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INFLUENCE OF THE MARANGONI EFFECT ON THE EMERGENCE OF FLUID ROTATION IN A THERMOGRAVITATIONAL BOUNDARY LAYER

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Abstract: Steady axisymmetric regimes of the fluid flow in a thermogravitational boundary layer near the free surface with a nonuniform temperature distribution on this boundary are calculated for equations of fluid motion in the Oberbeck–Boussinesq approximation, where the viscosity and thermal diffusivity are small. With due allowance for the thermocapillary effect, it is demonstrated that nonhomogeneous fluid flow regimes with rotation can arise in the boundary layer owing to a bifurcation in the case of local cooling of the free surface, whereas rotation outside this layer is absent.

Keywords: free boundary, nonhomogeneous fluid, boundary layer, bifurcation, thermocapillary effect, rotation.

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In a homogeneous fluid with small diffusion coefficients, there may emerge thin boundary layers in the vicinity of the free surface under the action of shear stresses on this surface. These stresses arise, e.g., if there is a temperature gradient along the free boundary. As a result, a thermocapillary fluid flow is formed (Marangoni effect). Napolitano [1] was the first researcher who obtained self-similar solutions that describe thermocapillary flows of a homogeneous fluid. An important cycle of investigations of the Marangoni effect was performed by Pukhnachev and his team. For example, the properties of unsteady and steady Marangoni boundary layers were considered in [2, 3].

Nonlinear boundary layers may be formed in the vicinity of the free boundary if there are no shear stresses on this boundary. This phenomenon occurs in the absence of the Marangoni effect in a nonhomogeneous fluid whose flow is described by the equations of motion in the Oberbeck–Boussinesq approximation. Owing to the influence of the temperature field on the velocity field, a thermogravitational boundary layer is formed in the fluid near the free boundary. Depending on the values of the problem parameters, the Marangoni effect can produce a small or finite action on the nonhomogeneous fluid dynamics. The influence of the Marangoni effect on the thermogravitational fluid flow in a thin layer was investigated in [4]. The present study is aimed at considering the Marangoni effect on the thermogravitational flow in the boundary layer near the free surface of a nonhomogeneous fluid, which occupies a finite-thickness layer. Conditions at which a fluid flow with rotation can arise near the free boundary are determined.

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1. MATHEMATICAL MODEL

A steady flow of a nonhomogeneous fluid under axial symmetry conditions is considered. The fluid occupies a horizontal layer of finite thickness h, which is bounded by a solid wall from below and by a free boundary Γ from above; the free boundary is subjected to the condition of a nonhomogeneous temperature distribution. The fluid flow is calculated on the basis of the equations of motion in the Oberbeck–Boussinesq approximation:

$$(\boldsymbol{v}, \nabla)\boldsymbol{v} = -\rho^{-1}\nabla p + \nu\Delta\boldsymbol{v} - \boldsymbol{g}\beta T, \qquad \boldsymbol{v}\nabla T = \chi\Delta T, \qquad \operatorname{div}\boldsymbol{v} = 0.$$

Here $\boldsymbol{v} = (v_r, v_\theta, v_z)$ is the velocity vector in cylindrical coordinates r, θ, z, $\boldsymbol{g} = (0, 0, -g_t)$ (g_t is the acceleration due to gravity), β is the thermal expansion coefficient, and T is the fluid temperature; the kinematic viscosity ν and thermal diffusivity χ are assumed to be small. The axial symmetry conditions mean that the velocity, pressure, and temperature are independent of the circumferential coordinate θ.

The boundary conditions on the free surface include the dynamic conditions for the normal and shear stresses, kinematic condition, and prescribed temperature T_Γ are

$$p = 2\nu\rho\boldsymbol{n}\Pi\boldsymbol{n} - \sigma(k_1 + k_2) + \rho gz + p_*, \qquad \boldsymbol{vn} = 0, \qquad (r, \theta, z) \in \Gamma,$$

$$2\nu\rho(\Pi\boldsymbol{n} - (\boldsymbol{n}\Pi\boldsymbol{n})\boldsymbol{n}) = \nabla_\Gamma\sigma, \qquad T = T_\Gamma, \qquad (r, \theta, z) \in \Gamma.$$

Here Π is the strain rate tensor, p is the dynamic pressure caused by convective motion of the fluid [5], ρgz is the hydrostatic pressure, $\boldsymbol{n}$ is the unit vector of the external normal to the surface Γ, k_1 and k_2 are the principal curvatures of the free surface, and σ is the surface tension coefficient, which is assumed to be linearly dependent on temperature [$\sigma = \sigma_0 - |\sigma_T|(T - T_\sigma)$, where σ_0, σ_T, and T_σ are constants], ∇_Γ is the gradient along the boundary Γ; the pressure p_* on the free surface is assumed to be constant. The solid wall is subjected to the no-slip condition $\boldsymbol{v} = \boldsymbol{0}$ and constant temperature condition $T = T_S$.

