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NONLINEAR AND QUANTUM OPTICS

Quantum Dynamics of Intracavity Third-Subharmonic Generation

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Abstract—The quantum dynamics of the mean number of photons and quantum entropy of interacting modes, as well as the Wigner function of the stationary state of the fundamental mode and the third subharmonic mode has been investigated for the intracavity third-subharmonic generation. It is shown that the quantum dynamics of the system depends strongly on the nonlinear coupling coefficient between the modes. It is also demonstrated that, in the steady-state limit, depending on the intermodal coupling coefficient, the fundamental mode can be either in a pure coherent state, or in a squeezed state, or in a pure vacuum state. The third subharmonic mode in the subthreshold regime of generation of this mode is in the vacuum state. The Wigner function is squeezed over three sides of an equilateral triangle (squeezed vacuum). The quantum entropy of this state is nonzero. It is also shown that the third subharmonic mode, depending on the nonlinear coupling coefficient in the steady-state limit, can be localized in the three-component state with the same probability of detecting a field in each coherent component of the state and with the presence of quantum-mechanical interference between the state components. The mean number of photons in this state is smaller than unity. Depending on the nonlinear coupling coefficient, the third subharmonic mode can also be localized in the three-component state, which is a statistical mixture of three squeezed states.

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INTRODUCTION

The intracavity generation of subharmonics is a simple and very interesting phenomenon for studying the problems of quantum physics in optical systems. From this point of view, the degenerate parametric oscillator (or generation of the second subharmonic) was thoroughly investigated by Wolinski and Carmichael [1]. They showed (in the positive P representation [2], using adiabatic exclusion of the fundamental mode from the Langevin equations of motion) that, in the steady-state domain of interaction in this system, one can obtain light either in a squeezed state or in a superposition state of even type formed by two coherent components. Then the quantum behavior of this system was investigated by the Monte Carlo method [3] in [4], where it was shown that entangled states of light over a variable number of photons can be obtained in this system by studying the correlation of the fluctuations of the number of photons between interacting modes. It was also shown in [4] (by analyzing the dynamics of the quantum entropy and the Wigner function of the state of field modes) that a two-component state of the field of the second subharmonic mode (with the same probability of detecting field in each component of the state) can be implemented in this system at long interaction times. The quantum entropy of this state is lower than the maximum entropy of the two-component state ln2, which indicates the presence of quantum-mechanical interference between the coherent components of the state of the subharmonic mode.

In this study, we investigate (using the Monte Carlo method) the quantum dynamics of intracavity third-subharmonic generation. The quantum dynamics of the number of photons, the quantum entropy, and the Wigner function of the interacting field modes is analyzed. It is shown that the dynamics of these quantities depends strongly on the intermodal coupling coefficient. It is also demonstrated that three-component state of the field of the third subharmonic mode with the same probability of detecting field in each component of the state can be implemented in this system. It is shown that the quantum entropy of this state is lower than the maximum entropy on the three-component state ln3, which indicates the presence of quantum-mechanical interference between the components of the state of the third subharmonic mode.

NONLINEAR SYSTEM, BASIC EQUATIONS, AND CALCULATION ALGORITHMS

We model the generation of the third subharmonic in a two-mode cavity. A nonlinear medium is placed in the cavity, which is tuned to fundamental mode frequency ω_1 and third subharmonic mode frequency ω_2 ($\omega_1 = 3\omega_2$). The fundamental mode is resonantly perturbed by an external classical field. The equation for the density matrix of this system can be presented in the form

$$\frac{\partial \rho}{\partial t} = (i\hbar)^{-1}[H_{\rm sys}, \rho] + L(\rho), \qquad (1)$$

where

$$H_{\rm sys} = \frac{i\hbar\chi}{2}(a_1 a_2^{+3} - a_1^+ a_2^3) + i\hbar E(a_1^+ - a_1), \qquad (2)$$

$$L(\rho) = \frac{\gamma_1}{2}(2a_1\rho a_1^+ - \rho a_1^+ a_1 - a_1^+ a_1 \rho) + \frac{\gamma_2}{2}(2a_2\rho a_2^+ - \rho a_2^+ a_2 - a_2^+ a_2 \rho). \qquad (3)$$

