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LOCAL SOLVABILITY OF FREE BOUNDARY PROBLEMS IN IDEAL COMPRESSIBLE MAGNETOHYDRODYNAMICS WITH AND WITHOUT SURFACE TENSION

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Abstract: Results are presented on local-in-time solvability of free boundary problems for a system of ideal compressible magnetohydrodynamics. A free plasma-vacuum interface problem and a free boundary problem with boundary conditions on a contact discontinuity are considered. An approach is given for proving the local existence and uniqueness of smooth solutions of these problems with and without surface tension.

Keywords: magnetohydrodynamics, free boundary problem, surface tension, local theorem of existence and uniqueness.

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1. FORMULATION OF A FREE BOUNDARY PROBLEM

Equations of magnetohydrodynamics (MHD) are considered, which describe a flow of an inviscid compressible ideally conducting fluid (in particular, plasma) in a magnetic field [1]:

$$\partial_t \rho + \operatorname{div}(\rho \boldsymbol{v}) = 0; \tag{1}$$

$$\partial_t (\rho \boldsymbol{v}) + \operatorname{div}(\rho \boldsymbol{v} \otimes \boldsymbol{v} - \boldsymbol{H} \otimes \boldsymbol{H}) + \nabla q = 0; \tag{2}$$

$$\partial_t \boldsymbol{H} - \nabla \times (\boldsymbol{v} \times \boldsymbol{H}) = 0; \tag{3}$$

$$\partial_t (\rho e + |\boldsymbol{H}|^2/2) + \operatorname{div}((\rho e + p)\boldsymbol{v} + \boldsymbol{H} \times (\boldsymbol{v} \times \boldsymbol{H})) = 0. \tag{4}$$

Here $\partial_t = \partial/\partial t$; $\nabla = (\partial_1, \partial_2, \partial_3)$; $\partial_i = \partial/\partial x_i$; t is the time; $\boldsymbol{x} = (x_1, x_2, x_3)$ are the spatial coordinates; ρ is the density; $\boldsymbol{v} = (v_1, v_2, v_3)$ is the velocity; $\boldsymbol{H} = (H_1, H_2, H_3)$ is the magnetic field strength; $q = p + |\boldsymbol{H}|^2/2$ is the total pressure; p is the pressure; S is the entropy; $E = E(\rho, S)$ is the internal energy; $e = E + |\boldsymbol{v}|^2/2$. With account for the thermodynamic identity

$$\vartheta\, dS = dE - (p/\rho^2)\, d\rho,$$

where ϑ is the absolute temperature, system (1)–(4) is a closed system of eight equations of conservation laws. The vector of unknowns can be, for example, $\boldsymbol{U} = (q, \boldsymbol{v}, \boldsymbol{H}, S)$. In this case, the expression $\rho = \rho(p, S)$ is considered as an equation of the state of the medium. Moreover, system (1)–(4) is complemented by the divergence constraint

$$\operatorname{div} \boldsymbol{H} = 0 \tag{5}$$

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of initial data $\boldsymbol{U}|_{t=0} = \boldsymbol{U}_0(\boldsymbol{x})$ (if the condition (5) is fulfilled at $t = 0$, then, as suggested by Eq. (3), it is also valid at $t > 0$).

Equation (5) can be used to write system (1)–(4) in nonconservative form

$$\frac{\rho_p}{\rho}\Big(\frac{dq}{dt} - \boldsymbol{H}\cdot\frac{d\boldsymbol{H}}{dt}\Big) + \operatorname{div}\boldsymbol{v} = 0, \qquad \rho\frac{d\boldsymbol{v}}{dt} - (\boldsymbol{H}\cdot\nabla)\boldsymbol{H} + \nabla q = 0,$$

$$\frac{d\boldsymbol{H}}{dt} - (\boldsymbol{H}\cdot\nabla)\boldsymbol{v} - \frac{\rho_p}{\rho}\boldsymbol{H}\Big(\frac{dq}{dt} - \boldsymbol{H}\cdot\frac{d\boldsymbol{H}}{dt}\Big) = 0, \qquad \frac{dS}{dt} = 0.$$

