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PHYSICAL OPTICS

Stokes Eigenvectors and Evolution of the Polarization of Light in an Anisotropic Medium

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Abstract—The problem of evolution of the Stokes vector of a wave upon its transmission through an arbitrary homogeneous anisotropic medium with a non-Hermitian dielectric tensor has been solved in the general form. Explicit expressions for the Stokes vectors of eigenwaves have been obtained.

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INTRODUCTION

In [1, 2] it has been shown that the spatial dynamics of the polarization transformation of light in an arbitrary anisotropic medium has much in common with the dynamics of spin in a magnetic field. The evolution of the normalized Stokes vector of a wave with distance is described by an equation that resembles the Landau–Lifshits equation [2]. A substantial difference is that, in the general case, the two vectors of the effective fields that are responsible for the precession and decay can be noncollinear.

This situation leads to nonorthogonal polarizations of eigenwaves and can occur in crystals with dichroism that belong to lowest crystallographic systems [3, 4], in gyrotropic crystals [5], in magnetic orthoferrites [6], and so on. In such crystals, the ordinary notion of the optical axis looses its meaning; however, a singular optical axis can exist [7]. In [2, 8], the consideration of the evolution of the Stokes vector of the wave has been restricted only to the solution of the simplest cases of orthogonal eigenpolarizations. In the most general case (except for the case of the singular axis), the problem has been considered in [9]. However, no explicit connection with the dielectric tensor of the medium has been established (initial phenomenological parameters were used), and no explicit expression for the Stokes vector of the wave has been derived.

Therefore, it is of interest to obtain a solution that describes the evolution of the Stokes vector of a wave propagating in an inhomogeneous medium with an arbitrary, non-Hermitian, in the general case, dielectric tensor and to find Stokes vectors of eigenwaves, which is the subject of this work.

BASIC EQUATION

The notion of optics of anisotropic media that the propagation of waves along a chosen direction (z direction) is determined by the components of the inverse dielectric permittivity tensor in the plane of the front, $\hat{\eta} = \varepsilon^{-1}_{\alpha\beta}$ (α and β take values of x and y) [7, 10], remains valid at an arbitrary (in the general case, complex and non-Hermitian) dielectric tensor. Namely, for two-dimensional vector $\mathbf{D}$ of the electric induction, the following equation holds [10]:

$$\partial \mathbf{D}/\partial z = -ik\hat{\eta}^{-1/2}\mathbf{D}. \quad (1)$$

Matrix $\hat{\eta}^{-1/2}$ can be calculated explicitly (see [11]),

$$\hat{\eta}^{-1/2} = (n_+ + n_-)\hat{\sigma}_0/2 - \eta_j\hat{\sigma}_j n_+^2 n_-^2/(n_+ + n_-), \quad (2)$$

where the decomposition $\hat{\eta} = \eta_0\hat{\sigma}_0 + \eta_j\hat{\sigma}_j$ over Pauli matrices $\hat{\sigma}_j$ and unit matrix $\hat{\sigma}_0$ is used, $n_\pm = (\eta_0 \pm \eta)^{-1/2}$ are the refractive indices of eigenwaves, $\eta = (\eta_j\eta_j)^{1/2}$ is the semidifference of the eigenvalues of matrix $\hat{\eta}$, and the summation over the repeated indices from 1 to 3 is performed. Here and elsewhere, we will enumerate Pauli matrices as is accepted in polarization optics: $\hat{\sigma}_1 = \hat{\sigma}_z$, $\hat{\sigma}_2 = \hat{\sigma}_x$, and $\hat{\sigma}_3 = \hat{\sigma}_y$.

