

TWO-LAYER STEADY CREEPING THERMOCAPILLARY FLOW IN A THREE-DIMENSIONAL CHANNEL

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Abstract: We study the problem of three-dimensional steady creeping flow of two immiscible liquids in a channel with solid parallel walls, one of which a given temperature distribution is maintained and the other is heat-insulated. Thermocapillary forces act on the flat interface. Temperature in the liquids depends quadratically on the horizontal coordinates, and the velocity field has a special form. The resulting conjugate problem for the Oberbeck–Boussinesq model is inverse and reduces to the system of ten integro-differential equations. The total energy condition on the interface is taken into account. The problem has up to two solutions, and if the heat fluxes are equal, it has one solution. Characteristic flow structures are constructed for each of the solutions. The influence of dimensionless physical and geometric parameters on the flows is investigated.

Keywords: interface, thermocapillarity, inverse problem.

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INTRODUCTION AND FORMULATION OF THE PROBLEM

Hiemenz [1] flows can be observed both on a macroscale (hydraulic fracturing in the oil industry) and a microscale (liquid biochips in medicine). Such flows, also known as flows near the critical point, are characterized by the presence of zones in which the pressure and temperature are higher than in the surrounding area. Studying the characteristics of these flows is required to evaluate the technological parameters and predict the dynamics and evolution of the liquid layer. The most effective way to study processes in liquids and evaluate their characteristics is to obtain exact solutions of constitutive equations. In the literature, one can find solutions of problems describing Hiemenz flows in various geometries: axisymmetric [2] and three-dimensional [3, 4] analogs of the Hiemenz solution, including flows in cylindrical geometry [5, 6]. A brief review of exact solutions close to the Hiemenz solution is given in [7]. In the present paper, three-dimensional flow of two viscous incompressible heat-conducting liquids with a Hiemenz type velocity field is considered within the framework of a model problem. At the inner interface, the energy balance condition taking into account the change in the internal energy of the interface is specified. As shown in [8], accounting for energy consumption for surface deformation can have a significant effect on the characteristics of liquid flows with low viscosities or under microconvection conditions. In this work, to evaluate this effect on emerging flows, we study a model linear problem in which the only nonlinear term is the term in the energy balance condition at the interface.

Consider three-dimensional plane steady flow of two viscous heat-conducting liquids in a layer where $|x| < \infty$, $|y| < \infty$, and $0 < z < l_2$. The liquids are in contact through a common interface $z = l_1 < l_2$ (l_j are constants). Liquid 1 occupies a layer $0 < z < l_1$, and liquid 2 occupies a layer $l_1 < z < l_2$. The planes $z = 0$ and $z = l_2$ are

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solid fixed walls, and the gravity is normal to the layers. As a mathematical model of liquid motion we use the Oberbeck–Boussinesq equations, whose solutions are sought in the following form (here and below $j = 1, 2$):

$$u_j(x, z) = (f_j(z) + h_j(z))x, \quad v_j(y, z) = (f_j(z) - h_j(z))y, \quad w_j(z) = -2 \int_{z_0}^z f_j(\xi) d\xi, \quad (1)$$

$$\frac{1}{\rho_j} p_j(x, y, z) = b_j(z)x^2 + d_j(z)y^2 + q_j(z), \quad T_j(x, y, z) = a_j(z)x^2 + c_j(z)y^2 + \theta_j(z).$$

Here $u_j(x, z)$, $v_j(y, z)$, and $w_j(z)$ are the projections of the velocity vectors onto the x , y , and z axes, respectively, $p_j(x, y, z)$ are the pressures, $T_j(x, y, z)$ are the absolute temperatures, and ρ_j are the constant densities. Studying flows (including unsteady ones) with such a velocity field for the Navier–Stokes system was proposed in [9, 10]. However, Lin [11] was apparently the first to advance the idea of searching for exact solutions of the Navier–Stokes equations with a linear dependence of the velocity components on two spatial variables.

