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EXPERIMENTAL AND ANALYTICAL STUDY OF GEOMETRIC NONLINEAR BENDING OF A CANTILEVER BEAM UNDER A TRANSVERSE LOAD

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Abstract: Exact and approximate analytical solutions are compared with experimental data on the geometrically nonlinear bending of a thin elastic cantilever beam under the action of a transverse concentrated load at its free end.

Keywords: geometrically nonlinear bending, large deformations, experiment, cantilever beam, thin elastic rod.

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INTRODUCTION

The solution of problems of designing modern complex transformable rod structures in aerospace engineering and microelectromechanical systems requires the high-precision modeling of deformation of thin flexible elastic rods. The problem of determining the equilibrium state of a bent rod is often reduced to solving the Euler — Bernoulli equation with initial or boundary conditions corresponding to a load applied to the rod. In the Cartesian coordinates, this equation has the form [1]

$$\frac{M(x)}{EJ} = \frac{d^2y}{dx^2}\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{-3/2}, \tag{1}$$

where $M(x)$ denotes the force moment acting on the rod, E is the elastic modulus, J is the inertia moment of the cross section of the rod, and y and x are the coordinates of the rod points.

As Eq. (1) is nonlinear, it is a nontrivial task to obtain its exact solution. In this case, it is required to consider large deformations of the bent rod, at which the deflection is comparable to the rod length. Equation (1) in the original nonlinear formulation yields an analytical solution expressed in elliptic functions and integrals [2] and in some cases in hypergeometric functions [3].

Small deformations of rods are considered in some classical applied problems of deformable solid mechanics, which makes it possible to linearize the equation (1):

$$\frac{M(x)}{EJ} = \frac{d^2y}{dx^2}. \tag{2}$$

This approach underlies the theory of strength of materials, and it is believed that it is applicable in cases where deflection does not exceed 3–5% of the rod length [4].

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Currently, there are approximate analytical solutions to the problem of transverse bending of a rod, obtained using various methods: power expansions of exact solutions [5, 6], power expansions of the original equation (1) [7], and approximation by polynomials [8]. There are also approximate solutions obtained using other approaches, for example, using modified linear expressions [9].

It is a relevant task to compare analytical solutions with experimental data to determine load intervals within which some approximate solution is applicable. The geometrically nonlinear bending of a cantilever by a transverse load is experimentally studied in [10–12].

A cantilever bending under the action of a transverse load was experimentally studied in [10]. The resulting data could be used to train a neural network determining the cantilever deflection under various loads. Only the axial displacement δx and deflection f were recorded in that experiment. The load range considered in that study was $\mu < 0.96$ ($\mu = PL^2/(2EJ)$ denoted the dimensionless load parameter, P was the force acting on the rod, and L was the rod length). An experiment with a cantilever made of spring steel was carried out in [11]. The experiment was carried out in a horizontal plane in order to exclude the influence of the gravitational load. A deflection and an axial displacement were obtained for a dimensionless load $\mu < 1.47$ in [11]. A laboratory experiment on the study of geometrically nonlinear bends was carried out in [12]. The case of a combined load was considered: transverse at the free end and uniformly distributed along the length of the rod (gravity load). In [12], the cantilever deflection was determined in the range of transverse loads $\mu < 1.75$.

In this paper, a number of exact [2] and approximate [9] analytical solutions for the problem of bending of a rod loaded with a transverse concentrated force at the free end are compared with the experimental data obtained under loads that lead to large deformations of the rod (geometrically nonlinear bending).

The analytical study of the dynamic stability of flexible rods in a geometrically nonlinear formulation was carried out in [13].

FORMULATION OF THE EXPERIMENT

Figure 1 illustrates the bending of a rod having length L and being under the action of a transverse concentrated force P at the free end. As a result of bending, the points of the rod are displaced relative to the x and y axes. The rod end is displaced along the y axis (deflection f) and along the x axis (displacement δx). The deflection, the axial displacement under various applied loads, and the bend shapes are recorded in the experiment.

A thin rectangular steel strip is used as an experimental sample, and its parameters are as follows: total length 332 mm, width 28 mm, thickness 0.8 mm, sample weight 48.7 g, elastic modulus $E = 200$ GPa, and the length of the section used in the experiment $L = 300$ mm.

