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DETERMINING STRESSES IN AN ADHESIVE JOINT WITH A LONGITUDINAL UNADHERED REGION USING A SIMPLIFIED TWO-DIMENSIONAL THEORY

S. S. Kurennov*

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Abstract: A simplified two-dimensional model of an overlap adhesive joint is proposed. The problem of a stress state of an adhesive joint, along the surfaces of which unadhered regions are located, is solved analytically under the assumption that the cross-sectional displacements of carrier layers vanish. The resulting solution is a functional series, with eigenfunctions being nonorthogonal. It is shown that the presence of unadhered regions may significantly increase stresses near the edge of the adherend.

Keywords: adhesive joint, analytical solution, two-dimensional model.

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INTRODUCTION

Most mathematical models of adhesive overlap joints, which allow determining the stress–strain state of a joint in analytical form, are one-dimensional [1]. These models assume a uniform stress distribution over the width of a joint and specify a stress distribution over the layer thickness in advance (usually uniform or linear). However, in a number of cases, a stress state of joints cannot be determined without accounting for the nonuniformity of a stress–strain state of adhered plates over the joint width. The examples of such structures could be the joints of load-bearing elements with paneling, patches, etc. Constructing an analytical solution of the problem of a two-dimensional stress state in a general formulation is extremely difficult, so no analytical solution of this problem exists at the moment [2]. Usually numerical methods are used to study a two-dimensional stress state of joints [3–5]. A simplified two-dimensional model of an overlap joint of two rectangular plates, based on the hypothesis of great stiffness of adherends in a direction perpendicular to the applied load, is described in [6]. This model is used to obtain an approximate analytical solution of the problem of a stress state of an adhesive joint of plates of varying width [7]. The adequacy of this model is validated by comparing the results of calculations performed with those carried out using the finite element method and also with experimental data [8].

The purpose of this paper is to solve the problem of the stress state of a joint containing unadhered regions located along the lateral boundaries of the adhesion region. This problem is being solved for the first time.

National Aerospace University–Kharkiv Aviation Institute, Kharkiv, 61000 Ukraine; *kurennov.ss@gmail.com.
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*Corresponding author.

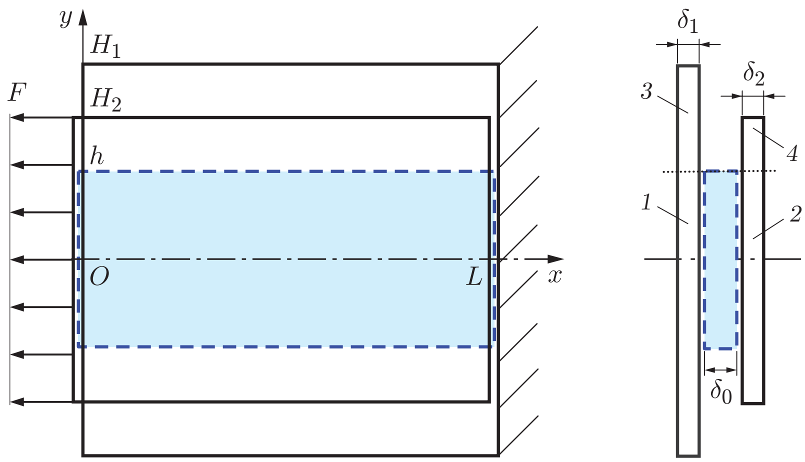

Fig. 1. Diagram of the adhesive joint: the plate regions adjacent to the adherend (*1* and *2*) and the plate regions located beyond the adherend (*3* and *4*).

FORMULATION OF THE PROBLEM AND EQUILIBRIUM EQUATIONS

A symmetric adhesive joint of two rectangular plates is under consideration (Fig. 1). The Ox axis is an axis of symmetry. Due to symmetry of the problem, there is no bending in the joint plane. The joint length is L, the widths of the jointed plates are $2H_1$ and $2H_2$, the width of the adhesion region is $2h$, and the thicknesses of the jointed plates and the adherend are δ_1, δ_2, and δ_0, respectively. It is assumed that displacements and stresses are uniformly distributed over the layer thickness. The jointed plates are assumed to be absolutely stiff in the direction of the Oy axis. This model occupies an intermediate position between beam models in structural mechanics and more accurate models of the elasticity theory and is first described by V. V. Vasil'ev (see, e.g., [9, 10]). Consequently, the elements of both load-bearing layers displace only in a longitudinal direction. The displacements of the elements of the first and second layers are denoted by U_1 and U_2 in the adhesion region and U_3 and U_4 in protruding sections, respectively.