Substituting solution (1) into the Oberbeck–Boussinesq system of equations, we write the resulting equations in the dimensionless form

$$\text{Mn} \left(F_j^2 + H_j^2 - 2F_j \int_{z_0/l_2}^{\xi} F_j(\zeta) d\zeta \right) + \text{Gr}_j \int_{z_0/l_2}^{\xi} (A_j(\zeta) + C_j(\zeta)) d\zeta = \text{Pr}_j l^2 \varepsilon_j F_j \xi \xi + N_{j1}; \quad (2)$$

$$2 \text{Mn} \left(F_j H_j - H_j \int_{z_0/l_2}^{\xi} F_j(\zeta) d\zeta \right) + \text{Gr}_j \int_{z_0/l_2}^{\xi} (A_j(\zeta) - C_j(\zeta)) d\zeta = \text{Pr}_j l^2 \varepsilon_j H_j \xi \xi + N_{j2}; \quad (3)$$

$$2 \text{Mn} \left(A_j(F_j + H_j) - A_j \int_{z_0/l_2}^{\xi} F_j(\zeta) d\zeta \right) = l^2 \varepsilon_j A_j \xi \xi; \quad (4)$$

$$2 \text{Mn} \left(C_j(F_j - H_j) - C_j \int_{z_0/l_2}^{\xi} F_j(\zeta) d\zeta \right) = l^2 \varepsilon_j C_j \xi \xi; \quad (5)$$

$$-2 \text{Mn} Q_j \int_{z_0/l_2}^{\xi} F_j(\zeta) d\zeta = l^2 \varepsilon_j Q_j \xi \xi + 2\varepsilon_j (A_j + C_j), \quad (6)$$

where Pr_j , Gr_j , and Mn are the Prandtl, Grashof, and Marangoni numbers, respectively. In the integral expressions, we have $z_0 = 0$ for $j = 1$ and $z_0 = l_1$ for $j = 2$, and hence $0 < \xi < l$ in the first layer and $l < \xi < 1$ in the second layer. In Eqs. (2) and (3), N_{j1} and N_{j2} are arbitrary constants.

On the solid boundaries, the no-slip conditions for velocities are imposed; as a result, we obtain the equalities

$$F_1(0) = H_1(0) = 0, \quad F_2(1) = H_2(1) = \int_l^1 F_2(\xi) d\xi = 0. \quad (7)$$

Temperature is specified on the lower solid wall, and the upper solid wall is heat-insulated:

$$A_1(0) = A_1^s, \quad C_1(0) = C_1^s, \quad Q_1(0) = Q_1^s, \quad A_{2\xi}(1) = C_{2\xi}(1) = Q_{2\xi}(1) = 0. \quad (8)$$

At the interface $z = l$, the velocities and temperatures are equal:

$$F_1(l) = F_2(l), \quad H_1(l) = H_2(l), \quad Q_1(l) = Q_2(l), \quad A_1(l) = A_2(l), \quad C_1(l) = C_2(l). \quad (9)$$

We assume that the surface tension depends linearly on temperature: $\sigma(T) = \sigma_0 - \varkappa(T - T_0)$ [σ_0 , \varkappa , and T_0 are given positive constants, and $T(x, y, l_1)$ is the temperature at this boundary]. Then, the condition for tangential stresses reduces to two relations:

$$F_{2\xi}(l) - \mu F_{1\xi}(l) = -\text{Mn} (A_1(l) + C_1(l)), \quad H_{2\xi}(l) - \mu H_{1\xi}(l) = -\text{Mn} (A_1(l) - C_1(l)). \quad (10)$$

The kinematic condition for the fixed non-deformable interface is equivalent to the integral equality

$$\int_0^l F_1(\xi) d\xi = 0. \quad (11)$$

The complete energy condition taking into account (1) reduces to the relations

$$\begin{aligned} A_{2\xi}(l) - kA_{1\xi}(l) &= 2EA_1(l)F_1(l), & C_{2\xi}(l) - kC_{1\xi}(l) &= 2EC_1(l)F_1(l), \\ Q_{2\xi}(l) - kQ_{1\xi}(l) &= 2EQ_1(l)F_1(l). \end{aligned} \quad (12)$$