Furthermore, we will pass to the notation of [2]:

$$\partial \mathbf{D}/\partial z = -i(G_0\hat{\sigma}_0 + G_j\hat{\sigma}_j)\mathbf{D}/2, \quad (3)$$

where three-dimensional parameter vector $\mathbf{G}$ is connected explicitly with the inverse dielectric tensor of the homogeneous medium without assuming its weak anisotropy, namely,

$$G_j = -[2kn_+^2n_-^2/(n_+ + n_-)]\eta_j. \quad (4)$$

In accordance with [2], the evolution of the normalized Stokes vector (here, we will restrict our consideration only to the three-dimensional part of the Stokes vector, since we assume that the light is completely polarized)

$$s_j = \langle \mathbf{D}|\hat{\sigma}_j|\mathbf{D}\rangle / \langle \mathbf{D}|\mathbf{D}\rangle \quad (5)$$

is described by the following equation:

$$\partial \mathbf{s}/\partial z = [\mathbf{P}, \mathbf{s}] + [\mathbf{s}, [\mathbf{Q}, \mathbf{s}]], \quad (6)$$

where $\mathbf{P} = \mathrm{Re}\mathbf{G}$ and $\mathbf{Q} = \mathrm{Im}\mathbf{G}$.

As was noted above, this equation coincides with the Landau–Lifshits equation only if **P** and **Q** are collinear and, as a consequence, the Stokes eigen (stationary) vectors are collinear.

STOKES EIGENVECTORS

Because the polarization of the eigenwave does not vary with distance, the eigenvectors should satisfy the equation

$$[\mathbf{P}, \mathbf{s}] + [\mathbf{s}, [\mathbf{Q}, \mathbf{s}]] = 0. \quad (7)$$

In order to find eigenvectors, we will use the following evident fact: if vector parameter **G** of a medium is multiplied by an arbitrary complex constant, eigenvectors of this medium will remain unchanged. If vectors **P** and **Q** are nonorthogonal, i.e., if $(\mathbf{P}, \mathbf{Q}) \neq 0$, then, dividing **G** by $G \equiv (\mathbf{G}, \mathbf{G})^{1/2} = [P^2 - Q^2 + 2i(\mathbf{P}, \mathbf{Q})]^{1/2}$, one can orthogonalize the real and imaginary parts. It can be easily verified that, for new vector parameter $\mathbf{g} \equiv \mathbf{G}/G = \mathbf{p} + i\mathbf{q}$, the equality $(\mathbf{p}, \mathbf{q}) = 0$ will hold and, in addition, $p^2 - q^2 = 1$. The transformation is defined by the following formulas:

$$\mathbf{p} = a\mathbf{P} + b\mathbf{Q},$$
$$\mathbf{q} = -b\mathbf{P} + a\mathbf{Q}, \quad (8)$$

where $a = [(P^2 - Q^2 + \Delta)/2]^{1/2}/\Delta$, $b = [(Q^2 - P^2 + \Delta)/2]^{1/2}/\Delta$, and $\Delta = [(P^2 - Q^2)^2 + 4(\mathbf{P}, \mathbf{Q})^2]^{1/2}$.

Eigenvectors that satisfy the equation $[\mathbf{p}, \mathbf{s}] + [\mathbf{s}, [\mathbf{q}, \mathbf{s}]] = 0$ will be sought in the form $\mathbf{s} = c_1\mathbf{p} + c_2\mathbf{q} + c_3[\mathbf{p}, \mathbf{q}]$. Its substitution and equating of components to zero yield the following result: $c_1 = \pm 1/p^2$, $c_2 = 0$, and $c_3 = 1/p^2$. Finally, we obtain

$$\mathbf{s}^{\pm} = (\pm\mathbf{p} + [\mathbf{p}, \mathbf{q}])/p^2, \quad (9)$$

thereby solving the posed problem on finding eigenvectors. In the general case, they are always oriented symmetrically with respect to $[\mathbf{p}, \mathbf{q}]$ in the plane that is perpendicular to vector **q** and form an angle with vector **p**, the tangent of which is equal to $\pm q$.