These equations are a symmetric system of the form

$$A_0(\boldsymbol{U})\,\partial_t\boldsymbol{U} + \sum_{j=1}^{3} A_j(\boldsymbol{U})\,\partial_j\boldsymbol{U} = 0, \tag{6}$$

where $\rho_p = \partial\rho/\partial p$; $d/dt = \partial_t + (\boldsymbol{v}\cdot\nabla)$; A_α ($\alpha = \overline{0.3}$) denotes the symmetric matrices. The hyperbolicity condition $A_0 > 0$ of the symmetric system (6) is satisfied under the following constraints having physical meaning:

$$\rho(p,S) > 0, \qquad \rho_p(p,S) > 0. \tag{7}$$

The second of the above-mentioned constraints is a requirement that the square of the sound velocity is positive: $a^2 = 1/\rho_p > 0$.

First, the classical formulation [2] of the problem with a free plasma — vacuum boundary is considered. Let $\Omega^+(t)$ and $\Omega^-(t)$ be the regions occupied by plasma and vacuum, respectively. The plasma flow in $\Omega^+(t)$ is described by the system of MHD equations (1)–(4), and the magnetic field strength $\boldsymbol{H}^-(t,\boldsymbol{x})$ in vacuum satisfies the system of equations of pre-Maxwellian dynamics

$$\operatorname{div}\boldsymbol{H}^- = 0, \qquad \nabla\times\boldsymbol{H}^- = 0 \qquad \text{in } \Omega^-(t). \tag{8}$$

In this case, the free boundary $\Gamma(t) = \{F(t,\boldsymbol{x}) = 0\}$ moves at a velocity equal to the velocity of plasma particles on it:

$$\frac{dF}{dt} = 0 \quad \text{on } \Gamma(t). \tag{9}$$

The rest of the boundary conditions imply that the total pressure has a zero jump, and the magnetic fields with strengths $\boldsymbol{H}$ and $\boldsymbol{H}^-$ at the boundary are parallel to it:

$$[q] = 0; \tag{10}$$

$$\boldsymbol{H}\cdot\boldsymbol{N} = 0; \tag{11}$$

$$\boldsymbol{H}^-\cdot\boldsymbol{N} = 0 \quad \text{on } \Gamma(t) \tag{12}$$

($\boldsymbol{N} = \nabla F$ is the normal to Γ; $[q] = q|_\Gamma - |\boldsymbol{H}^-|_\Gamma|^2/2$). In this case, one can show that Eq. (11) is bounded by the initial data of the problem (see [3]).

The problem with the free plasma — vacuum boundary is used to simulate the magnetic confinement of the plasma (see, e.g., [4]). In astrophysics, this problem can simulate the motion of a star (for example, a solar corona), in the case where it is necessary to account for the influence of magnetic fields.

Let the vacuum region $\Omega^-(t)$ have an outer motionless boundary Γ_- with a standard boundary condition on it:

$$\boldsymbol{H}^-\cdot\boldsymbol{\nu} = 0 \quad \text{on } \Gamma_- \tag{13}$$

($\boldsymbol{\nu}$ is the normal to Γ_-). If the region $\Omega^-(t)$ is simply connected, then the elliptic problem (8), (12), (13) has a unique solution $\boldsymbol{H}^- = 0$. This paper touches upon the local-in-time well-posedness of the problem, rather than the stability of a flow with a free boundary, so the geometry of the regions is not essential and we can assume that $\Gamma(t) = \{x_1 = \varphi(t,\boldsymbol{x}')\}$, where $\boldsymbol{x}' = (x_2, x_3)$. Then, for a simply connected vacuum region, the free boundary problem under consideration is reduced to a problem for the system of MHD equations (1)–(4) in the region $\Omega(t) = \Omega^+(t) = \{x_1 > \varphi(t,\boldsymbol{x}')\}$ with boundary conditions

$$q = 0, \qquad \partial_t\varphi = \boldsymbol{v}\cdot\boldsymbol{N} \quad \text{on } \Gamma(t). \tag{14}$$