In Eqs. (2)–(6) and boundary conditions (7)–(12), the following dimensionless variables and parameters are introduced:

$$\begin{aligned} \xi &= \frac{z}{l_2}, & \chi &= \frac{\chi_1}{\chi_2}, & \mu &= \frac{\mu_1}{\mu_2}, & k &= \frac{k_1}{k_2}, & l &= \frac{l_1}{l_2} < 1, & \varepsilon_j &= \frac{\chi_j}{\chi_1}, & \text{Pr}_j &= \frac{\nu_j}{\chi_j}, \\ N_{jk} &= \frac{l_1^4 n_{jk}}{\chi_1^2 \text{Mn}}, & F_j(\xi) &= \frac{l_1^2}{\chi_1 \text{Mn}} f_j(z), & H_j(\xi) &= \frac{l_1^2}{\chi_1 \text{Mn}} h_j(z), & A_j(\xi) &= \frac{a_j(z)}{a^* \text{Mn}}, \\ C_j(\xi) &= \frac{c_j(z)}{a^* \text{Mn}}, & Q_j(\xi) &= \frac{\theta_j(z)}{a^* l_1^2 \text{Mn}}, & \text{Gr}_j &= \frac{a^* l_2 l_1^4 g \beta_j}{\chi_1^2}, & \text{Mn} &= \frac{\varkappa a^* l_1^2 l_2}{\mu_2 \chi_1}, & E &= \frac{\varkappa^2 a^* l_2^2}{\mu_2 k_2}. \end{aligned}$$

Here ν_j , μ_j , χ_j , k_j , and β_j are the constant kinematic and dynamic viscosities, thermal diffusivities, heat conductivities, and the volume expansion coefficient, respectively, N_{jk} are the dimensionless pressure gradients along the horizontal coordinates ($j = 1, 2$ and $k = 1, 2$), and E is the energy parameter that characterizes the energy expended for deformation of the interface by thermocapillary forces. Under the assumption that $a^* = \max\{|a_1(0)|, |c_1(0)|\} > 0$ and the characteristic temperature at the interface is $\theta^* = a^* l_1^2$, the values Gr_j , Mn , and E are positive numbers. Note that a local change in the internal energy of the interface due to absorption (release) of heat generates inhomogeneity of the temperature field (thermocapillary effect). The influence of the heat of formation of the interface on the occurrence of temperature gradients and additional heat flows in the problem of bubble motion was found in [8].

Remark 1. The formulated initial-boundary value problem (2)–(12) is inverse since the constants N_{j1} and N_{j2} should be found along with its solution. To completely define this problem, we should specify two more conditions:

$$\int_0^l H_1(\xi) d\xi = 0, \quad \int_l^1 H_2(\xi) d\xi = 0, \quad (13)$$

which, along with the integral equalities (7) and (11), imply that the flows in the layers are closed [12].

Remark 2. Using the well-known dimensional functions a_j and c_j , the functions b_j and d_j are given by the quadratures

$$b_j(z) = g\beta_j \int_{z_0}^z a_j(\xi) d\xi - n_{j1}, \quad d_j(z) = g\beta_j \int_{z_0}^z c_j(\xi) d\xi - n_{j2},$$

and the pressures in the liquids are defined as

$$\begin{aligned} \frac{1}{\rho_j} p_j &= \left(g\beta_j \int_{z_0}^z a_j(\xi) d\xi - n_{j1} \right) x^2 + \left(g\beta_j \int_{z_0}^z c_j(\xi) d\xi - n_{j2} \right) y^2 - 2\nu_j f_j - gz \\ &+ g\beta_j \int_{z_0}^z \theta_j(\xi) d\xi - 2 \left(\int_{z_0}^z f(\xi, t) d\xi \right)^2 + q_{j0} \end{aligned}$$

(q_{j0} are arbitrary constants).

SOLUTION OF THE MODEL PROBLEM FOR CREEPING FLOWS

Flows at small Marangoni numbers are usually called creeping. The smallness of the parameter can be achieved due to the physical parameters of the liquid as well as due to the channel thickness [13, 14]. Let $Mn \rightarrow 0$, then, Eqs. (2)–(6) are linear, and the right sides of boundary conditions (10) are equal to zero. However, relations (12) remain nonlinear. Performing simple but rather cumbersome transformations, we obtain

$$\begin{aligned} A_1^0(\xi) &= \alpha_1 \xi + A_1^s, & A_2^0(\xi) &= \alpha_2 \equiv \alpha_1 l + A_1^s, \\ C_1^0(\xi) &= \gamma_1 \xi + C_1^s, & C_2^0(\xi) &= \gamma_2 \equiv \gamma_1 l + C_1^s, \\ Q_1^0(\xi) &= -\frac{2}{l^2} \left(\frac{\alpha_1 + \gamma_1}{6} \xi^3 + \frac{A_1^s + C_1^s}{2} \xi^2 \right) + R_1 \xi + Q_1^s, \\ Q_2^0(\xi) &= -(\alpha_2 + \gamma_2) l^{-2} \xi^2 + R_2 \xi + R_3; \end{aligned} \tag{14}$$