Here, a_i and a_i^+ $(i = 1, 2)$ are the annihilation and creation operators of photons of the fundamental mode and third subharmonic mode, respectively; χ is the intermodal coupling coefficient, which is proportional to the nonlinear susceptibility $\chi^{(3)}$ of the medium; E is the classical amplitude of the perturbing field at frequency ω_1; and γ_i $(i = 1, 2)$ are the damping coefficients of interacting modes via cavity mirrors. The perturbing-field phase is omitted for simplicity in expression (2).

To study the quantum properties of the optical system, we will calculate the Wigner functions of the field-mode state. These functions are calculated in polar coordinates $x = r\cos(\theta)$, $y = r\sin(\theta)$ using the formula [5]

$$W_i(r, \theta) = \sum_{m,n} \rho_{i,mn} w_{mn}(r, \theta) \quad (i = 1, 2). \qquad (4)$$

Here, $\rho_{i,mn}$ are the elements of the density matrices of the interacting modes in the Fock basis. The expression for $w_{mn}(r, \theta)$ has the form

$$w_{mn}(r,\theta) = \begin{cases} \frac{2}{\pi}(-1)^n \left(\frac{n!}{m!}\right)^{1/2} \exp(i(m-n)\theta) \\ \times \exp(-2r^2)(2r)^{m-n} L_n^{m-n}(4r^2), \quad m \ge n, \\ \frac{2}{\pi}(-1)^m \left(\frac{m!}{n!}\right)^{1/2} \exp(i(m-n)\theta) \\ \times \exp(-2r^2)(2r)^{n-m} L_m^{n-m}(4r^2), \quad m \le n. \end{cases} \qquad (5)$$

In the last expression, L_p^q are Laguerre polynomials.

We will also investigate the quantum dynamics of the mean number of photons of the field modes and the dynamics of the quantum entropy $S_i(t) = -{\rm Tr}(\rho_i(t)\ln\rho_i(t))$ $(i = 1, 2)$ of the fundamental and subharmonic modes. The quantum entropy of the field can be calculated using the numerical diagonalization of the density matrices of these modes in the Fock basis.

Equation (1) for the density matrix of an optical system can be solved by the numerical Monte Carlo method [3]. In this method, the density matrix of the system is presented as the mathematical expectation of the density matrices of quantum trajectories, and each of these matrices characterizes a pure state, which can be found using some calculation algorithm:

$$\rho(t) = M\{|\varphi^{(\alpha)}(t)\rangle\langle\varphi^{(\alpha)}(t)|\} = \lim_{N\to\infty} \frac{1}{N} \sum_{(\alpha)}^{N} |\varphi^{(\alpha)}(t)\rangle\langle\varphi^{(\alpha)}(t)|. \qquad (6)$$

Here, α is the trajectory number and N is the number of independent quantum trajectories.

The algorithm of this method for calculating one quantum trajectory in the case of intracavity third-harmonic generation was reported in [6]. Since the algorithm for calculating one quantum trajectory in our problem is similar to the algorithm for calculating the quantum trajectory in the case of intracavity third-harmonic generation, it is omitted here.

The dynamics of the system is investigated for identical values of the mode damping coefficients in the cavity ($\gamma_1 = \gamma_2 = \gamma$) in dimensionless time $\tau = \gamma t$ and with dimensionless parameters of the system, $\varepsilon = E/\gamma$, $k = \chi/\gamma$. All calculations are performed for the evolution of the system from the initial vacuum state of the fundamental mode and the third subharmonic mode. Each dynamics of the mean number of photons, mode quantum entropy, and Wigner function is obtained using 1000 independent quantum trajectories of the optical system. The dynamics of the system is analyzed for the amplitude of fundamental mode perturbation $\varepsilon = 1.5$.

THE DYNAMICS OF THE SYSTEM IN THE CASE OF WEAK INTERMODAL COUPLING

In this section, we consider the quantum dynamics of the system for a small intermodal coupling coefficient: $k = 0.03$.