As noted above, the condition (11) is a constraint on the initial data, so it is not included in Eq. (14). Moreover, the first condition in Eq. (14) implies that $p|_\Gamma = 0$. In the case of a gas (for example, polytropic gas), this equality means that $\rho|_\Gamma = 0$. In other words, if it is required that the hyperbolicity conditions (7) should be satisfied in the entire region up to the boundary, then it is assumed that, as in [5, 6], the case of a compressible fluid is considered, for which $\rho|_\Gamma > 0$.

However, even in the case of the simply connected region $\Omega^-(t)$, the magnetic field in vacuum is not identically zero if the plasma — vacuum system is not isolated from the environment because the surface current J is set on boundary Γ_- (see [4]). In this case, the boundary condition (13) in the above-given elliptic problem for $\boldsymbol{H}^-$ is replaced by the condition

$$(\boldsymbol{\nu} \times \boldsymbol{H}^-)\Big|_{\Gamma_-} = J \quad \text{on } \Gamma_-. \tag{15}$$

For $\Gamma(t) = \{x_1 = \varphi(t, \boldsymbol{x}')\}$, the boundary conditions (9), (10), and (12) take the form

$$[q] = 0, \quad \partial_t \varphi = \boldsymbol{v} \cdot \boldsymbol{N}, \quad \boldsymbol{H}^- \cdot \boldsymbol{N} = 0 \quad \text{on } \Gamma(t), \tag{16}$$

where $\boldsymbol{N} = (1, -\partial_2 \varphi, -\partial_3 \varphi)$.

If the surface tension is taken into account, the boundary conditions (14) and (16) are respectively replaced by the conditions

$$q = \mathfrak{s}\mathcal{H}(\varphi), \quad \partial_t \varphi = \boldsymbol{v} \cdot \boldsymbol{N} \quad \text{on } \Gamma(t); \tag{17}$$

$$[q] = \mathfrak{s}\mathcal{H}(\varphi), \quad \partial_t \varphi = \boldsymbol{v} \cdot \boldsymbol{N}, \quad \boldsymbol{H}^- \cdot \boldsymbol{N} = 0 \quad \text{on } \Gamma(t), \tag{18}$$

where

$$\mathcal{H}(\varphi) = \nabla' \cdot \Big(\frac{\nabla' \varphi}{\sqrt{1 + |\nabla' \varphi|^2}} \Big),$$

$\nabla' = (\partial_2, \partial_3)$; $\mathfrak{s} > 0$ is the surface tension coefficient. It should be noted that it is necessary to account for the effect of surface tension in simulating MHD flows of liquid metals (see, e.g., [7, 8] and references therein). At the same time, even when it comes to MHD simulation of large-scale phenomena, such as processes in astrophysical plasma where the effect of surface tension (and diffusion) may be neglected, the surface tension should be taken into account as a stabilizing parameter in the numerical simulation of the Rayleigh — Taylor magnetic instability [9, 10].

We consider flows with a strong discontinuity surface $\Gamma(t) = \{x_1 = \varphi(t, \boldsymbol{x}')\}$, on both sides of which (in the regions $\Omega^\pm(t) = \{\pm(x_1 - \varphi(t, \boldsymbol{x}')) > 0\}$), the vectors $U^\pm$ are smooth solutions of system (1)–(4). In an ideal compressible MHD fluid, there are two types of strong discontinuities, the mass flow through which is equal to zero ($\partial_t \varphi = (\boldsymbol{v}^\pm \cdot \boldsymbol{N})|_\Gamma$): tangential and contact discontinuities [1, 11]. From a mathematical point of view, these discontinuities are free characteristic surfaces for the hyperbolic system (6).