$$\begin{aligned} F_1^0(\xi) &= \frac{1}{Pr_1 l^2} \left[Gr_1 \left(\frac{\alpha_1 + \gamma_1}{24} \xi^4 + \frac{A_1^s + C_1^s}{6} \xi^3 \right) - \frac{N_{11} \xi^2}{2} \right] + D_1 \xi, \\ F_2^0(\xi) &= \frac{\chi}{Pr_2 l^2} \left[Gr_2 (\alpha_2 + \gamma_2) \left(\frac{\xi^3 - 1}{6} - \frac{l}{2} (\xi^2 - 1) \right) - \frac{N_{21}}{2} (\xi^2 - 1) \right] + D_2 (\xi - 1), \\ H_1^0(\xi) &= \frac{1}{Pr_1 l^2} \left[Gr_1 \left(\frac{\alpha_1 - \gamma_1}{24} \xi^4 + \frac{A_1^s - C_1^s}{6} \xi^3 \right) - \frac{N_{12} \xi^2}{2} \right] + D_3 \xi, \\ H_2^0(\xi) &= \frac{\chi}{Pr_2 l^2} \left[Gr_2 (\alpha_2 - \gamma_2) \left(\frac{\xi^3 - 1}{6} - \frac{l}{2} (\xi^2 - 1) \right) - \frac{N_{22}}{2} (\xi^2 - 1) \right] + D_4 (\xi - 1). \end{aligned} \tag{15}$$

The constants D_1, \dots, D_4 are found from the integral equalities (7), (11), and (13). In strength (14), we have $\alpha_2 + \gamma_2 = (\alpha_1 + \gamma_1)l + A_1^s + C_1^s$ and $\alpha_2 - \gamma_2 = (\alpha_1 - \gamma_1)l + A_1^s - C_1^s$. To determine the unknowns $\alpha_1, \gamma_1, R_1, R_2, R_3, N_{11}, N_{21}, N_{12}$, and N_{22} , we use the last condition of (8), the first three conditions of Eqs. (9), and conditions (10) and (12). The form of the constants $\alpha_1, \gamma_1, R_1, R_2, R_3, N_{11}, N_{21}, N_{12}$, and N_{22} is not given due to their cumbersome nature. Note that the unknown parameter α_1 satisfies the quadratic equation. Therefore, the creeping flow problem can have two solutions. Obviously, for $E = 0$, there is a unique solution. Analysis of the obtained solutions (14) and (15) leads to the following conclusions: for $A_1^s = C_1^s = 0$, the only solution of the problem is the solution corresponding to the state of rest; $N_{12} = N_{22} = 0, N_{11} \neq 0$, and $N_{21} \neq 0$ for $A_1^s = C_1^s$; $N_{12} = -N_{11}$ and $N_{22} = -N_{21}$ for $A_1^s = 0$ and $C_1^s \neq 0$; $N_{12} = N_{11}$ and $N_{22} = N_{21}$ for $A_1^s \neq 0$ and $C_1^s = 0$.

The calculations were carried out for the transformer oil–formic acid system with six dimensionless parameters $\mu = 11.4, \chi = 0.71, k = 0.41, \beta = \beta_1 \beta_2^{-1} = 1.46, Pr_1 = 308.2$, and $Pr_2 = 14.2$. Figures 1 and 2 show the results of calculations for $E = 0$, i.e., for the case where there is no influence of a change in the interfacial energy and there is a unique solution of the creeping flow problem. Figure 1 shows the temperature and velocity fields in layers of different thicknesses in the plane (y, ξ) (y, ξ , and T are dimensionless variables). For $l \leq 0.4$, one vortex occurs in each layer, and in the second layer, there is reverse flow (the liquid moves opposite to the z direction). At $l = 0.5$, the flow in the first layer becomes two-vortex, and with an increase in l , it becomes fully reverse. In the second layer, the flow structure also changes: at $l > 0.8$, one vortex occurs (the liquid moves in the positive z direction). With a change in the thickness of the layers, the temperature field also changes: at $l = 0.4$, a thermocline is formed in the lower layer, and at $l > 0.8$, the temperature is equally distributed in the layers.