The origin of the coordinate system is chosen to be on the free surface. Here we find the problem solution, where the free boundary in the principal approximation is not deformed and is described by the equation $z = 0$. The solid boundary is determined by the equation $z = -h$. Let us assume that the temperature on the boundary Γ behaves according to the law $T_\Gamma = T_S + T_m(r^2/L^2 - 1)$ at $r \leqslant L$ and $T_\Gamma = T_S$ at $r > L$ ($T_m = T_\Gamma|_{r=L} - T_\Gamma|_{r=0}$ is the temperature difference on the segment $r \in [0, L]$). Obviously, the free boundary in the vicinity of the axis of symmetry is cooled at $T_m > 0$ ($r < L$) and is heated at $T_m < 0$.

Let us introduce the dimensionless temperature difference α on the surface Γ by the formula $\alpha = T_m/T_S$. The equations of motion and the boundary conditions are also non-dimensionalized. The scales of length, temperature, velocity, and pressure are taken to be L, T_S, U_a, and ρU_a^2 ($U_a = \sqrt{g_t\beta L T_S}$). Assuming that the temperature difference changes in the course of cooling ($0 < T_m < T_S$), we have $0 < \alpha < 1$. If the parameter $\sqrt{\alpha}$ is finite, quantities $\sqrt{T_S}$ and $\sqrt{T_m}$ are of the same order and, hence, the scale U_a has the order $U_a \sim \sqrt{g_t\beta L T_m}$. The expression for the temperature on the surface Γ is presented as $T_\Gamma = 1 + \alpha(r^2 - 1)$ at $r \leqslant 1$ and $T_\Gamma = 1$ at $r > 1$.

Let us introduce the dimensionless parameter $\varepsilon = \nu/(LU_a)$, which is small for small values of the kinematic viscosity and finite values of the parameter L. The equations of motion in the dimensionless form are written as

$$(\boldsymbol{v}, \nabla)\,\boldsymbol{v} = -\nabla p + \varepsilon\Delta\boldsymbol{v} + T\boldsymbol{e}_z, \qquad \boldsymbol{v}\nabla T = (\varepsilon/\mathrm{Pr})\,\Delta T, \qquad \operatorname{div}\boldsymbol{v} = 0, \tag{1.1}$$

where $\boldsymbol{e}_z$ is the vector with the components $(0, 0, 1)$, and Pr is the Prandtl number.

2. ASYMPTOTIC APPROACH

In the dimensionless variables, the parameter ε is located at the higher derivatives in the equations of motion (1.1). This means that a thermogravitational boundary layer domain D_Γ is formed in this case near the free boundary. Outside this domain, an external flow is formed, which is described by diffusionless equations in the principal approximation. Let us consider the case with an identical order of the velocity vector components in the boundary layer domain and in the external flow as $\varepsilon \to 0$. Let us also assume that the external flow is not swirled, i.e., $v_\theta = 0$ outside D_Γ. Then we find the order of the velocity vector and boundary layer thickness. For this purpose, we introduce the stretching transformation $s_1 = z/\varepsilon^k$ in the domain D_Γ and estimate the orders of velocity, pressure, and temperature in the equations and boundary conditions. As a result, we find that the velocity vector has the order equal to $O(\varepsilon^{1/5})$. The temperature and pressure have the finite order $O(1)$. The order of the boundary layer thickness is $O(\varepsilon^{2/5})$. The deviation of the free boundary from the undisturbed level $z = 0$ has the order $O(\beta T_m \varepsilon^{2/5})$ and is significantly smaller than the boundary layer thickness.

The boundary-value problem is solved by using the boundary layer equations [6]. As $\varepsilon \to 0$, the asymptotic expansions of the velocity, temperature, and pressure field are constructed on the basis of the fractional powers of the parameter ε:

$$v_r = \varepsilon^{1/5}(v_{r1} + h_{r1}) + \ldots, \qquad v_z = \varepsilon^{1/5} v_{z1} + \varepsilon^{3/5}(v_{z2} + h_{z2}) + \ldots,$$

$$v_\theta = \varepsilon^{1/5} h_{\theta 1} + \varepsilon^{3/5} h_{\theta 2} + \ldots, \qquad p = p_0 + \varepsilon^{2/5}(p_1 + q_1) + \ldots, \tag{2.1}$$