The equilibrium equations of the elements of the load-bearing layers in the adhesion region $y \in [-h; h]$ have the form

$$\tau + \frac{\partial N_1}{\partial x} + \frac{\partial q_1}{\partial y} = 0, \qquad -\tau + \frac{\partial N_2}{\partial x} + \frac{\partial q_2}{\partial y} = 0, \tag{1}$$

where N_m and q_m ($m = 1, 2$) denote normal (in the longitudinal direction) and tangential forces in the load-bearing layer m, and τ refers to tangential stresses in the adherend in the longitudinal direction.

Cauchy's relations in the case of vanishing of longitudinal displacements take the form

$$N_m = \delta_m E_m \frac{\partial U_m}{\partial x}, \qquad q_m = \delta_m G_m \frac{\partial U_m}{\partial y}, \qquad m = 1, 2,$$

where E_m denotes the elastic modulus of the layer m in the direction of the Ox axis, and G_m is the shear modulus of the layer m in the plane xOy.

Tangential stresses in the adherend are proportional to the displacement difference:

$$\tau = P_0(U_2 - U_1). \tag{2}$$

Here P_0 denotes the shear stiffness of the adherend, calculated in the expression

$$P_0 = \frac{G_0}{\delta_0} \quad \text{or} \quad P_0 = \Big(\frac{\delta_0}{G_0} + \frac{\delta_1}{2G_1^T} + \frac{\delta_2}{2G_2^T}\Big)^{-1},$$

G_0 is the elastic modulus of the adherend, and G_m^T is the shear modulus of the load-bearing layers in the transverse direction.

The above-given relations are substituted into Eqs. (1), which yields the system [6]

$$\alpha_1\Big(\frac{\partial^2 U_1}{\partial x^2} + \mu_1 \frac{\partial^2 U_1}{\partial y^2}\Big) - U_1 + U_2 = 0, \qquad \alpha_2\Big(\frac{\partial^2 U_2}{\partial x^2} + \mu_2 \frac{\partial^2 U_2}{\partial y^2}\Big) + U_1 - U_2 = 0, \tag{3}$$

where $\alpha_m = E_m\delta_m/P_0$ and $\mu_m = G_m/E_m$.

The layer displacements beyond the adhesion region are described in view of the above-given hypotheses by the following equations [9, 10]:

$$\frac{\partial^2 U_3}{\partial x^2} + \mu_1 \frac{\partial^2 U_3}{\partial y^2} = 0, \qquad \frac{\partial^2 U_4}{\partial x^2} + \mu_2 \frac{\partial^2 U_4}{\partial y^2} = 0. \tag{4}$$

With account for symmetry, the boundary conditions can be formulated as follows:

$$\frac{\partial U_1}{\partial y}\Big|_{y=0} = \frac{\partial U_2}{\partial y}\Big|_{y=0} = \frac{\partial U_3}{\partial y}\Big|_{y=H_1} = \frac{\partial U_4}{\partial y}\Big|_{y=H_2} = 0; \tag{5}$$

$$N_2\Big|_{x=0} = E_2\delta_2 \frac{\partial U_1}{\partial x}\Big|_{x=0} = F(y), \qquad N_4\Big|_{x=0} = E_2\delta_2 \frac{\partial U_4}{\partial x}\Big|_{x=0} = F(y); \tag{6}$$

$$\frac{\partial U_1}{\partial x}\Big|_{x=0} = \frac{\partial U_3}{\partial x}\Big|_{x=0} = \frac{\partial U_2}{\partial x}\Big|_{x=L} = \frac{\partial U_4}{\partial x}\Big|_{x=L} = 0; \tag{7}$$

$$U_1\Big|_{x=L} = U_3\Big|_{x=L} = 0. \tag{8}$$

Matching conditions have the form

$$\frac{\partial U_m}{\partial y}\Big|_{y=h} = \frac{\partial U_{m+2}}{\partial y}\Big|_{y=h}, \qquad U_m\Big|_{y=h} = U_{m+2}\Big|_{y=h}, \qquad m = 1, 2. \tag{9}$$

The conditions (5) are the conditions of vanishing of tangential stresses in the load-bearing layers on the axis of symmetry and along the free edges.