It is also of interest to consider the inverse problem: to find the parameters of a medium from known eigenvectors. From formula (9), we can easily obtain

$$\mathbf{p} = (\mathbf{s}^+ - \mathbf{s}^-)/[1 - (\mathbf{s}^+, \mathbf{s}^-)],$$
$$\mathbf{q} = [\mathbf{s}^-, \mathbf{s}^+]/[1 - (\mathbf{s}^+, \mathbf{s}^-)]. \quad (10)$$

If desired, one can express the eigenvectors via **P** and **Q**, using relationships (8),

$$\mathbf{s}^{\pm} = \{\pm(a\mathbf{P} + b\mathbf{Q}) + (a^2 + b^2)[\mathbf{P}, \mathbf{Q}]\}/\{(a^2P^2 + b^2Q^2 + 2ab(\mathbf{P}, \mathbf{Q})\}. \quad (11)$$

The case $(\mathbf{P}, \mathbf{Q}) = 0$ with identical moduli $P = Q \neq 0$ requires special consideration. Here, we have the so-called "singular optical axis" and coinciding eigenvectors $\mathbf{s}^{\pm} = [\mathbf{P}, \mathbf{Q}]/P^2$.

Now, let us return to the very beginning and consider how eigenvectors could be found without using Eq. (6). First of all, we note that vector parameter **g** is closely related to tensor $\hat{\eta}$. It can be easily shown that $g_j = -\eta_j/\eta$ (clearly, except for the case of singular axis $\eta \neq 0$). Eigenvectors of waves $\mathbf{D}^{\pm}$ coincide with eigenvectors of matrix $\hat{\eta}$, which can be represented in the form

$$\mathbf{D}^{\pm} = \left\| \begin{matrix} g_1 \pm 1 \\ g_2 + ig_3 \end{matrix} \right\|. \quad (12)$$

In accordance with formula (5), for the components of the Stokes eigenvectors, the following expressions are obtained:

$$s_1^{\pm} = (|g_1 \pm 1|^2 - |g_2 + ig_3|^2)/(|g_1 \pm 1|^2 + |g_2 + ig_3|^2),$$
$$s_2^{\pm} = 2\mathrm{Re}[(g_1 \pm 1)^*(g_2 + ig_3)]/(|g_1 \pm 1|^2 + |g_2 + ig_3|^2), \quad (13)$$
$$s_3^{\pm} = 2\mathrm{Im}[(g_1 \pm 1)^*(g_2 + ig_3)]/(|g_1 \pm 1|^2 + |g_2 + ig_3|^2).$$

It is interesting to note that, by cyclically rearranging the indices, any of the three formulas (13) can be used to calculate the Stokes eigenvectors. By substituting real and imaginary parts g_j and performing rather cumbersome transformations, these bulky formulas can be reduced to the single formula (9), which reflects the symmetry of the problem and the role played by the decomposition over the Pauli matrices.

SOLUTION OF THE EQUATION

The solution of matrix equation (3) at $G \neq 0$ can be written in the following form (see [11]):

$$\mathbf{D}(z) = \exp[-iz(G_0\hat{\sigma}_0 + G_j\hat{\sigma}_j)/2]\mathbf{D}_0$$
$$= (\hat{\sigma}_0\cos\delta/2 + ig_j\hat{\sigma}_j\sin\delta/2)\exp(i\varphi)\mathbf{D}_0, \quad (14)$$

where $\varphi = -G_0z/2$ and $\delta = -Gz$, with $\varphi \pm \delta/2$ being the accumulated phase shifts of the eigenwaves. The matrix that appears in front of vector $\mathbf{D}_0$ in (14) is none other than the Jones matrix, which describes the transformation of vector **D** upon propagation of the wave through the medium. The components of the Müller matrix, which relate the values of the Stokes vectors at the input and output of the medium, can be

