 + 0.01)$, $t_{\rm abs} = 0.09515, -1.5706, -1.5706, -1.5706\ldots$, and $K = 273, 24, 24, \ldots$ [for $\mu = 2/(7\pi)(1 - 0.0001)$ and $v_0 = \mu(1 + 0.01)$: $t_{\rm abs} = 0.09515, 0.6740, 0.6740, 0.6740\ldots$, and $K = 268, 285, 285\ldots$]; (b) $\mu = \frac{2}{7\pi}(1 + 0.0001)$, $v_0 = \mu(1 + 0.01)$, and $K = 273, 7, 0, 0, \ldots$; and (c) $\mu = \frac{2}{7\pi}(1 - 0.001)$, $v_0 = \mu(1 + 0.01)$, and $K = 273, 8, 0, 0\ldots$.

This regime of the absolute sticking in the two-mirror scheme is indeed possible under certain conditions. In order for the situation of an almost complete sticking to be repeated over N oscillation periods of the mirror, the particle reflected at time moment $\pi/2$ at distance L_0 from the stationary mirror with a velocity of $\mu_{cr,N}$ should collide with the mirror, in the ideal case, over time interval $\pi(2N - 1)$, i.e., $2L_0/\mu_{cr,N} = \pi(2N - 1)$, so that

$$\mu_{cr,N} = 2L_0/\left(\pi(2N - 1)\right). \tag{13}$$

The described situation is demonstrated in Fig. 7, which shows the dependences of the particle velocity on the number of reflection from the stationary mirror for different values of the parameters, as well as of quantity K—the number of rereflections in the region of mirror oscillations during the period of these oscillations, including ranges up to moment of absolute sticking $t_{\rm abs}$. Time moments are given in a reference system that is fixed to the time interval in which the particle has already entered the region of mirror oscillations; in this case, $t = -\pi$ at the moment when $z_w(t) = 1 - \mu$.

The initial phase of the process was chosen such that the first collision of the particle with the oscillating mirror would occur at moment $t_0 = -\pi/2$. Figure 7 presents calculation results for $\mu = \mu_{cr,4}$ and close values with the restoration coefficient being $q = 0.95$.

The situation of the absolute sticking and its repeating takes place if initial velocity v_0 is rather close to μ (slightly exceeds μ). Thus, for $q = 0.95$ at $v_0 = \mu(1 + 0.01)$ and at $\mu = \mu_{cr,4}$ (Fig. 7a), we have the absolute sticking and its repeating. In this case, $t_{\text{abs}} = 0.09515, -1.5706, -1.5706, -1.5706\ldots$ and $K = 273, 24, 24, \ldots$.

Small deviations of the values of μ toward $\mu < \mu_{cr,N}$ ensure the condition under which the particle that was reflected from the stationary mirror comes into contact with the oscillating mirror at $t > -\pi/2 + 2\pi N$ (i.e., at $t = -\pi/2 + \delta + 2\pi N$), after which the sticking is possible. At $\mu = \mu_{cr,4}(1 - 0.0001)$ and $v_0 = \mu(1 + 0.01)$, the plot of the dependence of the velocity of the particle after its reflection from the stationary mirror on the number of the contact with it has a shape that is similar to that in Fig. 7a. Here, $t_{\text{abs}} = 0.09515$, 0.6740, 0.6740, 0.6740, ..., while $K = 268, 285, 285$, ... If $\mu > \mu_{cr,N}$, the occurrence of the absolute sticking to the oscillating mirror after the first contact with the stationary mirror leads to a situation in which the particle escapes from the near-mirror region at a velocity of $v = \mu > \mu_{cr,N}$ and, after the second reflection from the stationary mirror, the collision of the particle with the oscillating mirror occurs at $t < -\pi/2 + 2\pi N$ (i.e., at $t = -\pi/2 - \delta + 2\pi N$). This leads to breakdown of the absolute sticking phenomenon. Our calculations confirm the absence of the absolute sticking at $\mu = \mu_{cr,4}(1 + 0.001)$ and $v_0 = \mu(1 + 0.01)$ ($K = 287, 0, 0, \ldots$) (Fig. 7b). An increase in the deviation of μ toward $\mu < \mu_{cr,N}$ leads to a situation in which there is no absolute sticking. Figure 7c presents calculation results for $\mu = \mu_{cr,4}(1 - 0.001)$, $v_0 = \mu(1 + 0.01)$. It is seen from this figure that there is no repeating of the absolute sticking; in this case, $K = 277, 8, 0, 0\ldots$. A periodic regime is established.