In [12], a sufficient condition was found for the well-posedness of the tangential discontinuity problem, whose local-in-time solvability in Sobolev spaces was proven in [13] with satisfaction of that condition at the initial time. This paper deals with contact discontinuities. Unlike the tangential discontinuity at each point of the contact discontinuity surface $\Gamma(t)$, the magnetic field is not parallel to it: $(\boldsymbol{H}^\pm \cdot \boldsymbol{N})|_\Gamma \neq 0$. From the general relations on the strong discontinuity studied in MHD [1, 11], the following boundary conditions on the contact discontinuity are derived:

$$[p] = 0, \quad [\boldsymbol{v}] = 0, \quad [\boldsymbol{H}] = 0, \quad \partial_t \varphi = \boldsymbol{v}^+ \cdot \boldsymbol{N} \quad \text{on } \Gamma(t). \tag{19}$$

Here $[\,\cdot\,]$ denotes the variation in the values on the discontinuity. In other words, at a contact MHD discontinuity, the pressure, the velocity, and the magnetic field are continuous. At the same time, density, entropy, and temperature can have an arbitrary jump: $[\rho] \neq 0$, $[S] \neq 0$, and $[\vartheta] \neq 0$. The boundary conditions for contact discontinuities are typical for simulating the processes occurring in space plasma. The contact discontinuities are observed, for example, behind the shock waves that limit supernova remnants.

If the effect of surface tension is taken into account, the boundary conditions (19) are replaced by the conditions [14–16]

$$[p] = \mathfrak{s}\mathcal{H}(\varphi), \quad [\boldsymbol{v}] = 0, \quad [\boldsymbol{H}] = 0, \quad \partial_t \varphi = \boldsymbol{v}^+ \cdot \boldsymbol{N} \quad \text{on } \Gamma(t). \tag{20}$$

At the same time, it is actually assumed that the regions $\Omega^+(t)$ and $\Omega^-(t)$ are filled by two nonviscous compressible ideally conductive liquids.

2. LOCAL SOLVABILITY OF PROBLEMS WITH NO ACCOUNT FOR SURFACE TENSION

Free plasma — vacuum boundary problems were actively studied in the 1950s and 1970s. However, the main theoretical studies were devoted to searching for criteria for the stability of equilibrium states [2]. The mathematical study of the local-in-time solvability of these problems was begun in [3], in which two alternative conditions for the well-posedness of the free boundary problem (1)—(4), (8), (15), (16) were proposed: the condition of noncollinearity of magnetic fields at each point of the free boundary

$$\left|\boldsymbol{H}\times\boldsymbol{H}^{-}\right|_{\Gamma}\geq\delta_0>0 \tag{21}$$

and the generalized Rayleigh — Taylor condition for the jump of the normal derivative of the total pressure

$$\left[\frac{\partial q}{\partial\boldsymbol{\mathcal{N}}}\right]\leq-\delta_1<0, \tag{22}$$

where $\boldsymbol{\mathcal{N}}=-\boldsymbol{N}$ is the outer normal to Γ; δ_0 and δ_1 are the fixed constants. Note that the Rayleigh — Taylor condition in classical hydrodynamics (for $\boldsymbol{H}\equiv 0$ and $\boldsymbol{H}^-\equiv 0$) is the only condition for well-posedness of the corresponding free boundary problem (see [5, 6, 17–21] and references therein). Condition (21) means that the magnetic field can perform a stabilizing function.

In [22], based on an example of an ill-posed linearized problem with frozen coefficients, it is shown similar to an example of Hadamard that the simultaneous violation of both conditions (21) and (22) causes the Rayleigh — Taylor instability. Note that the proof of the ill-posedness of the original nonlinear problem is difficult (see [20, 21, 23], where problems in the absence of a magnetic field are considered). In [22], a natural conjecture is formulated according to which problem (1)–(4), (8), (15), (16) is locally well-posed if and only if condition (21) or condition (22) at each point of the initial boundary is satisfied. This hypothesis has not yet been proven. Moreover, even if the well-posedness of the problem with fulfillment of the Rayleigh — Taylor condition (22) at each point of the initial boundary is proven, there are certain difficulties that have not yet been resolved (see [22]). At the same time, the local solvability of the problem is shown in [24] under the noncollinearity condition (21) on an initial free boundary. Note that the noncollinearity condition was taken in [3] as a requirement of solvability of the boundary conditions (11) and (12) (for $F=x_1-\varphi$) for $\nabla'\varphi$. As for the simpler free boundary problem (1)–(4), (14) with a zero magnetic field in vacuum, its local solvability is proven in [25] under the Rayleigh — Taylor condition (22) for $\boldsymbol{H}^-\equiv 0$.