Figure 1 shows the dynamics of the numbers of photons of the fundamental mode (curve *a*) and the third subharmonic mode (curve *b*) for the above-mentioned case. The number of photons of the fundamental mode significantly increases and greatly exceeds unity for long interaction times (the steady-state value of the number of photons of the fundamental mode in this case is approximately 9). At the same time, the number of photons of the third subharmonic barely increases and is much smaller than unity for long interaction times (the steady-state value of the number of photons of the third subharmonic mode in this case is approximately 0.06). One can state that the system is in the subthreshold regime of subharmonic genera-

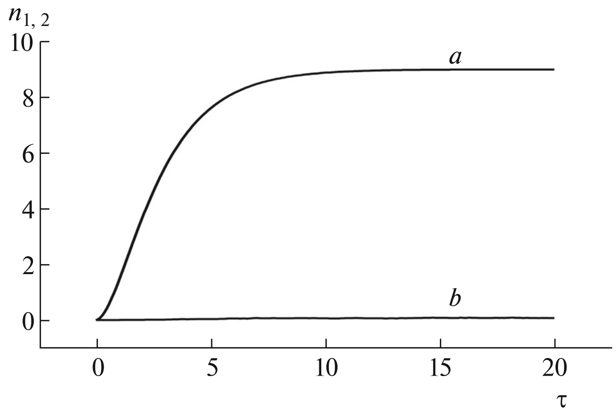

Fig. 1. Dynamics of the mean numbers of photons of (curve *a*) the fundamental mode and (curve *b*) the third subharmonic mode for coupling coefficient $k = 0.03$.

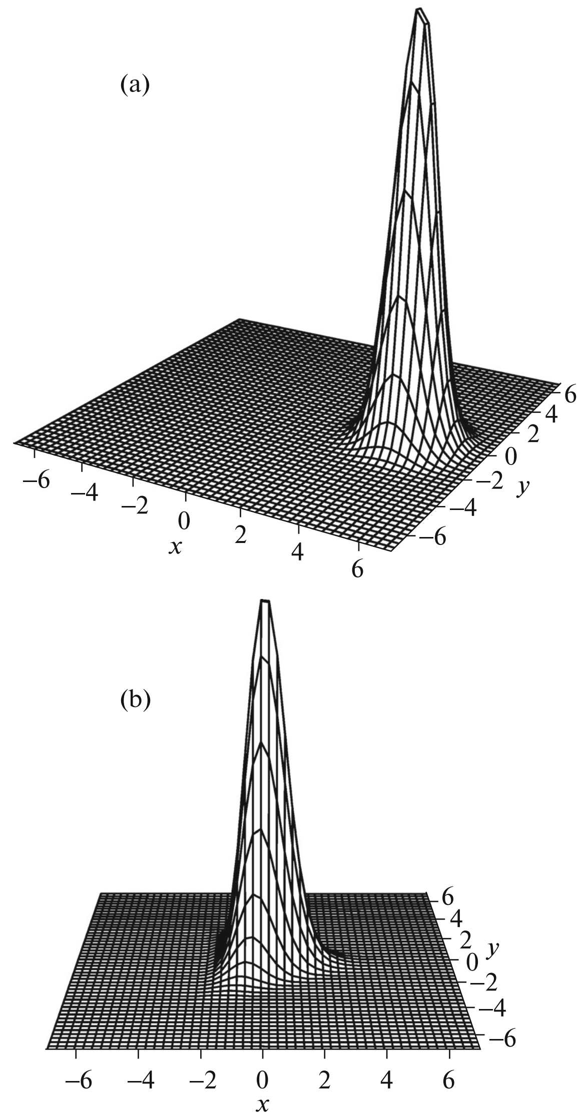

Fig. 2. Wigner functions of the stationary states of (a) the fundamental mode and (b) the third subharmonic for intermodal coupling coefficient $k = 0.03$.

tion, and the third subharmonic mode is close to the vacuum state.