The diagram and general view of the experimental device are shown in Fig. 2.

The sample under study is fixed in a clamp. A hook of mass m_r is placed at the free end of the sample to suspend the load. The bend shape is recorded using a digital camera located at some distance from the setup. Photographs are digitized using the Graph2Digit software. A reference mark is placed in the plane of the bend to determine the scale when processing photographs.

The purpose of the experiment is to determine the deflection, axial displacement, and bend shapes under loads that lead to geometrically nonlinear bending. At the beginning of the experiment, the test sample is placed

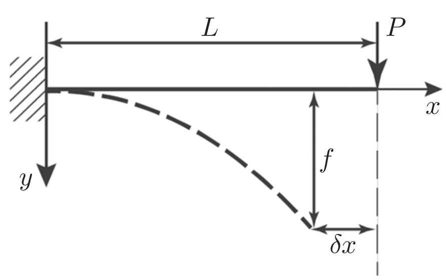

Fig. 1. Bending of the rod: the solid line is the equilibrium position of the rod (no load), and the dashed line is the position of the rod deformed under the action of load P.

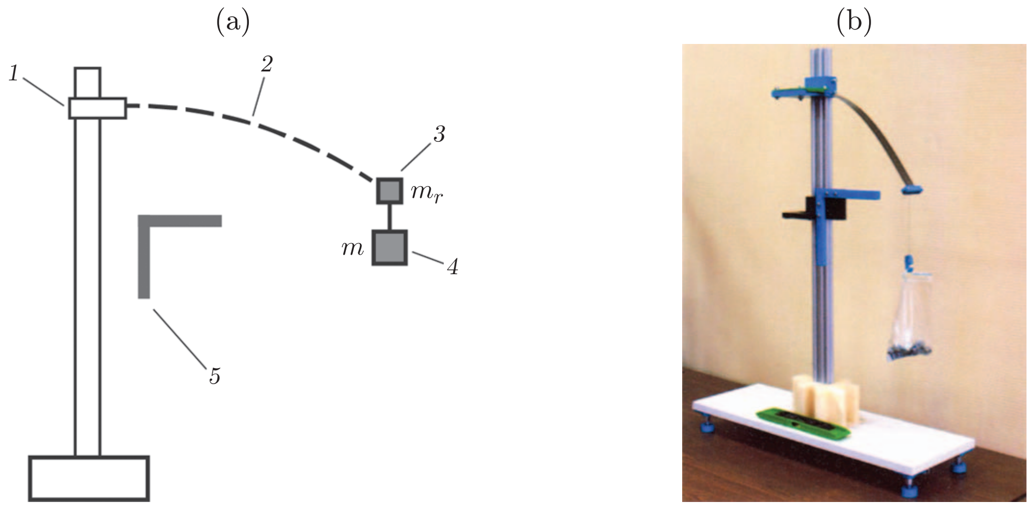

Fig. 2. Diagram (a) and general view (b) of the experimental device: (*1*) clamp, (*2*) test sample, (*3*) hook, (*4*) cargo, and (*5*) reference marks.

in the experimental device. The equilibrium position of this cantilever is fixed to determine the deflection under the action of its own weight (distributed gravity load), which is $f_w = 8.6$ mm. Then, a hook with a mass $m_r = 11.5$ g is attached to the free end of the sample. The bent state of the cantilever under the action of this load is recorded. Further, the load is increased step by step by hanging weights ($m = 40$ g) on the hook. The state of the cantilever is recorded at each loading step. The data describing the shape of the bent profile of the cantilever are processed using the Graph2Digit software: the position of the pinch point of the cantilever and the reference mark are fixed, and then points along the profile of the bent rod are plotted for digitization. The resulting data array is presented in the form of spreadsheets.

The coordinates of points x and y and the load P are recalculated into the dimensionless values ξ, η and μ according to the expressions

$$\xi = \frac{x}{L}, \quad \eta = \frac{y}{L}, \quad \mu = \frac{PL^2}{2EJ}.$$

Error minimization is achieved by performing the experiment eight times, each time with digitization and data processing. The resulting data are averaged. The bend shapes are determined by averaging the data via the third-order least squares method.