Figure 2 shows the velocity profiles $W(\xi)$ in the layers for different values of C_1^s (the situation for A_1^s is similar). Here and below, the function $W(\xi)$ coincides with the functions $W_j(\xi)$ ($j = 1, 2$) in their domains of definition. Note that for the case $A_1^s = -C_1^s$, the only solution of the problem is the solution corresponding to the state of rest. We also studied the influence of Grashof numbers (Gr_j) on the emerging flow: as Gr_j increases, the modulus of the dimensionless velocity $W(\xi)$ increases.

For the case $E \neq 0$, two solutions are found. Figure 3 shows the velocity profiles $W^1(\xi)$ and $W^2(\xi)$ corresponding to each solution for $E = 0.5$ (superscript denotes the solution number). The velocity profile $W^1(\xi)$ is similar to the velocity profile $W(\xi)$ for $E = 0$, $\left| \max_{\xi \in [0,1]} |W^1(\xi)| - \max_{\xi \in [0,1]} |W(\xi)| \right| \approx 10^{-5}$, and the influence of the parameter E on the emerging flow corresponding to the first solution is negligible. The velocity profiles $W^2(\xi)$ in the layers for different values of the dimensionless parameter E are given in Fig. 4. It can be seen that as E increases, the dimensionless velocity $W^2(\xi)$ decreases.