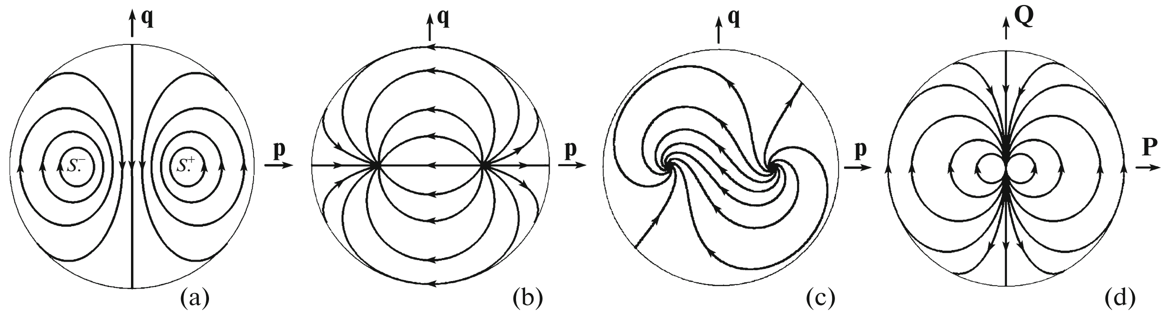

Fig. 1. Possible variants of the evolution of a normalized Stokes vector on the Poincaré sphere upon propagation of waves through a medium: (a) the case of equal absorptions of eigenwaves, (b) the case of equal velocities of eigenwaves, (c) the general case, and (d) the case of a singular optical axis. The Poincaré sphere is rotated such that vectors **p** and **q** would lie in the plane of the figure.

found using the well-known relations with the components of the Jones matrix [9]:

$$\begin{aligned} M_{00} &= p^2c'' - q^2c', \\ M_{0a}, M_{a0} &= -s''p_a - s'q_a \pm ce_{ajk}q_jp_k, \qquad (15) \\ M_{ab} &= \delta_{ab}(p^2c' - q^2c'') + e_{abj}(s'p_j - s''q_j) + c(p_ap_b + q_aq_b), \end{aligned}$$

where the following notation is introduced: $s' = \sin\delta'$, $s'' = \sinh\delta''$, $c' = \cos\delta'$, $c'' = \cosh\delta''$, $c = c'' - c'$, $\delta = \delta' + i\delta''$, and e_{jkl} is the third-rank antisymmetric tensor. In formulas (15) and, further, (17), we omitted insignificant factor $\exp(z\mathrm{Im}G_0)$, which is responsible for the overall decay.

Found Müller matrix (15) makes it possible to write the solution of Eq. (6); it is only necessary to perform the corresponding normalization of the Stokes vector at the output. As a result, we obtain

$$s(z) = \frac{(p^2c' - q^2c'')\mathbf{s} + s'[\mathbf{s},\mathbf{p}] - s''[\mathbf{s},\mathbf{q}] + c(\mathbf{s},\mathbf{p})\mathbf{p} + c(\mathbf{s},\mathbf{q})\mathbf{q} + c[\mathbf{p},\mathbf{q}] - s''\mathbf{p} - s'\mathbf{q}}{p^2c'' - q^2c' - c(\mathbf{s},[\mathbf{p},\mathbf{q}]) - s''(\mathbf{s},\mathbf{p}) - s'(\mathbf{s},\mathbf{q})}, \qquad (16)$$

where $\mathbf{s} \equiv \mathbf{s}(z = 0)$. It is remarkable that, at $q = 0$, the found solution yields the solution to the Landau–Lifshits equation [12].

The case of a singular optical axis requires special consideration because $\delta \to 0$, and $|\mathbf{p}|, |\mathbf{q}| \to \infty$. By performing the passage to the limit at $G \to 0$ in (14) and (15) (note that the components of complex vector **G** are not necessarily equal to zero), we obtain the Jones matrix

$$\hat{J} = (\hat{\sigma}_0 - iG_j\hat{\sigma}_j z/2)\exp(i\varphi)$$

and the Müller matrix

$$\begin{aligned} M_{00} &= 1 + P^2z^2/2, \\ M_{0a}, M_{a0} &= Q_az \pm e_{ajk}Q_jP_kz^2/2, \qquad (17) \\ M_{ab} &= \delta_{ab}(1 - P^2z^2/2) - e_{abj}P_jz + (P_aP_b + Q_aQ_b)z^2/2. \end{aligned}$$

In this case, writing the expression for $\mathbf{s}(z)$ is straightforward.