The Wigner functions of the stationary state of the fundamental and third subharmonic modes are presented in Figs. 2a and 2b, respectively, for interaction time $\tau = 20$. The fundamental mode for long interaction times is in the one-component state, the Wigner function of which resembles that of the coherent state. The quantum entropy of this state is much lower than unity (approximately 0.0005, see Fig. 4). One can state that the fundamental mode is in the pure coherent state.

The Wigner function of the third subharmonic mode resembles the Wigner function of the vacuum state, which is symmetrically squeezed over three sides of equilateral triangle. The quantum entropy of this state is approximately 0.1 (Fig. 5), a fact indicating that the state is nonpure. For comparison, Fig. 3 presents the Wigner function of the stationary state of the second subharmonic mode in the subthreshold regime of generation of this mode, which was obtained in [2]. These Wigner functions differ significantly in symmetry, although both describe the vacuum state of the modes in the subthreshold regime of generation of the corresponding subharmonics.

The dynamics of the quantum entropy of the fundamental and subharmonic modes is presented in Figs. 4 and 5, respectively.

DYNAMICS OF THE SYSTEM IN THE CASE OF STRONG INTERMODAL COUPLING

In this section, we investigate the quantum dynamics of the system for a nonlinear coupling coefficient exceeding that analyzed in the previous section by a factor of 10 ($k = 0.3$ and 0.03, respectively). The perturbation amplitude is retained the same ($\varepsilon = 1.5$). In this case, the quantum behavior of the system differs significantly from the behavior of the system described in the previous section.

Figure 6 shows the dynamics of the number of photons of the fundamental and subharmonic modes (curves *a* and *b*, respectively). One can see that, in this case (in contrast to that described in the previous section), the system is in the above-threshold regime of subharmonic mode generation. The number of photons of the subharmonic mode for long interaction times is approximately equal to the number of photons of the fundamental mode for the same interaction times.

Figure 7 shows the Wigner functions of the fundamental and subharmonic modes (panels a and b, respectively) in the stationary state of the system.

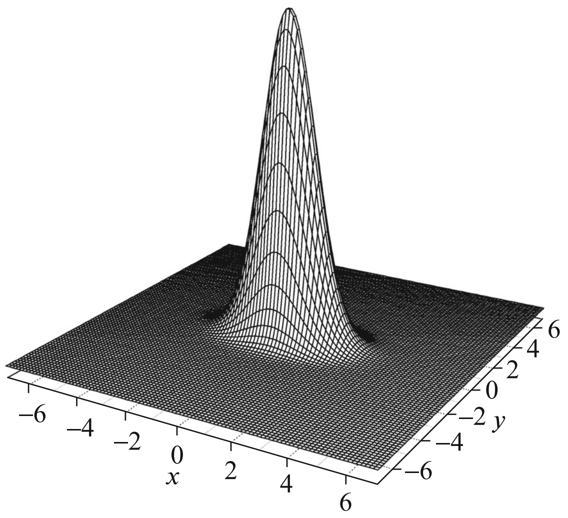

Fig. 3. Wigner function of the stationary state of the second subharmonic mode in the subthreshold regime of generation of this mode [4].

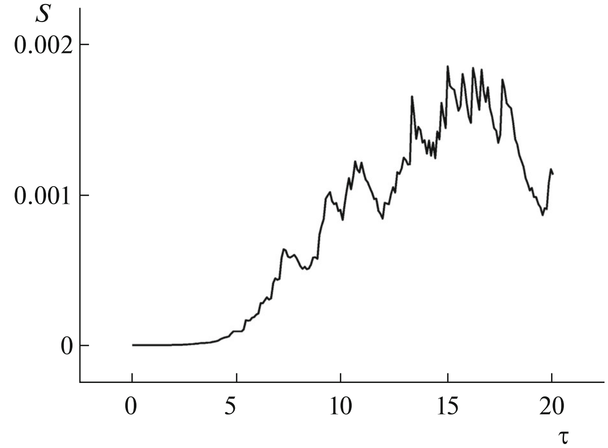

Fig. 4. Dynamics of the quantum entropy of the fundamental mode for intermodal coupling coefficient $k = 0.03$.