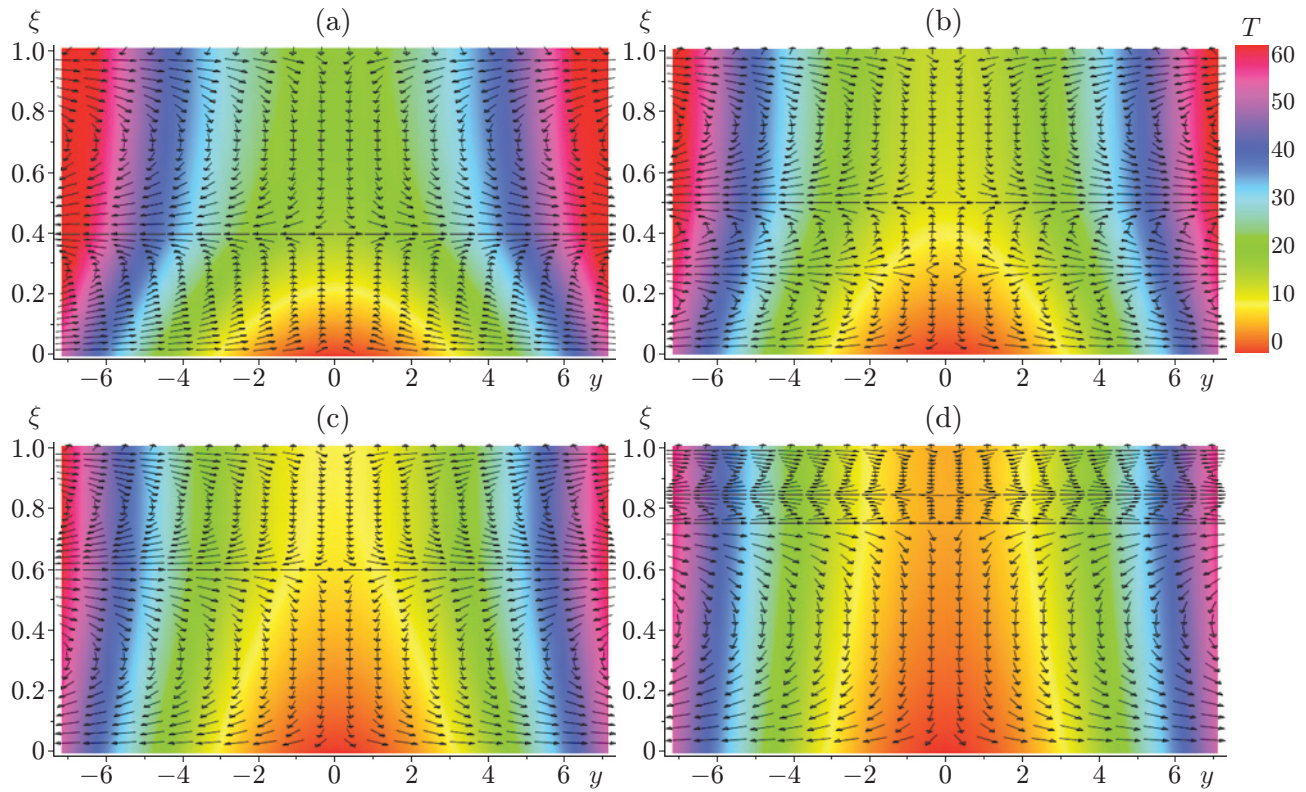


Fig. 1. Temperature and velocity fields in the layers at different thicknesses: $l = 0.4$ (a), 0.5 (b), 0.6 (c), and 0.75 (d).

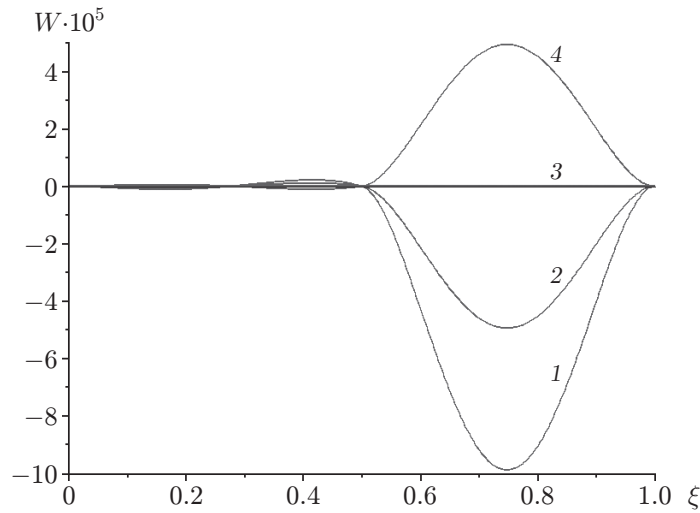


Fig. 2. Velocity profiles $W(\xi)$ in the layers for $A_1^s = 0.1$ and different values of C_1^s : $C_1^s = 0$ (1), -0.05 (2), -0.1 (3), and -0.15 (4).

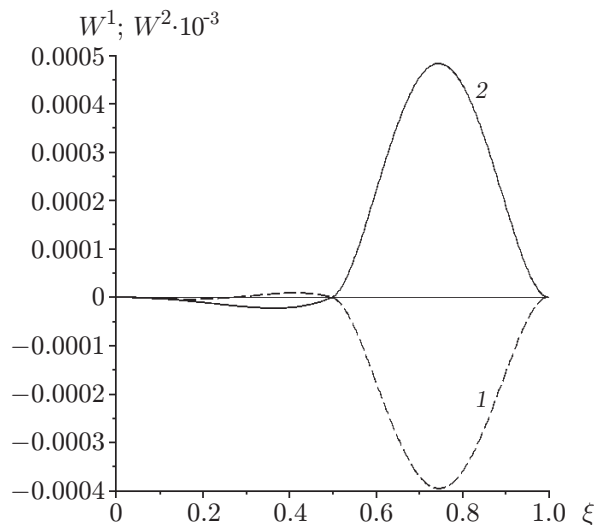


Fig. 3.

Fig. 3. Velocity profiles $W^1(\xi)$ (curve 1) and $W^2(\xi)$ (curve 2) in the layers for $E = 0.5$.

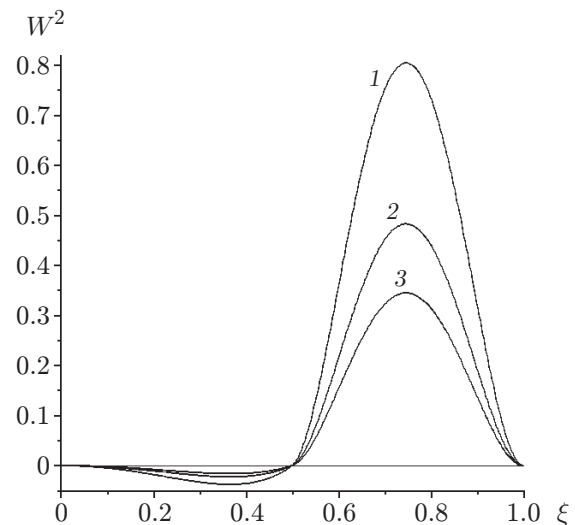


Fig. 4.