$$T = 1 + T_1(r, s) + \varepsilon^{2/5}(T_2(r, s) + T_{1b}(r, z)) + \ldots .$$

The functions h_{r1}, h_{z2}, $h_{\theta 1}$, $h_{\theta 2}$, T_1, and T_2 are determined in the boundary layer domain D_Γ, depend on the stretched variable $s = z/\varepsilon^{2/5}$, and tend to zero when leaving the domain D_Γ, i.e., as $s \to -\infty$. The function $q_1(r, s)$ is determined in the domain D_Γ and tends to a constant value at infinity. The functions v_{r1}, v_{z1}, v_{z2}, p_0, p_1, and T_{1b} (external solution) depend on the variables r and z, are determined in the entire fluid flow domain, and describe the problem solution outside the boundary layer domain D_Γ. The conditions $h_{\theta 0} \to 0$ and $h_{\theta 1} \to 0$ as $s \to -\infty$ mean that fluid rotation in the boundary layer does not induce rotation outside the domain D_Γ.

Let us substitute the asymptotic series (2.1) into the system of the equations of motion (1.1) and the boundary conditions and equate the sum of the coefficients at identical powers of ε to zero. We introduce the functions H_r and H_z by the formulas $H_r = h_{r1} + v_{r1}|_\Gamma$ and $H_z = h_{z2} + v_{z2}|_\Gamma + s\,\partial v_{z1}/\partial z|_\Gamma$. In the principal approximation, we obtain the following system of the boundary layer equations for calculating the thermogravitational fluid flow in the domain D_Γ:

$$H_r \frac{\partial H_r}{\partial r} + H_z \frac{\partial H_r}{\partial s} - \frac{h_{\theta 1}^2}{r} = -\frac{\partial q_1}{\partial r} + \frac{\partial^2 H_r}{\partial s^2} + v_{r1} \frac{\partial v_{r1}}{\partial r}\Big|_\Gamma,$$

$$H_r \frac{\partial h_{\theta 1}}{\partial r} + H_z \frac{\partial h_{\theta 1}}{\partial s} + \frac{H_r h_{\theta 1}}{r} = \frac{\partial^2 h_{\theta 1}}{\partial s^2}, \qquad -\frac{\partial q_1}{\partial s} + T_1 = 0, \tag{2.2}$$

$$H_r \frac{\partial T_1}{\partial r} + H_z \frac{\partial T_1}{\partial s} = \frac{1}{\mathrm{Pr}} \frac{\partial^2 T_1}{\partial s^2}, \qquad \frac{\partial H_r}{\partial r} + \frac{H_r}{r} + \frac{\partial H_z}{\partial s} = 0.$$

The boundary conditions on the free boundary and the conditions at the exit from the domain D_Γ have the form

$$H_z = 0, \qquad \frac{\partial H_r}{\partial s} = -\gamma\tau r, \qquad \frac{\partial h_{\theta 1}}{\partial s} = 0, \qquad T_1 = \theta_\Gamma, \qquad s = 0,$$

$$H_r \to v_{r1}\Big|_\Gamma, \qquad \frac{\partial H_r}{\partial s} \to 0, \qquad h_{\theta 1} \to 0, \qquad T_1 \to 0, \qquad s \to -\infty, \qquad \tau = 2\alpha. \tag{2.3}$$

The function θ_Γ is converted to the form $\theta_\Gamma = 0.5\tau(r^2 - 1)$, and the parameter γ in the boundary conditions is determined by the formula $\gamma = |\sigma_T|\rho^{-1}(T_S^2 \nu^{-4} L^{-4} g_t^{-3} \beta^{-3})^{1/5}$, which takes into account the Marangoni effect. At $\gamma = 0$, the thermocapillary effect is ignored.

In contrast to Prandtl's boundary layer equations for a homogeneous fluid, system (2.2) contains derivatives of the pressure function q_1 with respect to both spatial coordinates r and s. This is caused by the influence of the temperature field on the velocity field in a nonhomogeneous fluid.

In solving the boundary-value problem for the Oberbeck–Boussinesq equations, the functions of the external solution v_{r1}, v_{z1}, v_{z2}, p_0, p_1, and T_{1b} are sought with the use of the first iterative process in the boundary layer method [6]. As a result, we determine p_0 and find that the vector $\boldsymbol{v}_1$ with the components v_{r1}, v_{z1}, and $v_{\theta 1} = 0$, as well as the functions p_1 and T_{1b} in expansions (2.1) satisfy system (1.1) at $\varepsilon = 0$. Let us assume that the radial component of the external flow velocity on the free boundary in the vicinity of the axis of symmetry is a linear function of the radial coordinate: $v_{r1}|_\Gamma = Ur$ ($U \geqslant 0$). The asymptotic value of the external flow velocity field near the free boundary is $v_{r1} \sim Ur$, $v_{z1} \sim -2Uz$, and $v_{\theta 1} = 0$ for $|z| \leqslant h_1$ (h_1 is the dimensionless thickness of the fluid layer and U is the amplitude of the external flow velocity on the free surface).