CONCLUSIONS

Therefore, upon inelastic reflections of particles from an oscillating mirror in a two-mirror trap, periodic, stochastic, or sticking regimes can be established, which are also well-known for the case of elastic reflections. The difference lies in the fact that the regimes become asymptotically stable, so that small initial deviations from their «ideal» values decrease with time. This means that, in the two-mirror scheme with inelastic reflections, a regime can be established in which not only the particle velocity is fixed (in magnitude) (i.e., the monochromatization of the atomic beam takes place), but also there is strict localization of particles, which is determined by the oscillating mirror. Thereby, dynamic traps provide additional possibilities in localization of peaks compared to static traps. Qualitatively, a similar behavior is known for parametric generation of radiation pulses in a cavity with a oscillating mirror [21–23], where localization of formed pulses is caused by matching with the phase of mirror oscillations.

The inelasticity of particle reflections from the mirror also makes it possible to obtain a periodically repeating sticking regime. In it, the particle is localized on the oscillating mirror within a half of the oscillation period, after which it flies away toward the stationary mirror, reflects off of this mirror, returns back to the oscillating mirror and is localized on it again within half of the oscillation period. In the case of elastic reflections, periodic repeating of the sticking regime is difficult, whereas an additional dissipative factor ensures stability of this regime.

ACKNOWLEDGMENTS

This work was supported by the Russian Science Foundation, project no. 14-12-00894.

REFERENCES

1. V. G. Minogin and B. C. Letokhov, *Laser Radiation Pressure on Atoms* (Nauka, Moscow, 1986) [in Russian].
2. A. P. Kazantsev, G. I. Surdutovich, and V. L. Yakovlev, *Mechanical Action of Light on Atoms* (World Scientific, Singapore, 1990; Nauka, Moscow, 1991).
3. S. Chu, Rev. Mod. Phys. **70**, 685 (1998); C. N. Cohen-Tannoudji, Rev. Mod. Phys. **70**, 707 (1998); W. D. Phillips, Rev. Mod. Phys. **70**, 721 (1998).
4. R. J. Cook and R. H. Hill, Opt. Commun. **43**, 258 (1982).
5. S. M. Tan and D. F. Walls, Phys. Rev. A **50**, 1561 (1994).
6. V. I. Balykin, V. S. Letokhov, Yu. B. Ovchinnikov, and A. I. Sidorov, JETP Lett. **45**, 353 (1987).
7. M. Kazevich, P. S. Weiss, and P. Chu, Opt. Lett. **15**, 607 (1990).
8. C. G. Aminoff, A. M. Steane, P. Bouyer, P. Desbiolles, and C. Cohen-Tannoudji, Phys. Rev. Lett. **71**, 3083 (1993).
9. J. Soding, R. Grimm, and Yu. B. Ovchinnikov, Opt. Commun. **119**, 652 (1995).
10. Ya. G. Sinai, Russ. Math. Surv. **25**, 137 (1970).
11. A. J. Lichtenberg and M. A. Lieberman, *Regular and Stochastic Motion* (Springer, New York, 1983).
12. L. D. Pustyl'nikov, Russ. Math. Surv. **50**, 145 (1995).
13. A. Yu. Loskutov, Phys. Usp. **50**, 939 (2007).
14. E. Fermi, Phys. Rev. **75**, 1169 (1949).

15. S. Ulam, in *Proceedings of the 4th Berkley Symposium on Mathematics, Statistics, and Probability* (Univ. of California Press, Berkeley, CA, 1961), Vol. 3, p. 315.
16. V. D. Vinokurova and N. N. Rosanov, Tech. Phys. Lett. **40**, 946 (2014).
17. V. D. Vinokurova, N. N. Rozanov, and E. G. Fedorov, Tech. Phys. **61**, 965 (2016).
18. Yu. B. Ovchinnikov, D. V. Laryushkin, V. I. Balykin, and V. S. Letokhov, JETP Lett. **62**, 113 (1995).
19. N. N. Rosanov, G. B. Sochilin, V. D. Vinokurova, and N. V. Vysotina, Phil. Trans. R. Soc. A **372** (2027), 20140012 (2014).
20. S. M. Targ, *Short Course of Theoretical Mechanics,* 10th ed. (Vyssh. Shkola, Moscow, 1986) [in Russian].
21. V. N. Krasil'nikov, *Parametric Wave Phenomena in Classical Electrodynamics* (SPb. Gos. Univ., St. Petersburg, 1996) [in Russian].
22. N. N. Rozanov, E. G. Fedorov, and A. A. Matskovskii, Quantum Electron. **46**, 13 (2016).
23. N. N. Rosanov and E. G. Fedorov, Opt. Spectrosc. **120**, 803 (2016).

Translated by V. Rogovoi