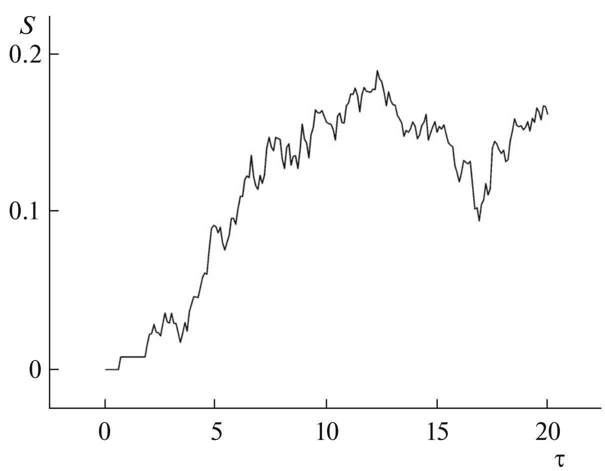

Fig. 5. Dynamics of the quantum entropy of the third subharmonic mode for intermodal coupling coefficient $k = 0.03$.

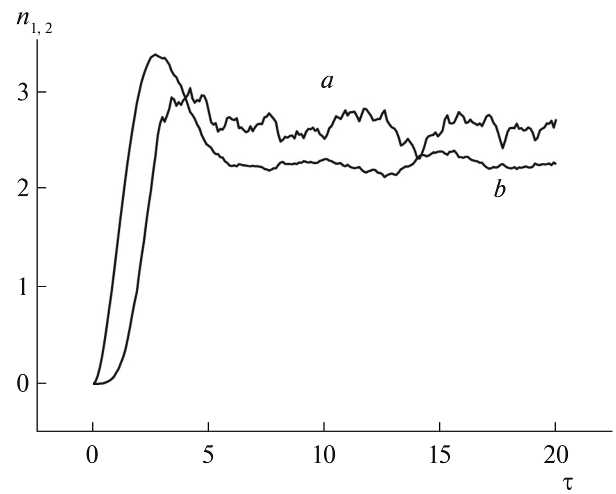

Fig. 6. Dynamics of the mean numbers of photons of (curve *a*) the fundamental mode and (curve *b*) the third subharmonic mode for coupling coefficient $k = 0.3$.

These functions were calculated for interaction time $\tau = 20$.

The Wigner function of the fundamental mode in the stationary state of the system differs significantly from the Wigner function of the stationary state of the same mode in the previous case: at $k = 0.03$, it is similar to the Wigner function of the coherent state, whereas at $k = 0.3$ it resembles the Wigner function of the squeezed state. The quantum entropy of this state is approximately 1.2 (Fig. 8), a fact indicative of nonpurity of this state. The form of the Wigner function of the subharmonic mode also sharply changes: at $k = 0.03$ it resembles the Wigner function of the vacuum state squeezed over three sides, whereas at $k = 0.3$ it is similar to the Wigner function of the state that is a statistical mixture of three components. Each component resembles the Wigner function of the squeezed state. The field of the third subharmonic mode can be found in each component of the state with equal probability. The quantum entropy of this state exceeds the maximum quantum entropy of the statistical mixture of the three-component state ln3 (Fig. 9). The latter indicates that the field state has a much more complicated structure, which does not follow from the analysis of the Wigner function.

The dynamics of the quantum entropy of the field state for the fundamental and third subharmonic modes is presented in Figs. 8 and 9, respectively. Both values become steady-state at long interaction times.