In view of the quadratic dependence of temperature on the radial coordinate on the free boundary, the exact solution of problem (2.2), (2.3) is constructed in the form

$$H_r = r(U - W'(\eta)), \qquad H_z = 2(\eta U - W), \qquad h_{\theta 1} = rG(\eta),$$
$$T_1 = 0.5r^2 T_{11}(\eta) + T_{12}(\eta), \qquad q_1 = -0.5r^2 q_{11}(\eta) - q_{12}(\eta), \qquad \eta = -s. \tag{2.4}$$

Formulas (2.4) define the problem solution only in the vicinity of the axis of symmetry; they do not extend outside this domain, in particular, they are invalid in the region $r > 1$, where the solution can be continued numerically.

Formulas (2.2) and (2.3) yield the following boundary-value problem for the functions W, G, T_{11}, and T_{12}:

$$W^{(4)} = 2(W - \eta U)W''' + 2GG' + T_{11}, \qquad G'' = 2(W - \eta U)G' + 2(U - W')G,$$
$$T''_{11} = 2\text{Pr}\ ((W - \eta U)T'_{11} + (U - W')T_{11}), \qquad T''_{12} = 2\text{Pr}\ (W - \eta U)T'_{12},$$
$$W = 0, \qquad W'' = -\gamma\tau, \qquad G' = 0, \qquad T_{11} = \tau, \qquad T_{12} = -0.5\tau \qquad (\eta = 0), \tag{2.5}$$
$$W' = W'' = G = T_{11} = T_{12} = 0 \qquad (\eta = +\infty).$$

By using formulas (2.2) and (2.4), the functions q_{11} and q_{12} can be expressed via T_{11} and T_{12}. After solving the boundary-value problem (2.5), we find the functions q_{11} and q_{12} by the formulas

$$q_{11} = \int_0^\eta T_{11}\, d\eta + \text{const}, \qquad q_{12} = \int_0^\eta T_{12}\, d\eta + \text{const}.$$

If $U = 0$ in problem (2.5), we have $v_{r1} = v_{z1} = v_{\theta 1} = 0$. The external flow velocity vector has the components v_{r2}, v_{z2}, and $v_{\theta 2} = 0$, which are induced by the boundary layer (by the functions W, G, T_{11}, and T_{12}); moreover, the kinematic condition for the external flow on the surface Γ takes the form $v_{z2}|_\Gamma = -2W(\infty)$.

3. FLUID FLOW REGIMES IN THE ABSENCE OF ROTATION

The functions W, G, T_{11}, and T_{12}, which describe the fluid flow in the boundary layer, are found numerically by means of solving problem (2.5) by the shooting method. The calculations are performed for the Prandtl number Pr $= 7$. We introduce the parameter $V = U - W'(0)$, which describes the amplitude of the radial component of velocity on the free boundary. Figure 1 shows the parameter V as a function of the amplitude of the external flow velocity U on the boundary Γ. It is seen that the fluid velocity on the free boundary increases monotonically with an increase in the external flow velocity U. The Marangoni effect at $\tau < 0$ leads to an increase in the fluid velocity on the free surface Γ.

In the case of cooling of the free boundary, the fluid flow regimes without rotation exist only if the amplitude of the external flow velocity U on the surface Γ is not smaller than a certain limiting value U_m. It should be noted that $U_m \approx 0.7029$ for $\gamma = 1$ and $U_m \approx 0.2489$ for $\gamma = 0$. In the presence of the Marangoni effect, the parameter U_m increases. The parameter U_m corresponds to the extreme left points ("vertices") of curves *3* and *4*. For $U > U_m$, each value of the parameter U corresponds to two solutions for the flow regime in the absence of rotation, which differ by the velocity profile shapes. These solutions coincide for $U = U_m$ and vanish for $U < U_m$. The analysis of the velocity profiles in the boundary layer shows that the flow domain with $V < 0$ is divided into the region where the fluid particles move away from the axis of symmetry ($v_r > 0$) and the region where they approach the axis of symmetry ($v_r < 0$). The region $v_r < 0$ adjacent to the free boundary is significantly narrower than the region $v_r > 0$. At $V > 0$ and $\tau > 0$, the boundary layer domain consists only of the region $v_r > 0$. In this case, the radial component of the fluid velocity on the surface Γ decreases owing to the Marangoni effect. At $V < 0$, the Marangoni effect on this component is non-unique. For some values of the parameter U, the absolute value of this component on the surface Γ can increase with increasing γ, while this value decreases for some other values of U (for small values of $|V|$).