CONSTRUCTION OF THE SOLUTION

The first equation of system (3) can be applied to express displacements of the second layer via displacements of the first one:

$$U_2 = U_1 - \alpha_1\Big(\frac{\partial^2 U_1}{\partial x^2} + \mu_1 \frac{\partial^2 U_1}{\partial y^2}\Big). \tag{10}$$

Equation (10) is substituted into the second equation of system (3), which yields

$$\beta_1 \frac{\partial^4 U_1}{\partial x^4} + \beta_2 \frac{\partial^4 U_1}{\partial x^2\,\partial y^2} + \beta_3 \frac{\partial^4 U_1}{\partial y^4} - \beta_4 \frac{\partial^2 U_1}{\partial x^2} - \beta_5 \frac{\partial^2 U_1}{\partial y^2} = 0, \tag{11}$$

where $\beta_1 = \alpha_1\alpha_2$, $\beta_2 = (\mu_1 + \mu_2)\alpha_1\alpha_2$, $\beta_3 = \mu_1\mu_2\alpha_1\alpha_2$, $\beta_4 = \alpha_1 + \alpha_2$, and $\beta_5 = \alpha_1\mu_1 + \alpha_2\mu_2$.

In [6], the variable separation method was used to obtain the general solution of Eq. (11) (provided that $\mu_1 = \mu_2$) and to show that it could be represented in the form of the sum

$$U_m = W_m(x) + V_m(x, y),$$

where $W_m(x)$ is the classic one-dimensional Volkersen solution [1]. In turn, the function $V_m(x, y)$ is a superposition of partial solutions (10) and (11), which could be represented in the form of linear combinations of functions $e^{\pm\lambda x}\sin ky$ and $e^{\pm\lambda x}\cos ky$ (as well as the solution of Eq. (4) [9]). The solution of this problem is constructed using the same solution structure.

The partial solution $e^{\pm\lambda x}\sin ky$ is substituted into Eq. (11), which results in an algebraic equation that relates λ and k:

$$\beta_3 k^4 + (\beta_5 - \beta_2\lambda^2)k^2 + \beta_1\lambda^4 - \beta_4\lambda^2 = 0. \tag{12}$$

It follows from Eq. (12) that each value of $\pm\lambda$ corresponds to four values of k, which can be written as $\pm k_1(\lambda)$ and $\pm k_2(\lambda)$. Consequently, the partial solution of Eq. (11), which corresponds, for example, to a positive value of λ, has the form

$$V_1^* = e^{\lambda x}(S_1 \sin k_1 y + C_1 \cos k_1 y + S_2 \sin k_2 y + C_2 \cos k_2 y),$$

where C_m and S_m are arbitrary constants.

Equation (10) yields

$$V_2^* = \mathrm{e}^{\lambda x}(S_1\gamma_1 \sin k_1 y + C_1\gamma_1 \cos k_1 y + S_2\gamma_2 \sin k_2 y + C_2\gamma_2 \cos k_2 y),$$

where $\gamma_m = 1 - \alpha_1(\lambda^2 - \mu_1 k_m^2)$, $m = 1, 2$.

It is assumed that the partial solutions of expressions (3) and (4) satisfy the boundary conditions (5) on boundaries $y = 0$, $y = H_1$, and $y = H_2$ and the boundary condition (9) on a boundary $y = h$. The first two conditions (5) yield $S_1 = S_2 = 0$. Considering that the partial solutions are also valid for negative values of λ, the partial solutions can be written as

$$V_1^* = [A\cosh\,\lambda x + B\cosh\,\lambda(x - L)]Y^{(1)}, \qquad V_2^* = [A\cosh\,\lambda x + B\cosh\,\lambda(x - L)]Y^{(2)},$$

$$Y^{(1)} = C_1 \cos k_1 y + C_2 \cos k_2 y, \qquad Y^{(2)} = C_1\gamma_1 \cos k_1 y + C_2\gamma_2 \cos k_2 y,$$

where A, B, C_1, and C_2 are arbitrary constants.