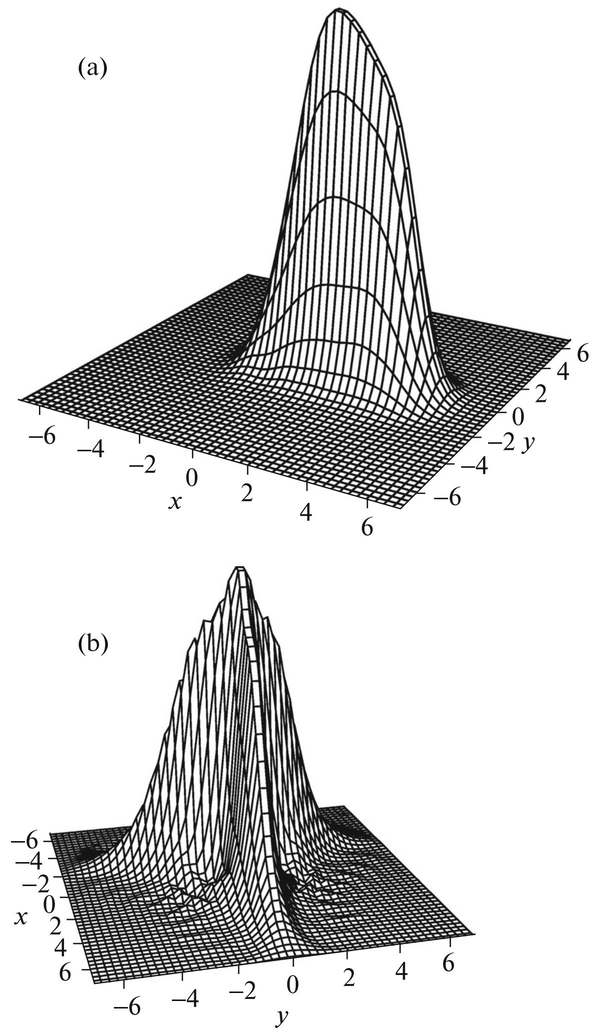

Fig. 7. Wigner functions of the stationary states of (a) the fundamental mode and (b) the third subharmonic for intermodal coupling coefficient $k = 0.3$.

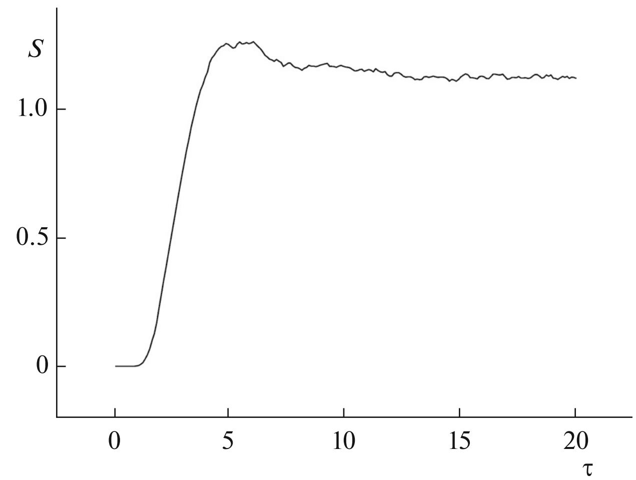

Fig. 8. Dynamics of the quantum entropy of the fundamental mode for intermodal coupling coefficient $k = 0.3$.

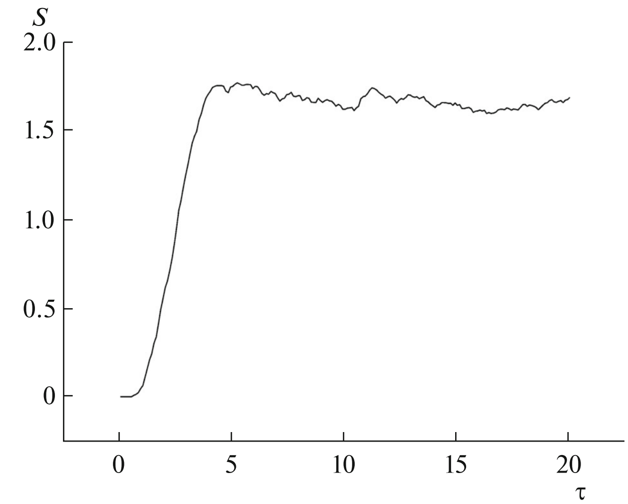

Fig. 9. Dynamics of the quantum entropy of the third subharmonic mode for intermodal coupling coefficient $k = 0.3$.