Then, the deflection value resulting from the influence of the cantilever weight f_w is subtracted from the deflection value obtained in the experiment. The bend shapes are also corrected to exclude the effect of the gravity load. For this purpose, the theoretically determined coordinates of the bend shape due to the effect of the gravity load are subtracted from the coordinates of the bend shape obtained in the experiment [14]:

$$\eta = \frac{1}{12}\frac{WL^2}{2EJ}\left(\xi^4 - 4\xi^3 + 6\xi^2\right)$$

(W is the cantilever weight).

ANALYTICAL SOLUTIONS

Modified analytic linear expressions are given in [9]. Within the framework of the classical linear theory, one can determine the bending of the cantilever using a transverse load by solving Eq. (2) with the initial conditions $y(0) = 0$ and $y'(0) = 0$. In this case, the acting moment is equal to $M(x) = Px$, and the shape of the bend in dimensionless coordinates is determined by the expression

$$\eta = \eta(\xi, \mu) = \mu(3\xi^2 - \xi^3)/3. \quad (3)$$

In the classical linear theory, the deflection is determined by assuming that there is no axial displacement δx at small deformations, so the deflection is determined at the end point of the rod in the equilibrium state:

$$f = f(\mu) = \eta(1, \mu) = 2\mu/3. \tag{4}$$

However, the real situation is that, as the load increases, so does the axial displacement δx, which cannot be neglected at large deformations. Thus, it is necessary to determine the deflection at point $L - \delta x$ (the abscissa of the rod end point). In general, the dependence of the deflection on the axial displacement and load can be written in dimensionless variables as

$$f = \eta(1 - \delta\xi, \mu), \tag{5}$$

where $\delta\xi = \delta x/L$ is the dimensionless axial displacement.

$\delta\xi$ is calculated from the condition of conservation of the curvilinear length in dimensionless variables:

$$\int\limits_0^{1-\delta\xi} \Big[1 + \Big(\frac{d\eta(\xi,\mu)}{d\xi}\Big)^2\Big]^{1/2} d\xi = 1. \tag{6}$$

It is possible to use expressions (3)–(6) to obtain a deflection in a load range corresponding to geometrically nonlinear bending, which significantly exceeds the range of applicability of the classical linear theory.

The study described in [2] presents an exact analytical solution that describes the shapes of deformable thin elastic rods under various types of loading, as well as solutions and critical loads at which there is a transition to new forms of the equilibrium state of a bent rod.

In the case of a transverse end load of the cantilever, the equations for the bent axis of the rod have the parametric form:

$$\begin{aligned} \xi &= \frac{x}{L} = \frac{2k}{pK - F_1}\Big[\Big(1 - \frac{1}{(2k)^2}\Big)^{1/2} - \mathrm{cn}\,(u)\Big], \\ \eta &= \frac{y}{L} = t - \frac{2}{pK - F_1}\,[E(\mathrm{am}\,(u), k) - E_1]. \end{aligned} \tag{7}$$

Here $F_1 = F[\arcsin(\sqrt{2}/(2k)), k]$; $K(k)$ and $F(\varphi, k)$ are the complete and incomplete elliptic integrals of the first kind, respectively; $u = [(2n-1)K - F_1]t + F_1$; n is the mode number; cn (u) is the Jacobi elliptic cosine; am (u) is the Jacobi elliptic amplitude; $E_1 = E[\arcsin(\sqrt{2}/(2k)), k]$; $E(\varphi, k)$ is an incomplete elliptic integral of the second kind. The modulus of the elliptic integral k is the dimensionless load parameter ($1/2 \leqslant k^2 \leqslant 1$). In the case under consideration, $n = 1$ as the first static mode of the equilibrium state of the console is being investigated.

As a result of the experiment, the deflection f, the axial displacement $\delta\xi$ (Fig. 3) and the bend shape (Fig. 4) are determined.

COMPARISON OF EXPERIMENTAL DATA AND ANALYTICAL SOLUTIONS

The experimental data are compared with the analytical exact solution [2] and the approximate solution using modified linear expressions [9]. Figure 5 shows the dependences of the deflection on the dimensionless load parameter μ. The differences between theoretical and experimental deflections are shown in Fig. 6.

Figure 5 shows that the calculation results obtained using the modified linear equations and the exact solution are consistent with the experimental data over the entire load range. It follows from Fig. 6 that the exact solution deviates from the experimental data over most of the load interval by less than 1–2%. In the case of calculations carried out using the modified linear equations, this value is greater (3–4%).