Let us use $V_r = U - W'(\eta)$ to denote the amplitude of the radial component of velocity in the principal approximation inside the boundary layer domain D_Γ. The calculated results show that the velocity component V_r for flow regimes without rotation monotonically decreases with distance from the free boundary in the case of heating of the free boundary and monotonically increases in the case of cooling of the free boundary. At the exit from the domain D_Γ, the amplitude V_r tends to a limiting value equal to U.

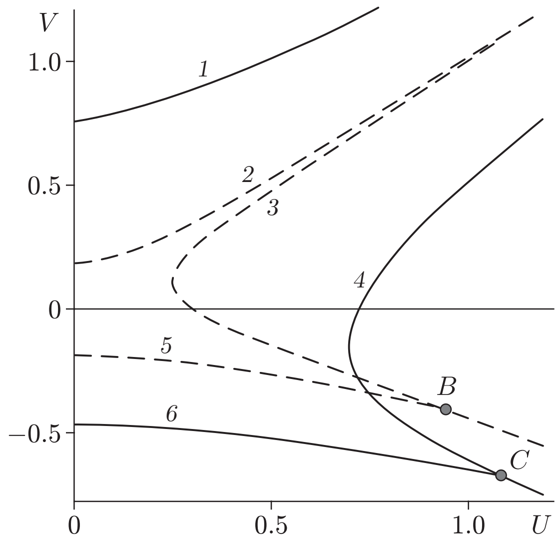

Fig. 1. Parameter V versus the amplitude of the external flow velocity U on the free boundary for flow regimes without rotation ($G = 0$) (curves *1–4*) and with rotation ($G \neq 0$) (curves *5* and *6*) at $\tau = -0.75$ (local heating) (*1* and *2*) and 0.75 (local cooling) (*3–6*): the solid and dashed curves show the results for $\gamma = 1$ and 0, respectively; the bifurcation points are denoted by B and C.

4. BIFURCATIONS. FLUID FLOW REGIMES WITH ROTATION

In the case of local cooling of the free boundary and $U < U_m$, there are no fluid flow regimes without rotation ($G = 0$) in the boundary layer, and the boundary-value problem (2.5) has solutions that describe flow regimes with rotation at $G \neq 0$, which appear as a result of a bifurcation of flow regimes without rotation. The bifurcation points are determined in the course of the numerical solution of the boundary-value eigenvalue problem obtained by means of linearization of problem (2.5) near the solution that describes flow regimes in the absence of rotation. If the eigenfunctions are denoted by W_c, G_c, T_{c1}, and T_{c2}, the boundary-value problem for determining these functions and eigenvalues can be presented in the form

$$W_c^{(4)} = 2W'''W_c + 2(W - \eta U)W_c''' + T_{c1}, \qquad G_c'' = 2(W - \eta U)G_c' - 2(W' - U)G_c,$$

$$T_{c1}'' = 2\mathrm{Pr}\,(W_c T_{11}' - W_c' T_{11} + (W - \eta U)T_{c1}' - (W' - U)T_{c1}),$$

$$T_{c2}'' = 2\mathrm{Pr}\,(W_c T_{12}' + (W - \eta U)T_{c2}'), \qquad (4.1)$$

$$W_c = W_c'' = T_{c1} = T_{c2} = G_c' = 0 \quad (\eta = 0), \qquad W_c' = W_c'' = T_{c1} = T_{c2} = G_c = 0 \quad (\eta = +\infty).$$

The eigenvalue problem (4.1) is solved numerically. The parameters U, τ, and γ are varied in the limited range

$$M = \{(U, \tau, \gamma); \quad U \in [0, 2], \quad \tau \in [-1, 1], \quad \gamma \in [0, 1]\}.$$

In the case of local heating of the free boundary ($\tau < 0$), no eigenvalues were obtained. In the case of cooling of the free boundary ($\tau > 0$), we calculated the branch of simple eigenvalues $U_*(\gamma, \tau)$ of the parameter U in the domain M. The eigenfunctions are found in the form $G_c = c_g G_*(\eta)$ and $W_c = T_{c1} = T_{c2} = 0$ (c_g is an arbitrary constant not equal to zero). The function $G_*(\eta)$ satisfies the normalization condition $G_*(0) = 1$, monotonically decreases with increasing η, and tends to zero as $\eta \to +\infty$. Figure 2 shows the dependences of the bifurcation parameter U_* on τ. The numerical results show that the bifurcation values of U in the presence of the thermocapillary effect increase both with increasing τ at a fixed value of γ and with increasing γ at a fixed value of τ. The Marangoni effect is enhanced with increasing γ.