The partial solutions of the Poisson equation (4), which satisfy the conditions (5), have the form

$$V_3^* = [A^{(3)}\cosh\,\Lambda x + B^{(3)}\cosh\,\Lambda(x - L)]\cos\left(\Lambda(y - H_1)/\sqrt{\mu_1}\,\right),$$

$$V_4^* = [A^{(4)}\cosh\,\Omega x + B^{(4)}\cosh\,\Omega(x - L)]\cos\left(\Omega(y - H_2)/\sqrt{\mu_2}\,\right),$$

where Λ and Ω are variable separation constants, and $A^{(3)}$, $A^{(4)}$, $B^{(3)}$, and $B^{(4)}$ are arbitrary constants.

The one-dimensional solutions (3) and (4) have the form

$$W_m = A_0 x + B_0 + d_m(A_0^{(1)}\cosh\,(x/\sqrt{\beta_4}\,) + B_0^{(1)}\cosh\,((x - L)/\sqrt{\beta_4}\,)),$$

$$W_3 = a_0^{(1)}x + b_0^{(1)}, \qquad W_4 = a_0^{(2)}x + b_0^{(2)},$$

where A_0, B_0, a_0, and b_0 are arbitrary constants, $d_1 = 1$, $d_2 = -\alpha_1/\alpha_2$, and $m = 1, 2$.

It follows from the condition (9) that

$$W_1 = W_3, \qquad W_2 = W_4, \qquad V_1^*(x, h) = V_3^*(x, h), \qquad V_2^*(x, h) = V_4^*(x, h),$$

$$\frac{\partial V_1^*}{\partial y}\Big|_{y=h} = \frac{\partial V_3^*}{\partial y}\Big|_{y=h}, \qquad \frac{\partial V_2^*}{\partial y}\Big|_{y=h} = \frac{\partial V_4^*}{\partial y}\Big|_{y=h}.$$

Thus,

$$A_0^{(1)} = B_0^{(1)} = 0, \quad a_0^{(m)} = A_0, \quad b_0^{(m)} = B_0, \quad A^{(k)} = C_k A, \quad B^{(k)} = C_k B,$$

where C_3 and C_4 denotes the coefficients relating $A^{(3)}$, $B^{(3)}$ and $A^{(4)}$, $B^{(4)}$ with coefficients A and B. In order to determine $C_1, \ldots, C_4$, a system of homogeneous linear equations is obtained:

$$A\boldsymbol{C} = 0. \tag{13}$$

Here

$$A = \begin{pmatrix} c_1 & c_2 & -c_3 & 0 \\ \gamma_1 c_1 & \gamma_2 c_2 & 0 & -c_4 \\ k_1 s_1 & k_2 s_2 & -\lambda\mu_1^{-0,5} s_3 & 0 \\ k_1\gamma_1 s_1 & k_2\gamma_2 s_2 & 0 & -\lambda\mu_2^{-0,5} s_4 \end{pmatrix}, \qquad \boldsymbol{C} = \begin{pmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{pmatrix},$$

$$c_m = \cos k_m h, \quad s_m = \sin k_m h, \quad c_{m+2} = \cos\frac{\lambda(h - H_m)}{\sqrt{\mu_m}}, \quad s_{m+2} = \sin\frac{\lambda(h - H_m)}{\sqrt{\mu_m}}.$$

System (13) has a nontrivial solution provided that

$$\det A(\lambda) = 0. \tag{14}$$

Equation (14) has an infinite countable set of real roots λ_n, which correspond to $k_{m,n} = k_m(\lambda_n)$ and $\gamma_{m,n} = \gamma_m(\lambda_n)$ and constants $C_{1,n}$, $C_{2,n}$, $C_{3,n}$, and $C_{4,n}$, determined from system (13) with an accuracy of up to an arbitrary factor.

Thus, displacements $U_1, \dots, U_4$ can be written as

$$U_j = A_0 x + B_0 + \sum_{n=1}^{\infty} \Big[A_n X_n^{(1)}(x) + B_n X_n^{(2)}(x) \Big] Y_n^{(j)}(y), \tag{15}$$

where

$$j = 1, \dots, 4, \qquad X_n^{(1)} = \frac{\cosh\,(\lambda_n x)}{\lambda_n \sinh\,(\lambda_n L)}, \qquad Y_n^{(1)} = C_{1,n} \cos k_{1,n} y + C_{2,n} \cos k_{2,n} y,$$

$$X_n^{(2)} = \frac{\cosh\,(\lambda_n (x-L))}{\lambda_n \sinh\,(\lambda_n L)}, \qquad Y_n^{(2)} = C_{1,n}\gamma_{1,n} \cos k_{1,n} y + C_{2,n}\gamma_{2,n} \cos k_{2,n} y,$$

$$Y_n^{(3)} = C_{3,n} \cos\,(\lambda_n (y - H_1)/\sqrt{\mu_1}\,), \qquad Y_n^{(4)} = C_{4,n} \cos\,(\lambda_n (y - H_2)/\sqrt{\mu_2}\,);$$

the factors $\lambda_n \sinh\,(\lambda_n L)$ in the denominator are introduced to simplify the study of convergence solutions.