A brief scheme for proving the local-in-time existence and uniqueness of smooth solutions to the free boundary problems [24, 25] (the same scheme was used in [6] in the case of a free boundary problem for the Euler equations of a compressible fluid) is given below.

1. "Straightening" of the free boundary, i.e., reduction of the original problem to an initial boundary-value problem in a fixed region (in this case, the problem continues to be considered in Euler coordinates).

2. Derivation of the basic a priori estimate for the solutions of the linearized problem (it usually implies the uniqueness of the solution to the original nonlinear problem) under certain and often previously unknown well-posedness conditions imposed on an unperturbed flow.

3. Proof of the existence of solutions to the linearized problem under the revealed well-posedness conditions.

4. Derivation of so-called tame estimates of solutions to the linearized problem, which are necessary to prove the convergence of Nash — Moser iterations [26].

5. Proof of the existence of solutions to the nonlinear problem using the Nash — Moser iterations.

The need to use the Nash — Moser method is due to the fact that, at least within the framework of the approach used, it is impossible to derive a priori estimates for the solutions of the linearized problem for the free boundary problems under consideration without losing derivatives of the source terms and coefficients. Therefore, it is impossible to apply the contraction mapping principle, i.e., to prove the existence of solutions to a nonlinear problem using Picard iterations.

The free boundary can be "straightened" in the simplest way, i.e., via the change of variables $\tilde{x}_1=x_1-\varphi(t,\boldsymbol{x}')$, which reduces the problem under consideration to an initial boundary-value problem in a half-space $\mathbb{R}^3_+=\{\tilde{x}_1>0,\ \boldsymbol{x}'\in\mathbb{R}^2\}$. However, more complex substitutions were used in [24, 25]. For example, in the above-given replacement in [25], φ is preceded by a factor representing the infinitely smooth cutoff function $\chi(x_1)$. This makes it possible to avoid making an assumption about the finiteness of the initial data in the direction normal to the free boundary. It is assumed below that $\chi \equiv 1$. Then, following the change of variables and dropping tildes, system (6) is reduced to the equations

$$\mathbb{L}(\boldsymbol{U}, \varphi) = 0 \quad \text{in } \Omega_T, \tag{23}$$

where

$$\Omega_T = [0, T] \times \mathbb{R}^3_+, \qquad \mathbb{L}(\boldsymbol{U}, \varphi) = L(\boldsymbol{U}, \varphi)U,$$

$$L(\boldsymbol{U}, \varphi) := A_0(\boldsymbol{U})\,\partial_t + \tilde{A}_1(\boldsymbol{U}, \varphi)\,\partial_1 + A_2(\boldsymbol{U})\,\partial_2 + A_3(\boldsymbol{U})\,\partial_3,$$

$\tilde{A}_1$ is the boundary matrix:

$$\tilde{A}_1(\boldsymbol{U}, \varphi) := A_1(\boldsymbol{U}) - A_0(\boldsymbol{U})\,\partial_t\varphi - A_2(\boldsymbol{U})\,\partial_2\varphi - A_3(\boldsymbol{U})\,\partial_3\varphi.$$

It should be noted that, due to the second boundary conditions, $\det \tilde{A}_1|_{x_1=0} = 0$ in Eqs. (14) and (16), that is, the boundary is characteristic. The same change of variables is performed in system (8), and the boundary conditions remain the same, but are formulated at the boundary $\Sigma_T = [0, T] \times \{x_1 = 0\} \times \mathbb{R}^2$.