The solutions that describe flow regimes with rotation branch away from the solutions that describe flow regimes without rotation at the bifurcation points B and C (see Fig. 1). For each point belonging to curves *5* and *6*, we calculated two regimes that differ only by the rotation direction.

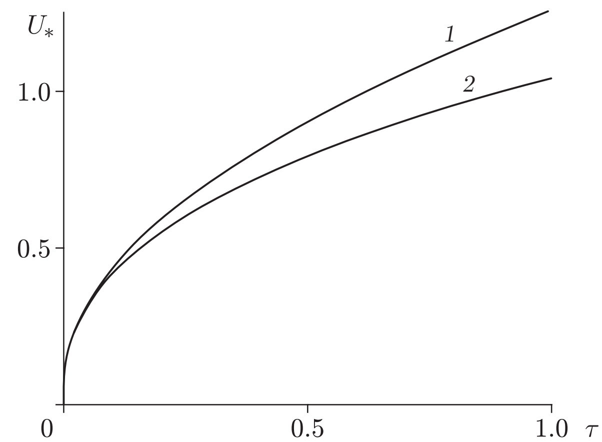

Fig. 2. Bifurcation values of U versus τ for $\gamma = 1$ (curve *1*) and 0 (curve *2*).

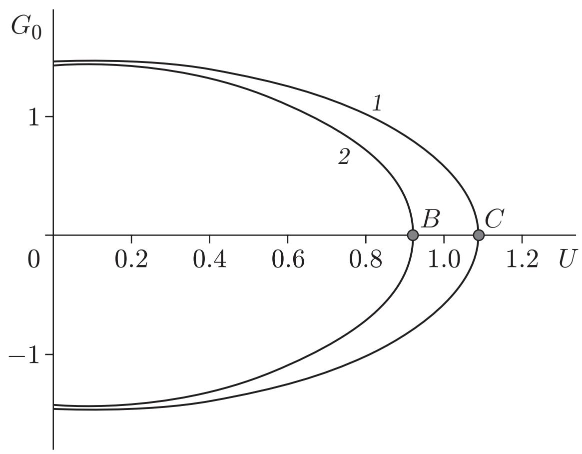

Fig. 3.

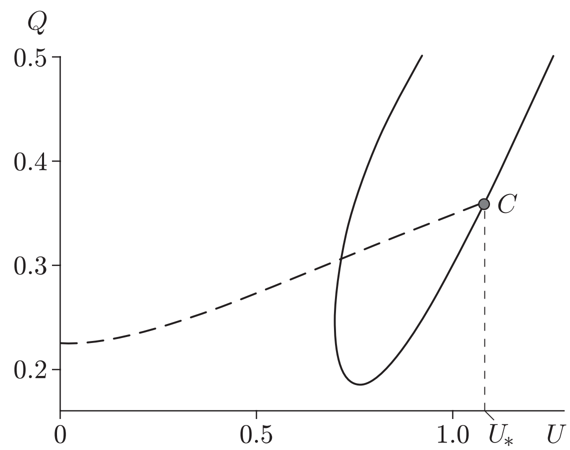

Fig. 4.

Fig. 3. Amplitude of the circumferential component of velocity G_0 versus the external flow velocity U on the free boundary for flow regimes with rotation for $\tau = 0.75$ and $\gamma = 1$ (*1*) and 0 (*2*); B and C are the bifurcation points.

Fig. 4. Dependence $Q(U)$ on the free surface for $\gamma = 1$, $\tau = 0.75$, and $r = 0.5$: the solid and dashed curves show the flow regimes without and with rotation, respectively; C is the bifurcation point.

For the free boundary points, Fig. 3 shows the amplitude of the circumferential component of velocity for flow regimes with rotation $G_0 = G(0)$ as a function of the external flow velocity U for $\tau = 0.75$. Owing to the thermocapillary effect, the circumferential component of velocity on the free boundary increases. For $U < U_*$, two regimes with symmetric values of $\pm|G_0|$ on the free boundary were calculated.