The coefficients $C_{1,n}$, $C_{2,n}$, $C_{3,n}$, and $C_{4,n}$ are determined with an accuracy of up to an arbitrary factor, so an additional condition is introduced for them (function normalization):

$$\int_0^h [Y_n^{(1)}]^2\,dx + \int_0^h [Y_n^{(2)}]^2\,dx + \int_h^{L_1} [Y_n^{(3)}]^2\,dx + \int_h^{L_2} [Y_n^{(4)}]^2\,dx = 1.$$

Thus, the boundary conditions (5) and (9) are fulfilled exactly. The coefficients A_0, A_n, B_0, and B_n are determined from the boundary conditions (6)–(8). These boundary conditions are used to obtain the following equations

$$A_0 - \sum_{n=1}^{\infty} B_n Y_n^{(1)} = 0, \qquad A_0 - \sum_{n=1}^{\infty} B_n Y_n^{(2)} = \frac{F(y)}{E_2\delta_2}, \qquad y \in (0; h),$$

$$A_0 - \sum_{n=1}^{\infty} B_n Y_n^{(3)} = 0, \qquad A_0 L + B_0 + \sum_{n=1}^{\infty} [A_n \theta_n + B_n \varkappa_n] Y_n^{(3)} = 0, \qquad y \in (h; H_1),$$

$$A_0 - \sum_{n=1}^{\infty} B_n Y_n^{(4)} = \frac{F(y)}{E_2\delta_2}, \qquad A_0 + \sum_{n=1}^{\infty} A_n Y_n^{(4)} = \frac{F(y)}{E_2\delta_2}, \qquad y \in (h; H_2),$$

$$A_0 + \sum_{n=1}^{\infty} A_n Y_n^{(1)} = 0, \qquad A_0 L + B_0 + \sum_{n=1}^{\infty} [A_n \theta_n + B_n \varkappa_n] Y_n^{(1)} = 0, \qquad y \in (0; h),$$

$$\theta_n = \frac{\cosh\,(\lambda_n L)}{\lambda_n \sinh\,(\lambda_n L)}, \qquad \varkappa_n = \frac{1}{\lambda_n \sinh\,(\lambda_n L)}, \qquad f(y) = \frac{F(y)}{E_2\delta_2}, \qquad y \in (0; H_2).$$

Summation is limited by some number of terms N, and the above-given boundary conditions are written in vector form

$$A_0 \boldsymbol{H}^{(1)} + B_0 \boldsymbol{H}^{(2)} + \sum_{n=1}^{N} A_n \boldsymbol{W}_n^{(1)}(y) + \sum_{n=1}^{N} B_n \boldsymbol{W}_n^{(2)}(y) - \boldsymbol{R} = 0, \tag{16}$$

where

$$\boldsymbol{H}^{(1)} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ L \\ 1 \\ 1 \\ 1 \\ L \end{pmatrix}, \ \boldsymbol{H}^{(2)} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \ \boldsymbol{W}_n^{(1)} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \theta_n Y_n^{(3)} \\ 0 \\ Y_n^{(4)} \\ Y_n^{(2)} \\ \theta_n Y_n^{(1)} \end{pmatrix}, \ \boldsymbol{W}_n^{(2)} = \begin{pmatrix} -Y_n^{(1)} \\ -Y_n^{(2)} \\ -Y_n^{(3)} \\ \varkappa_n Y_n^{(3)} \\ -Y_n^{(4)} \\ 0 \\ 0 \\ \varkappa_n Y_n^{(1)} \end{pmatrix}, \ \boldsymbol{R} = \begin{pmatrix} 0 \\ f \\ 0 \\ 0 \\ f \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Function systems $Y_n^{(1)}, \dots, Y_n^{(4)}$ are not orthogonal on the corresponding intervals. Using these functions can fulfill eight independent boundary conditions by accordingly choosing coefficients A_0, B_0, A_n, and B_n. The coefficients are determined from the condition that the left part of Eq. (16) is orthogonal to linearly independent vector functions $\boldsymbol{H}^{(1)}$, $\boldsymbol{H}^{(2)}$, $\boldsymbol{W}_n^{(1)}$, $\boldsymbol{W}_n^{(2)}$, $n = 1, \dots, N$. For this purpose, a scalar product is introduced:

$$(\boldsymbol{u}, \boldsymbol{v}) = \int_0^h u_1 v_1 \, dy + \int_0^h u_2 v_2 \, dy + \int_h^{H_1} u_3 v_3 \, dy + \int_h^{H_1} u_4 v_4 \, dy$$

$$+ \int_h^{H_2} u_5 v_5 \, dy + \int_h^{H_2} u_6 v_6 \, dy + \int_0^h u_7 v_7 \, dy + \int_0^h u_8 v_8 \, dy.$$

Here u_i and v_i denotes the corresponding components of the vectors $\boldsymbol{u}$ and $\boldsymbol{v}$.

This approach is applied to obtain a system of linear algebraic equations

$$\begin{pmatrix} \boldsymbol{M}^{(1,1)} & \boldsymbol{M}^{(1,2)} & \boldsymbol{M}^{(1,3)} \\ \boldsymbol{M}^{(2,1)} & \boldsymbol{M}^{(2,2)} & \boldsymbol{M}^{(2,3)} \\ \boldsymbol{M}^{(3,1)} & \boldsymbol{M}^{(3,2)} & \boldsymbol{M}^{(3,3)} \end{pmatrix} A = \begin{pmatrix} \boldsymbol{R}^{(1)} \\ \boldsymbol{R}^{(2)} \\ \boldsymbol{R}^{(3)} \end{pmatrix}, \tag{17}$$

where

$$A = (A_0, B_0, A_1, \dots, A_N,\ B_1, \dots, B_N)^{\mathrm{t}},$$

$$M_{i,j}^{(1,1)} = (\boldsymbol{H}^{(j)}, \boldsymbol{H}^{(i)}), \qquad M_{i,n}^{(1,2)} = (\boldsymbol{W}_n^{(1)}, \boldsymbol{H}^{(i)}), \qquad M_{i,n}^{(1,3)} = (\boldsymbol{W}_n^{(2)}, \boldsymbol{H}^{(i)}),$$

$$M_{k,n}^{(2,2)} = (\boldsymbol{W}_n^{(1)}, \boldsymbol{W}_k^{(1)}), \qquad M_{k,n}^{(2,3)} = (\boldsymbol{W}_n^{(2)}, \boldsymbol{W}_k^{(1)}), \qquad M_{k,n}^{(3,3)} = (\boldsymbol{W}_n^{(2)}, \boldsymbol{W}_k^{(2)}),$$

$$\boldsymbol{M}^{(2,1)} = [\boldsymbol{M}^{(1,2)}]^{\mathrm{t}}, \qquad \boldsymbol{M}^{(3,1)} = [\boldsymbol{M}^{(1,3)}]^{\mathrm{t}}, \qquad \boldsymbol{M}^{(3,2)} = [\boldsymbol{M}^{(2,3)}]^{\mathrm{t}}, \qquad i = 1, 2, \quad j = 1, 2,$$

$$R_i^{(1)} = (\boldsymbol{R}, \boldsymbol{H}^{(i)}), \quad R_k^{(2)} = (\boldsymbol{R}, \boldsymbol{W}_n^{(1)}), \quad R_k^{(3)} = (\boldsymbol{R}, \boldsymbol{W}_n^{(2)}), \quad n = 1, \dots, N,\ k = 1, \dots, N.$$

The functions $Y_n^{(m)}$ are linear combinations of trigonometric functions in which coefficients are limited due to the normalization condition introduced above. The arguments of trigonometric functions are expressions of the form $k_{m,n} y$, where $k_{m,n}$ with sufficient large values of m and n are proportional to n. Consequently, the nondiagonal elements of the matrix of system (17), located at a large distance from the diagonal, are proportional to $(nk)^{-1}$. Thus, the series comprised of squares of nondiagonal coefficients converges and the solution of the infinite system of linear equations (17) can be obtained using a reduction method.