Figure 1 shows possible variants of the evolution of the normalized Stokes vector on the Poincaré sphere as waves propagate through the medium. Without loss of generality, the Poincaré sphere is rotated such that vectors **p** and **q** would lie in the plane of the figure, while [**p**, **q**] would be directed toward the reader. Variant (a) shows the case of equal absorption of eigenwaves ($\delta'' = 0$), variant (b) is the case of equal velocities of eigenwaves ($\delta' = 0$), (c) is the general case, and (d) is the case of a singular optical axis. It can be shown that the trajectories in cases (a), (b), and (d) are circles (or arcs of circles) on the Poincaré sphere.

CONCLUSIONS

Let us list the sequence of steps to calculate in the general case the change in the polarization of light upon its transmission through a crystal with the known dielectric tensor.

(i) An inverse dielectric tensor is sought in a coordinate system the z axis of which is parallel to the wave vector.

(ii) A decomposition of the two-dimensional part of the tensor in the plane of the front over Pauli matrices is found.

(iii) The decomposition coefficients are normalized such that the sum of their squares would be unity.

(iv) The vectors that represent the real and imaginary parts of the obtained coefficients determine the Stokes vectors of eigenwaves by formula (8).

(v) The phase difference of eigenwaves is sought by multiplying the normalization constant (from step iii) into the path length, and the coefficient from formula

(4) (which is singled out in square brackets) is determined.

(vi) The sought normalized Stokes vector of the polarization of a wave upon its propagation through the crystal is given by formula (16).

It is of interest to apply this approach to the case of usual biaxial transparent crystal for the arbitrary orientation of the vector of the incident wave when it does not coincide with either optical axis. Using the known decomposition of the inverse dielectric tensor over components of vectors of optical axes [3], we obtain a zero value of vector **q** and the following components of vector **p**:

$$p_1 = \cos(\alpha_1 + \alpha_2), \quad p_2 = \sin(\alpha_1 + \alpha_2), \quad \text{and} \quad p_3 = 0,$$

where α_1 and α_2 are the azimuthal angles of the projections of the optical axes onto the plane of the front.

Stokes eigenvectors $\mathbf{s}^{\pm} = \pm\mathbf{p}$ are directed oppositely to each other, while the eigenpolarizations are always linear and are orthogonal to each other (we note that the Stokes eigenvectors written in the four-dimensional form are also orthogonal to each other). Trajectories of variation of the polarization on the Poincaré sphere are closed circles, which are perpendicular to the Stokes eigenvectors (as if the points that correspond to the Stokes eigenvectors in Fig. 1a would be diametrically opposite). The same pattern is obtained in the case of a gyrotropic crystal with the only difference that component p_3 differs from zero, and we have an elliptical polarization of eigenwaves.

In the case of a uniaxial crystal that possesses dichroism, the eigenpolarizations are also always linear and orthogonal; however, trajectories of the polarization variation are spiral-like and are similar to those shown in Fig. 1c with diametrically opposite eigenvectors. In the case of a partial polarizer with a zero real part of the phase shift, we have the situation of Fig. 1b with diametrically opposite eigenvectors.

Only in crystals that belong to lowest crystallographic systems with dichroism (including gyrotropic and magnetic crystals) can realize all the cases shown in the figure. The obtained results show that the behavior of the polarization of light that propagates through a medium with a non-Hermitian dielectric tensor substantially differs from the dynamics of spin in a magnetic field. First, the polarizations of eigenwaves can be nonorthogonal and stationary Stokes vectors can be noncollinear. Second, even in a medium with dissipation and nonorthogonal eigenpolarizations, the normalized Stokes vector of the wave can perform precession along a closed trajectory, without approaching a stationary state in the case of equal absorption of eigenwaves. Third, in the case of a singular optical axis, the occurrence of only one stationary state is possible.

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