DYNAMICS OF THE SYSTEM IN THE CASE OF VERY STRONG INTERMODAL COUPLING

In this section we investigate the dynamics of the system in the case of a large nonlinear coupling coefficient between the interacting modes: $k = 4$.

Figure 10 shows the dynamics of the number of photons of the fundamental and subharmonic modes (curve *a* and *b*, respectively). The number of photons of the stationary state for both modes is smaller than unity. The number of photons of the stationary state for the fundamental mode is $\langle n_1 \rangle \approx 0.02$, and one can state that this mode is close to the vacuum state. The number of photons of the stationary state for the subharmonic mode is $\langle n_2 \rangle \approx 0.5$.

The Wigner functions of the stationary state of the fundamental and third subharmonic modes are presented in Figs. 11a and 11b, respectively. The Wigner function of the fundamental mode resembles the

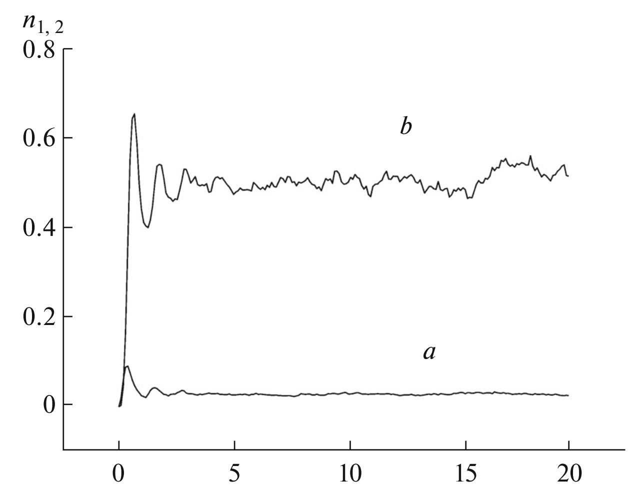

Fig. 10. Dynamics of the mean numbers of photons of (curve *a*) the fundamental mode and (curve *b*) the third subharmonic mode for coupling coefficient $k = 4$.

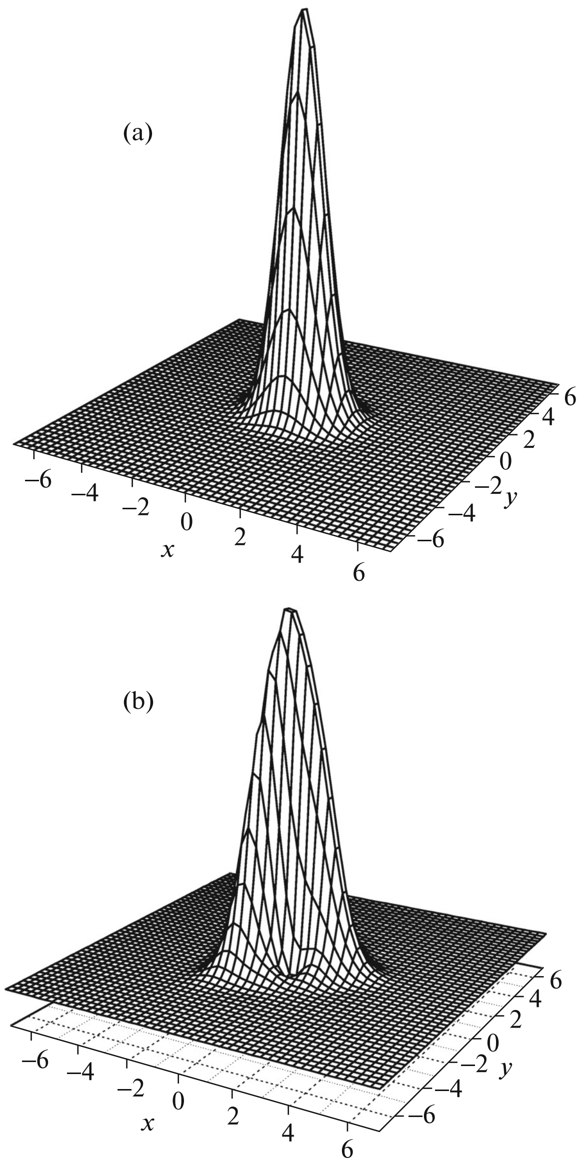

Fig. 11. Wigner functions of the stationary states of (a) the fundamental mode and (b) the third subharmonic for intermodal coupling coefficient $k = 4$.