In solving the boundary-value problem (2.5), we also calculated the heat flux on the free boundary; for calculating the heat flux component normal to the boundary, we used the formula $q_z = -\lambda\,\partial T/\partial z|_\Gamma$ transformed to $q_z = q_* Q(r, U, \tau, \gamma)$ [$q_* = \lambda T_S L^{-1} \varepsilon^{-2/5}$ is a dimensional parameter including the thermal conductivity λ, and $Q = 0.5r^2 T'_{11}(0) + T'_{12}(0)$ is a dimensionless function]. The calculated results are summarized in Fig. 4. It is seen that the heat flux in the case of flow regimes with rotation monotonically increases with increasing U in the interval $[0, U_*]$.

5. ASYMPTOTIC BEHAVIOR OF THE FLUID FLOW WITH ROTATION IN THE VICINITY OF THE BIFURCATION POINT

Let us introduce the parameter δ by the formula $\delta = U - U_*$, where U and U_* are the current and bifurcation values of the parameter U, respectively. Obviously, the parameter δ in a small vicinity of the bifurcation point, as $U \to U_*$, is small. Let us use W_*, T_{*1}, and T_{*2} to denote the fluid flow regime at the bifurcation point. Let us introduce the parameter ε_* by the formula $\varepsilon_* = G(\eta, U, \tau, \gamma)|_{\eta=0}$, where $G(0, U, \tau, \gamma)$ is the amplitude of the circumferential velocity component for the flow regime with rotation on the boundary Γ. It should be noted that the parameter $\varepsilon_*(U, \tau, \gamma)$ vanishes at $U = U_*$. In a small vicinity of the bifurcation point, as $U \to U_*$, the parameter ε_* is small. Flow regimes with rotation in the vicinity of the bifurcation point can be presented as

$$W = W_* + \Psi, \qquad G = \varepsilon_* g, \qquad T_{11} = T_{*1} + t_1, \qquad T_{12} = T_{*2} + t_2.$$

The expression $\varepsilon_* = G(0, U, \tau, \gamma)$ yields the boundary condition $g(0) = 1$.

Taking into account Eq. (2.5) we obtain the following boundary-value problem for determining the functions Ψ, g, t_1, and t_2:

$$L\Psi = t_1 - 2\delta\eta(\Psi''' + W_*''') + 2\varepsilon_*^2 g g' + 2\Psi\Psi''',$$

$$Nt_1 = 2\mathrm{Pr}\ (\Psi T_{*1}' - \Psi' T_{*1} + \Psi t_1' - \Psi' t_1 + \delta(T_{*1} - \eta T_{*1}' + t_1 - \eta t_1')),$$

$$t_2'' = 2\mathrm{Pr}\ ((W_* - \eta U_*) t_2' + \Psi t_2' + \Psi T_{*2}' - \delta\eta(t_2' + T_{*2}')),$$

$$Kg = 2\Psi g' - 2\Psi' g + 2\delta(g - \eta g'), \tag{5.1}$$

$$\Psi = 0, \qquad \Psi'' = 0, \qquad g' = 0, \qquad t_1 = 0, \qquad t_2 = 0 \qquad (\eta = 0),$$

$$\Psi' = 0, \qquad \Psi'' = 0, \qquad g = 0, \qquad t_1 = 0, \qquad t_2 = 0 \qquad (\eta = +\infty).$$

Here L, K, and N are the linear operators

$$L = D^4 - 2(W_* - \eta U_*)D^3 - 2W_*''' E,$$

$$K = D^2 - 2(W_* - \eta U_*)D + 2(W_*' - U_*)E, \qquad N = D^2 - 2\mathrm{Pr}\ ((W_* - \eta U_*)D - (W_*' - U_*)E),$$

$D = d/d\eta$ is the operator of differentiation, and E is the unit operator.

The solutions of the boundary-value problem (5.1) are constructed as asymptotic series in powers of the parameter δ:

$$\Psi = \delta\Psi_1 + \delta^2\Psi_2 + \ldots \qquad (\delta \to 0),$$

$$t_1 = \delta t_{11} + \delta^2 t_{12} + \ldots, \qquad t_2 = \delta t_{21} + \delta^2 t_{22} + \ldots, \qquad g = G_* + \delta G_1 + \ldots\ . \tag{5.2}$$

The properties of the function G_* are presented above. From the conditions $G_*(0) = 1$ and $g(0) = 1$ for the function G_1 in Eq. (5.2), we derive the boundary condition $G_1(0) = 0$.