MODEL PROBLEM

We determine the stress–strain state of an adhesive joint of identical (5×4)-cm plates ($H_1 = H_2 = H = 2$ cm, and $L = 5$ cm) with a thickness $\delta_0 = 4$ mm, made from an aluminum alloy with Young's moduli $E_1 = E_2 = 70$ GPa and shear moduli $G_1 = G_2 = 25.6$ GPa. The adherend parameters: $\delta_0 = 0.1$ mm and $G_0 = 0.5$ GPa. It is assumed that the second layer is subjected to a uniform load $F(y) = F = \text{const}$. Figure 2 illustrates the tangential stress distribution in the adherend 3 cm in width ($h = 1.5$ cm and $h/H = 0.75$). Clearly, there is an unadhered region along the lateral surface of the joint, which significantly increases the stress near the lateral surfaces of the adherend. The largest stress is observed in the angles of the adherend.

The calculations show that, as n increases, the coefficients A_n and B_n in expressions (15) nonmonotonously decrease and generally depend on the number of retained members of the series. The number of terms N in the series (15) significantly affects only the coefficients A_n and B_n with values of subscripts close to N. The calculations also show that the absolute values of A_n and B_n can be majorized by a numerical series whose coefficients decrease proportionally to n^{-2}. This indicates the stability of counting and rapid convergence of the solution.

In order to estimate the influence of relative width of the unadhered region h/H on the nonuniformity of stress distribution in the adherend over the joint width, it is necessary to calculate the ratio of the stresses in the

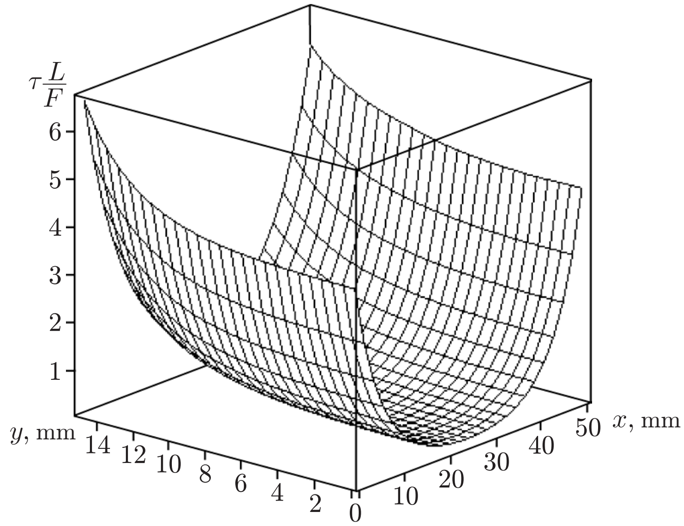

Fig. 2. Stress in the adherend.

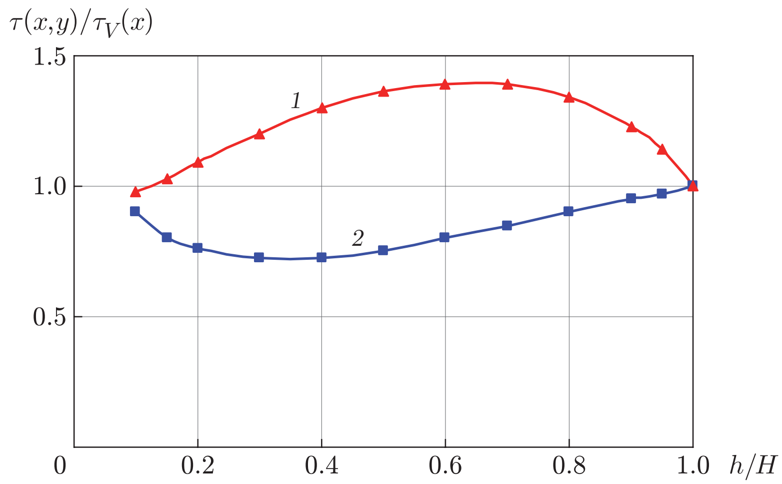

Fig. 3. Stresses near the lateral surface of the joint versus the relative width of the unadhered region: curves *1* and *2* refer to $\tau(0,h)/\tau_V(0)$ and) $\tau(0,0)/\tau_V(0)$, respectively.

adherend determined using the proposed two-dimensional model (2) to the stresses $\tau_V(x)$ obtained with the help of the one-dimensional Volkersen model. The dependence between $\tau(x,y)/\tau_V(x)$ and h/H at two points at the end of the joint is under consideration: on the axis of symmetry $(0;0)$ and in the angle of the adherend $(0;h)$.