As the Nash — Moser method is used below, it is necessary to linearize the problem about some given basic state, i.e., to consider the first variation of a nonlinear operator. Problem (1)–(4), (14) is considered. It is assumed that the basic state $(\hat{\boldsymbol{U}}(t, \boldsymbol{x}), \hat{\varphi}(t, \boldsymbol{x}'))$ is a sufficiently smooth bounded vector function satisfying the hyperbolicity conditions (7) and the second boundary condition in Eq. (14). The linearized system (23) has the form

$$\frac{d}{d\varepsilon}\,\mathbb{L}(\hat{\boldsymbol{U}} + \varepsilon\,\boldsymbol{U}, \hat{\varphi} + \varepsilon\,\varphi)\Big|_{\varepsilon=0} = L(\hat{\boldsymbol{U}}, \hat{\varphi})\boldsymbol{U} + \mathcal{C}(\hat{\boldsymbol{U}}, \hat{\varphi})\boldsymbol{U} - \{L(\hat{\boldsymbol{U}}, \hat{\varphi})\varphi\}\,\partial_1\hat{\boldsymbol{U}} = f,$$

where $\boldsymbol{U}$ and φ denotes perturbations, $\mathcal{C}$ is the matrix, and $f = f(t, \boldsymbol{x})$ is the artificially introduced source right-hand side whose inclusion is necessary for using linearization to prove the existence of solutions to the original nonlinear problem.

The term $\{L(\hat{\boldsymbol{U}}, \hat{\varphi})\varphi\}\,\partial_1\hat{\boldsymbol{U}}$ in the linearized system contains the derivatives of the function φ, which is undesirable as it is easier to study the linear system only for perturbation $\boldsymbol{U}$. This difficulty is overcome by using Alignac's unknown [27] of the form

$$\dot{\boldsymbol{U}} := \boldsymbol{U} - \varphi\,\partial_1\hat{\boldsymbol{U}}. \tag{24}$$

As a result of this shift of the unknown vector, the linearized equations take the form

$$L(\hat{\boldsymbol{U}}, \hat{\varphi})\dot{\boldsymbol{U}} + \mathcal{C}(\hat{\boldsymbol{U}}, \hat{\varphi})\dot{\boldsymbol{U}} - \varphi\,\partial_1\{\mathbb{L}(\hat{\boldsymbol{U}}, \hat{\varphi})\} = f. \tag{25}$$

The last term on the left-hand side of system (25) is equal to zero if the basic state $(\hat{\boldsymbol{U}}, \hat{\varphi})$ is a solution to the original nonlinear system (23). Then, if we eliminate this term and further considering the linear equations

$$L(\hat{\boldsymbol{U}}, \hat{\varphi})\dot{\boldsymbol{U}} + \mathcal{C}(\hat{\boldsymbol{U}}, \hat{\varphi})\dot{\boldsymbol{U}} = f \quad \text{in } \Omega_T, \tag{26}$$

it can be expected that the eliminated term, which should be considered as an additional error of the Nash — Moser iterations in the analysis of the nonlinear problem, tends to zero as the iteration index tends to infinity.

Similarly, the boundary conditions (14) are linearized and the result is written using the unknown vector (24). Thus, we obtain a linear problem for system (26) with boundary conditions

$$\begin{pmatrix} \dot{v}_N - \partial_t\varphi - \hat{v}_2\,\partial_2\varphi - \hat{v}_3\,\partial_3\varphi + (\partial_1\hat{v}_N)\varphi \\ \dot{q} + (\partial_1\hat{q})\varphi \end{pmatrix} = g \quad \text{on } \Sigma_T, \tag{27}$$