It follows from the relation $U = \delta + U_*$ that the parameter $\varepsilon_*(U, \tau, \gamma)$ depends on the parameter δ. The parameter ε_* is determined by solving the boundary-value problem (5.1) with allowance for the additional boundary condition $g(0) = 1$. As problem (5.1) involves the parameter ε_* only in the form of the quadratic function ε_*^2, we assume that the parameter ε_*^2, similar to the functions Ψ, t_1, t_2, and g, can be presented in the form of the asymptotic series in powers of the parameter δ:

$$\varepsilon_*^2 = \delta B_1 + \delta^2 B_2 + \ldots \qquad (\delta \to 0). \tag{5.3}$$

For determining the coefficients of the asymptotic expansions (5.2) and (5.3), we substitute these expansions into the boundary-value problem (5.1) and equate the sums of the coefficients at consecutive powers of the parameter δ to zero. The boundary-value problem for the principal terms of the asymptotic series (5.2) has the form

$$L\Psi_1 = t_{11} + 2B_1 G_* G_*' - 2\eta W_*''', \qquad Nt_{11} = 2\mathrm{Pr}\ (\Psi_1 T_{*1}' - \Psi_1' T_{*1} + T_{*1} - \eta T_{*1}'),$$

$$\Psi_1 = 0, \quad \Psi_1'' = 0, \quad t_{11} = 0 \quad (\eta = 0), \qquad \Psi_1' = 0, \quad \Psi_1'' = 0, \quad t_{11} = 0 \quad (\eta = +\infty). \tag{5.4}$$

The function t_{12} is found by solving the linear boundary-value problem after integrating problem (5.4).

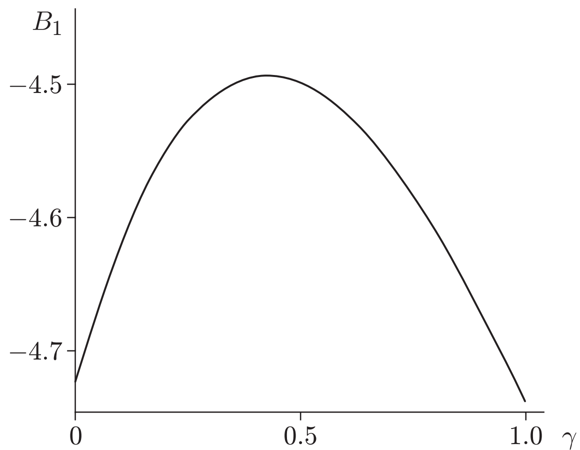

Fig. 5. Parameter B_1 versus γ for $\tau = 1$.

Problem (5.4) contains the unknown parameter B_1; therefore, the functions Ψ_1 and t_{11} in the course of integrating Eq. (5.4) can be presented as $\Psi_1 = \Psi_{b1} + B_1\Psi_{b2}$ and $t_{11} = t_{b1} + B_1 t_{b2}$. The boundary-value problems for the functions Ψ_{b1}, t_{b1}, Ψ_{b2}, and t_{b2} do not involve the parameter B_1 and are solved numerically. The parameter B_1 and the function G_1 are found by solving the linear boundary-value problem

$$KG_1 = 2(\Psi_{b1}G'_* - \Psi'_{b1}G_* + G_* - \eta G'_*) + 2B_1(\Psi_{b2}G'_* - \Psi'_{b2}G_*), \tag{5.5}$$

$$G'_1(0) = 0, \qquad G_1(+\infty) = 0.$$

Problem (5.5) with allowance for the additional condition $G_1(0) = 0$ is solved numerically; the parameters γ and τ are varied in the interval $[0, 1]$. Figure 5 shows the parameter B_1 as a function of the parameter γ for $\tau = 1$. It is found that the parameter B_1 for $\tau \in [0, 1]$ is negative. If the thermocapillary effect is absent ($\gamma = 0$), the parameter B_1 takes the values $B_1 \approx -4.7265\tau^{2/5}$.

Formula (5.3) yields the relation

$$\varepsilon_* = \pm\sqrt{\delta B_1} + O(\delta) \qquad (\delta \to 0),$$

from which it follows that solutions that describe two flow regimes with rotations, which differ only by the rotation direction, can branch away at the bifurcation point from the solutions that describe the flow regime in the absence of rotation in the boundary layer. These regimes arise at $U < U_*$.

CONCLUSIONS

The emergence of rotation of a nonhomogeneous fluid in the boundary layer near the free boundary under the condition that this boundary is locally nonuniformly cooled in the vicinity of the axis of symmetry is considered in the present study. The influence of the thermocapillary effect on the fluid flow regimes with and without rotation is calculated. In particular, it is demonstrated that the Marangoni effect enhances rotation in the boundary layer in situations without rotation outside the boundary layer. Fluid rotation arises due to a bifurcation of flow regimes in the absence of rotation. The Marangoni effect leads to an increase in the bifurcation values of external flow velocity.

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