Fig. 4. Velocity profiles $W^2(\xi)$ in the layers for $E = 0.3$ (1), 0.5 (2), and 0.7 (3).

CONCLUSIONS

The problem of three-dimensional two-layer motion with a special velocity field was solved by reducing it to a conjugate problem for a system of one-dimensional integro-differential equations. The resulting problem is inverse with respect to pressure gradients along horizontal coordinates. In the case of steady flow at low Marangoni numbers, the solution is found in analytical form. It is shown that, depending on the values of physical parameters, there can be up to two steady modes. For the transformer oil–formic acid system, the influence of dimensionless physical and geometric parameters on emerging flows was investigated.

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REFERENCES

1. K. Hiemenz, “Die Grenzschicht an Einem in den Gleichförmigen Flüssigkeitsstrom Eingetauchten Geraden Kreiszyylinder,” *Dinglers Politech. J.* **326**, 321–440 (1911).
2. F. Howann, “Der Einfluss Grosser Zähigkeit bei der Stromung um den Zylinder und um Die Kugel,” *Z. Angew. Math. Mech.* **16**, 153–164 (1936).
3. L. Howarth, “The Boundary Layer in Three-Dimensional Flow. 2. The Flow near a Stagnation Point,” *Philos. Mag.: Ser. 7* **42** (335), 1433–1440 (1951).
4. A. Davey, “Boundary-Layer Flow at a Saddle Point of Attachment,” *J. Fluid Mech.* **10** (4), 593–610 (1961).
5. C. Y. Wang, “Axisymmetric Stagnation Flow on a Cylinder,” *Quart. Appl. Math.* **32** (2), 207–213 (1974).
6. R. S. R. Gorla, “Unsteady Laminar Axisymmetric Stagnation Flow over a Circular Cylinder,” *Develop. Mech.* **9**, 286–288 (1977).
7. V. B. Bekezhanova, V. K. Andreev, and I. A. Shefer, “Influence of Heat Defect on the Characteristics of a Two-Layer Flow with the Hiemenz-Type Velocity,” *Interfacial Phenom. Heat Transfer* **7** (4), 345–364 (2019).
8. F. E. Torres and E. Helborzheimer, “Temperature Gradients and Drag Effects Produced by Convection of Interfacial Internal Energy around Bubbles,” *Phys. Fluids A* **5** (3), 537–549 (1993).

9. V. K. Andreev, Yu. A. Gaponenko, O. N. Goncharova, and V. V. Pukhnachev, *Mathematical Models of Convection* (De Gruyter, Berlin–Boston, 2020).
10. S. N. Aristov, D. V. Knyazev, and A. D. Polyanin, “Exact Solutions of the Navier–Stokes Equations with a Linear Dependence of the Velocity Components on Two Spatial Variables,” *Teor. Osn. Khim. Tekhnol.* **43** (5), 547–566 (2009).
11. C. C. Lin, “Note on a Class of Exact Solutions in Magneto Hydrodynamics,” *Arch. Rat. Mech. Anal.* **1**, 391–395 (1958).
12. V. K. Andreev, V. E. Zakhvataev, and E. A. Ryabitskii, *Thermocapillary Instability* (Nauka, Novosibirsk, 2000) [in Russian].
13. L. K. Antanovskii and B. K. Kopbosynov, “Nonstationary Thermocapillary Drift of a Drop of Viscous Liquid,” *Prikl. Mekh. Tekh. Fiz.* **27** (2), 59–64 (1986) [*J. Appl. Mech. Tech. Phys.* **27** (2), 208–213 (1986); <https://doi.org/10.1007/BF00914730>].
14. O. V. Admaev and V. K. Andreev, “Nonstationary Motion of a Bubble under the Action of Thermocapillary Forces,” in *Mathematical Modeling in Mechanics* (Comp. Center, Sib. Branch, Russian Acad. of Sci., Krasnoyarsk, 1997), pp. 4–9. Deposited at VINITI February 12, 1997, No. 446-B97.