Figure 7 shows the deviation of the axial displacement in the analytical solutions from the experimental data. The results of the comparison with the axial displacement in the linear solution are not shown because the axial displacement in this case is always zero. The calculation carried out according to the modified linear equations and the exact solution yield close results, and their deviation from the experimental data does not exceed 4%.

The significant difference between the results shown in Figs. 6 and 7 from the experimental data at low loads is explained by the small values of deflections and axial displacements at these loads, which leads to large errors in the processing of experimental data.

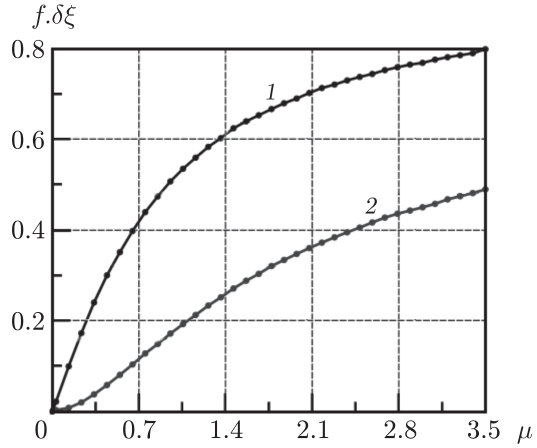

Fig. 3.

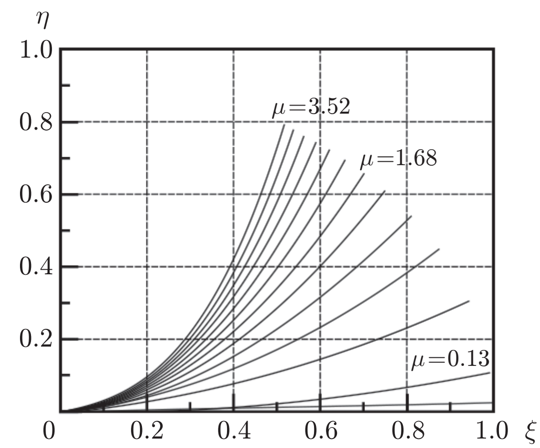

Fig. 4.

Fig. 3. Experimental dependences of the cantilever deflection f (*1*) and the dimensionless axial displacement $\delta\xi$ (*2*) on the dimensionless load μ.

Fig. 4. Experimental bend shapes at different values of the dimensionless loading $\mu = 0.13$–3.52.

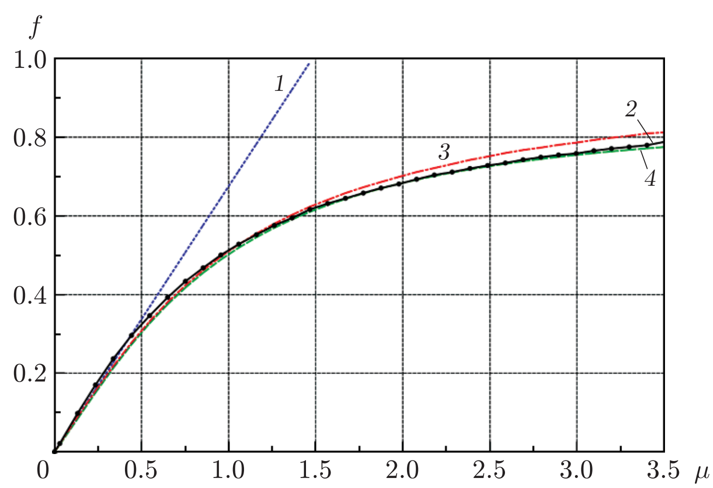

Fig. 5. Theoretical and experimental dependences of the deflection on the dimensionless load parameter μ: (*1*) linear solution (3), (4), (*2*) experimental data, (*3*) calculation by the modified linear expressions (5) and (6), (*4*) exact analytical solution (7).