where $\dot{v}_N = \dot{\boldsymbol{v}} \cdot \hat{\boldsymbol{N}}$; $\hat{v}_N = \hat{\boldsymbol{v}} \cdot \hat{\boldsymbol{N}}$; $\hat{\boldsymbol{N}} = (1, -\partial_2\hat{\varphi}, -\partial_3\hat{\varphi})$; $g = g(t, \boldsymbol{x}')$ is the artificially introduced right-hand side. The sign of the factor $\partial_1\hat{q}$ in (27) significantly affects the well-posedness of the problem (in "straightened" variables, the Rayleigh — Taylor condition $\partial q/\partial\boldsymbol{\mathcal{N}} < 0$, written on the basic state, takes the form $\partial_1\hat{q} > 0$). It is the Rayleigh — Taylor condition $\partial_1\hat{q} > 0$ that allows one to estimate norm φ in space L^2 and obtain the basic a priori estimate for perturbations $(\boldsymbol{U}, \varphi)$ in space L^2 (see [25]):

$$\|\boldsymbol{U}\|_{L^2(\Omega_T)} + \|\varphi\|_{L^2(\Sigma_T)} \le C\{\|f\|_{L^2(\Omega_T)} + \|g\|_{H^1(\Sigma_T)}\}. \tag{28}$$

Here, initial data for $(\boldsymbol{U}, \varphi)$ are chosen to be equal to zero, and the case of nonzero initial data is investigated in the nonlinear analysis (construction of a so-called approximation solution [6, 25]). Thus, one derivative of the right-hand side g is lost in the estimate (28).

The existence of solutions to the linearized problem (26), (27) is proven in [25] using the standard duality argument based on the application of an estimate in L^2 (of the form (28) for $g = 0$) for the original linear problem and a similar estimates in L^2 for the coupled problem (see, e.g., [28]). However, similar reasoning is not applicable in the case of the linearization of problem (1)–(4), (8), (15), (16) as it was in [29] possible to “close” the estimate only in space H^1 (more precisely, in the weighted anisotropic Sobolev space H^1_* (see [13] and references therein). This difficulty is overcome in [29] using some “hyperbolic” regularization of the problem proposed in [30], the secondary symmetrization of Maxwell’s equations in vacuum, and also using the limit with respect to a small regularization parameter (Maxwell’s equations for $\boldsymbol{H}^-$ and the artificially introduced electric field $\boldsymbol{E}$ are considered as the regularization of an elliptic system for $\boldsymbol{H}^-$ [29]).

The existence of solutions to the original nonlinear problems was proven in [24, 25] using the Nash — Moser iterations. The Nash — Moser method is described in detail in [26], and the related papers are given therein. The idea of this method is to solve the nonlinear equation $\mathcal{F}(u) = 0$ using the iterative scheme

$$\mathcal{F}'(S_{\theta_n} u_n)(u_{n+1} - u_n) = -\mathcal{F}(u_n),$$

where $\mathcal{F}'$ is the linearization (first variation) of functional $\mathcal{F}$; S_{θ_n} is a sequence of smoothing operators with property $S_{\theta_n} \to I$ as $n \to \infty$. This scheme is the classical Newton scheme if $S_{\theta_n} = I$. The use of a smoothing operator at each step of the Nash — Moser scheme allows one to compensate for the loss of derivatives not only from the right-hand sides (as in the estimate (28)), but also from the linearization coefficients (in this case, from the basic state $(\hat{\boldsymbol{U}}, \hat{\varphi})$).

The errors of the classical Nash — Moser scheme are the quadratic error of the Newton scheme and the “substitution” error caused by the use of smoothing operators S_{θ_n}. For problems (1)–(4), (14) and (1)–(4), (8), (15), (16), the Nash — Moser method is not very standard as it is necessary to account for an additional error arising upon the introduction of an intermediate (modified) state $u_{n+1/2}$, satisfying some nonlinear constraints. The main such constraint is the requirement that the second boundary conditions in Eqs. (14) and (16) should be satisfied (as it is assumed that $\partial_t \hat{\varphi} - \hat{v}_N|_{x_1=0} = 0$). The presence of another additional error is caused by eliminating the last term on the left-hand side of system (25).