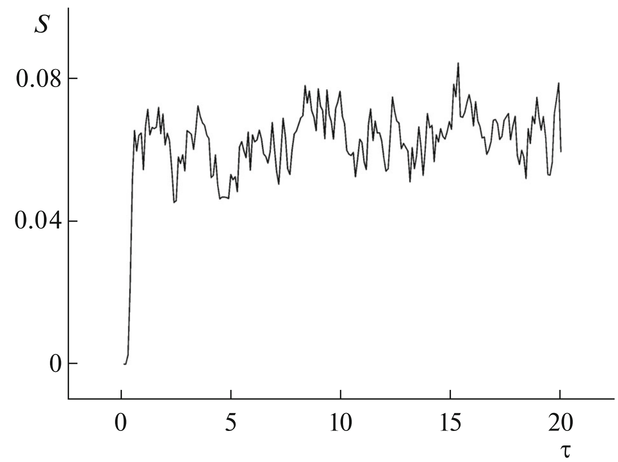

Fig. 12. Dynamics of the quantum entropy of the fundamental mode for intermodal coupling coefficient $k = 4$.

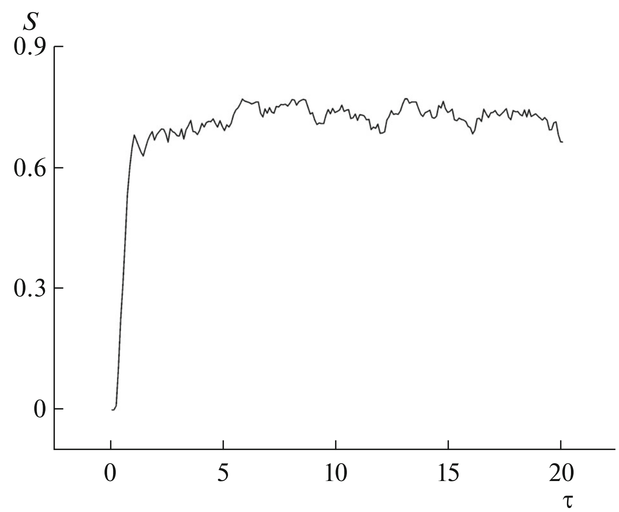

Fig. 13. Dynamics of the quantum entropy of the third subharmonic mode for intermodal coupling coefficient $k = 4$.

Wigner function of the vacuum state. The quantum entropy of this state is also close to zero (Fig. 12): it is approximately 0.05. Thus, one can state that in this case the fundamental mode is in the pure vacuum state at long interaction times.

The Wigner function of the stationary state of the third subharmonic mode has a three-component structure in this case. This mode is in the three-component state with the same probability of detecting field in each component of the state. The quantum entropy of this state is approximately 0.7 (Fig. 13). This value is smaller than the maximum quantum entropy of the three-component state ln3, which indicates the existence of a quantum-mechanical interference between the components of the field state for the third subharmonic mode in this interaction domain. The possibility of implementing three-component superposition states of light in optical systems was indicated for the first time in [7]. A mechanism of the occurrence of three-component states of light with quantum interference between the field state components in the range of small amplitudes of the state (the mean number of photons is smaller than unity) in a dissipative medium was proposed in [8].

The dynamics of the quantum entropy of the fundamental and third subharmonic modes for $k = 4$ is presented in Figs. 12 and 13, respectively. Both entropies become steady-state at long interaction times.

The steady-state entropy of the fundamental mode is much smaller than the steady-state entropy of the third subharmonic mode.

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