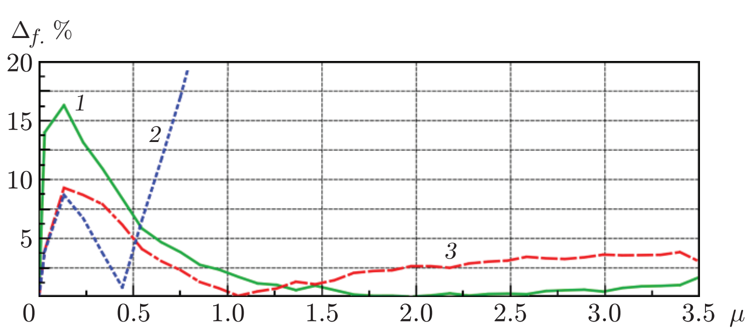

Fig. 6. Differences between the theoretical and experimental deflections Δ_f: (*1*) exact analytical solution (7), (*2*) linear solution (3), (4), (*3*) calculation by the modified linear expressions (5) and (6).

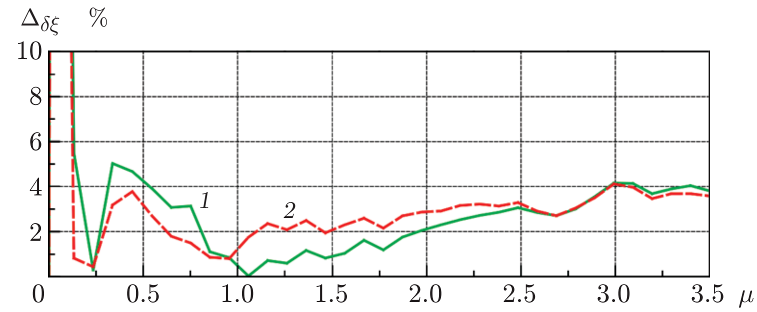

Fig. 7. Difference between the theoretical and experimental axial displacements $\Delta_{\delta\xi}$: (*1*) exact analytical solution (7), (*2*) calculation by the modified linear expressions (5) and (6).

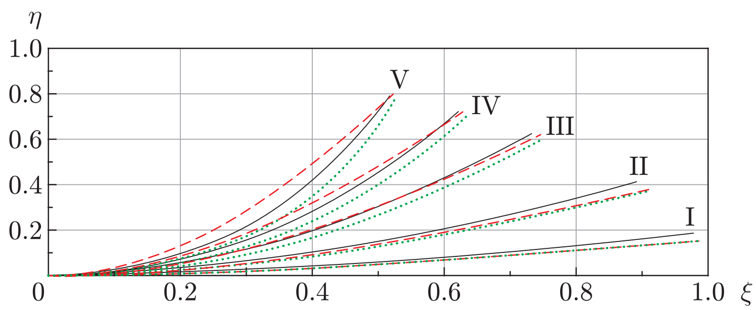

Fig. 8. Theoretical and experimentally obtained bending shapes: the solid lines denote the experimental data, the dashed lines show the calculation performed according to the modified linear expressions (3), (5), and (6), and the dotted lines refer to the exact analytical solution (7); I — $\mu = 0.24$, II — $\mu = 0.66$, III — $\mu = 1.49$, IV $\mu = 2.32$, V — $\mu = 3.56$.

Figure 8 shows bend shapes at various loads. It can be seen that the analytical solutions agree with the experimental data, but, as the load increases, the bend shape changes.

It should be noted that, unlike [10–12], this study presents the experimental data that characterize the geometrically nonlinear bending of thin elastic rods under the action of a transverse concentrated load at the free end over a larger range of dimensionless loads $\mu < 3.5$. The deflection and the axial displacement are determined, and the bend shapes are constructed via processing a large number of experimental points with a small step of a dimensionless load of 0.1μ.

CONCLUSIONS

This paper describes the experiment as a result of which data are obtained for the deflection and axial displacement of a thin elastic rod under the action of a transverse concentrated load at its free end with geometrically nonlinear bending. Various bend shapes are obtained as a result of processing a large number of experimental points.

The comparison of the resulting data with the analytical exact solution and the analytical approximate solution obtained using the modified linear equations shows that the analytical solutions are highly accurate (the difference does not exceed 4%). It should be noted that the approximate analytical solution obtained using the modified linear expressions is highly accurate and close to the accuracy of the analytical exact solution in elliptic functions. Thus, the approximate solution is applicable in a range of dimensionless loads $\mu < 3.5$, which significantly exceeds the validity range of the original linear solution ($\mu < 0.075$).

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