Basic a priori estimates of the form (28) are insufficient to prove the convergence of the Nash — Moser iterations. It is required to derive some more delicate a priori estimates for the linearized problem in the higher norms of the Sobolev spaces H^s, which take into account the number of “lost” derivatives not only of the right-hand sides, but also of the coefficients, i.e., of $(\hat{\boldsymbol{U}}, \hat{\varphi})$. The main properties of such estimates known as called tame estimates are linearity with respect to the higher norms and a fixed loss of derivatives, i.e., the number of “lost” derivatives should be the same for any superscript s of space H^s.

As for the general scheme for proving the local-in-time well-posedness of problem (1)–(4), (19) for a contact MHD discontinuity [32], it is the same as in [24, 25]. However, there are difficulties which were overcome in [31] in order to obtain a basic a priori estimate for the linearized problem in H^1. In [31, 32], a two-dimensional flow with a contact discontinuity was considered, and the local solvability of this problem in a three-dimensional case was not proven. Another difficulty arose in proving the existence of solutions to the linearized problem as it was not possible to obtain an a priori estimate for it in L^2. That difficulty was overcome in [31] by the “strictly dissipative” regularization of the problem, proving the existence of solutions to the regularized problem in space L^2 and passing to the limit with respect in terms of the small regularization parameter using an estimate that is uniform with respect to this parameter in H^1.

It should be noted that it is not the magnetohydrodynamic Rayleigh — Taylor condition (22) that naturally arises in the mathematical analysis of problem (1)–(4), (19), but rather its classical hydrodynamic version

$$\left[\frac{\partial p}{\partial \boldsymbol{\mathcal{N}}}\right] \leq -\delta_1 < 0 \tag{29}$$

for pressure p (and not for the total pressure q). Condition (29) is the main requirement in [32] for the initial data, which guarantees the local-in-time existence of a contact MHD discontinuity in the two-dimensional case.

3. LOCAL SOLVABILITY OF PROBLEMS WITH ACCOUNT FOR SURFACE TENSION

We consider the problem of proving the local solvability of free boundary problems with account for surface tension, formulated in Sec. **1**. Note that the existence and uniqueness theorems for these problems should be proven using the scheme described in Sec. **2**. The presence of term $\mathcal{H}(\varphi)$ in the boundary conditions makes it possible to use estimates for the derivatives with respect to the spatial variables of the function φ when deriving basic a priori estimates for the corresponding linearized problems. This, in turn, makes it possible in [16, 33] to obtain a priori estimates for the linearization of problems (1)–(4), (17) and (1)–(4), (20) without imposing the corresponding Rayleigh — Taylor conditions ((22) for $\boldsymbol{H}^- \equiv 0$ and (29)) for an unperturbed flow.

In the case of problems with account for surface tension, the main difficulty is proving the existence of solutions to the corresponding linearized problems. This difficulty was overcome in [16, 33] by certain regularization of these problems. We consider problem (1)–(4), (17). During the linearization of this problem in [33], it was possible to close the estimate only in space H^1_* (therefore, due to the absence of an estimate in L^2, the duality argument does not apply). System (26) is supplemented by a "strictly dissipative" term $(0, -\varepsilon\,\partial_1 \dot{v}_N, 0, \ldots, 0)$, which makes it possible to estimate $\dot{v}_N|_{x_1=0}$ and a fourth-order "parabolic" term $-\varepsilon(\partial_2^4 + \partial_3^4)\varphi$, where $\varepsilon > 0$ is a regularization parameter (see [33]). It is clear that the second boundary condition in Eq. (27) contains an additional term due to the linearization of $\mathcal{H}(\varphi)$. For such a regularized problem and for its coupled problem, one can derive estimates in L^2 and prove the existence of solutions using the duality argument for any fixed ε.

However, the a priori obtained estimate of the linearized problem in L^2 is not uniform with respect to ε. In order to pass to the limit $\varepsilon \to 0$ and prove the existence of solutions to the original linear problem, an estimate uniform in ε was derived in [33] in H^1_*. Similar reasoning was carried out in [16] for a contact MHD discontinuity with account for the surface tension.

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