The classical one-dimensional Volkersen model is described, for example, in [1]. In this case, with $E_1 = E_2 = E$ and $\delta_1 = \delta_2 = \delta$, the stresses in the adherend are calculated from the expression

$$\tau_V(x) = \frac{P_0 F H}{\xi E \delta h}\Big(\frac{1+\cosh \xi L}{\sinh \xi L}\cosh \xi x - \sinh \xi x\Big), \qquad \xi = \sqrt{\frac{2P_0}{E\delta}}\,.$$

The dependences of $\tau(x,y)/\tau_V(x)$ on h/H are shown in Fig. 3. Clearly, the stresses located on the axis of symmetry of the joint at a distance from lateral surfaces and calculated on the basis of the proposed model are smaller than the stresses calculated on the basis of the one-dimensional model, and the stresses located along lateral surfaces are larger. At the same time, the stresses near the lateral surface of the adherend can exceed the stresses calculated using the one-dimensional model by 40%. With $h \to 0$ or $h = H$, the stress distribution over the width of the adherend can be considered to be uniform.

CONCLUSIONS

The simplified two-dimensional mathematical model for the overlap adhesive joint was proposed, and the problem of the stress state in the adhesive joint of the plates of a varying width with unadhered regions in the longitudinal direction was solved. The solution of the model problem showed that the presence of unadhered regions near the lateral surface of the adhesive region significantly increased stresses in the angles of the adherend. The proposed model could be used for determining the stress state of an adherend in the presence of bending in the joint plane, calculating the stress state of overlap joints of force elements of structures with paneling, determining the stresses in the joints of elements of integral composite structures, etc.

REFERENCES

1. L. F. M. da Silva, P. J. C. das Neves, R. D. Adams, and J. K. Spelt, “Analytical Models of Adhesively Bonded Joints. 1. Literature Survey,” Int. J. Adhes. Adhesiv. **29**, 319–330 (2009).
2. N. G. Ryabenkov and Yu. P. Artyukhin, “Determining Adhesive Stresses in a Joint of Two Semi-Infinite Plates,” Issled. Teor. Plastin Obolochek **16**, 82–90 (1981).
3. P. Rapp, “Mechanics of Adhesive Joints As a Plane Problem of the Theory of Elasticity. 2. Displacement Formulation for Orthotropic Adherends,” Arch. Civil Mech. Eng. **15** (2), 603–619 (2015).
4. A. Barut, J. Hanauska, E. Madenci, and D. R. Ambur, “Analysis Method for Bonded Patch Repair of a Skin with a Cutout,” Compos. Struct. **55**, 277–294 (2002).
5. A. Chukwujekwu Okafor, N. Singh, U. E. Enemuoh, and S. V. Rao, “Design, Analysis and Performance of Adhesively Bonded Composite Patch Repair of Cracked Aluminum Aircraft Panels,” Compos. Struct. **71**, 258–270 (2005).
6. S. S. Kurennov, “A Simplified Two-Dimensional Model of Adhesive Joints. Nonuniform Load,” Mech. Composite Mater. **51** (4), 479–488 (2015).
7. S. S. Kurennov and E. V. Tanchik, “Calculation of the Stress State of an Adhesive Joint of Rectangular Plates of a Varying Width,” Vestn. Mosk. Aviats. Inst. **22** (2), 162–169 (2015).
8. S. S. Kurennov, “Stress State of Plates of a Varying Width. Approximate Theory and Experiment,” Visnik Zaporizkogo Nats. Univ., No. 1, 235–244 (2017).
9. V. V. Vasil’ev and Yu. V. Bokov, “Study of the Stress State of an Adhesive Joint of a Composite Material with a Metal Sheet,” in *Design, Calculation, and Testing of Structures Made of Composite Materials*, Vol. 7 (Central Aerohydrodynamic Institute, Moscow, 1979) [in Russian].
10. V. V. Vasil’ev and S. A. Lur’e, “Singularity in the Solution of a Plane Problem of the Elasticity Theory for a Cantilever Strip,” Izv. Ross. Akad. Nauk, Mekh. Tv. Tela, No. 4, 40–49 (2013).