ISSN 0021-8944, Journal of Applied Mechanics and Technical Physics, 2018, Vol. 59, No. 5, pp. 928–933.

SOLVING THE MOTION EQUATIONS OF A VISCOUS FLUID WITH A NONLINEAR DEPENDENCE BETWEEN A VELOCITY VECTOR AND SOME SPATIAL VARIABLES

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UDC 532.51

Abstract: It is shown that the classes of exact solutions of Navier–Stokes equations with a linear and inversely proportional dependence between velocity components and some spatial variables can be expanded by adding finite perturbations, being power and trigonometric series or their sections on one of the coordinates. An example of single integration of the three-dimensional motion equations a viscous fluid, reduced to an equation for the potential of two velocity components, is given.

Keywords: Navier–Stokes equations, exact solutions, separation of variables, overdetermined system.

DOI: 10.1134/S0021894418050218

INTRODUCTION

Currently, the exact solution (a class of exact solutions) of motion equations of a viscous incompressible fluid is usually understood as a representation of hydrodynamic fields, which reduces the Navier–Stokes equations to a closed system of differential equations with respect to unknowns that depend on one or two arguments, one of which is usually time.

A special place among viscous fluid dynamics equations is occupied by potential solutions or Beltrami solutions as the equations of the Navier–Stokes system are reduced in both cases to linear equations. However, the order of these linear equations is significantly lower than the order of original equations, which prevents from fulfilling necessary boundary conditions, e.g., a no-slip condition.

The greatest attention is paid to the classes of exact solutions that inherit the nonlinear properties of hydrodynamic equations and retain a high order. Most of these classes of solutions can be divided into three types: conical flows whose velocity vector is inversely proportional to the distance to the origin [1, 2], solutions with the linear dependence of some velocity vector components on two Cartesian coordinates [3, 4], and solutions with the linear dependences of some velocity vector components on one spatial variable (as a rule, an axial coordinate in a cylindrical system) [5–7]. Thus, these classes are such that one or two spatial variables are separated and the velocity vector components have the shape of a linear or inversely proportional dependence on these variables.

In this paper, it is shown that the above-mentioned classes can be supplemented so that some velocity vector components are written in a form of power and trigonometric series or their sections with respect to one of the variables.

The exact solutions of the Navier–Stokes equations were analyzed in [8, 9]. In [10], the construction of exact solutions of fluid dynamics equations using the method of group analysis of differential equations was described along with the concept of an exact solution from a theoretical-group point of view.

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1. SOLUTIONS LINEAR IN ONE COORDINATE

Viscous incompressible fluid equations in a Cartesian coordinate system allow for the representation of hydrodynamic fields with which one of the vector velocity components (V_x, V_y, V_z) is linearly dependent on a coordinate that corresponds to it 4]:

$$V_x = u(t,x,y), \quad V_y = v(t,x,y), \quad V_z = W(t,x,y) - zw(t,x,y),$$
$$P = p_0(t,x,y) + zp_1(t) + z^2 p_2(t)/2. \tag{1}$$

Here P is the pressure attributed to the medium density constant with kinematic viscosity ν, while the new unknown functions u, v, w, and p_0 satisfy the closed system

$$u_t + uu_x + vu_y = \nu\,\Delta u - p_{0x}, \quad v_t + uv_x + vv_y = \nu\,\Delta v - p_{0y}, \quad w = u_x + v_y,$$
$$w_t + uw_x + vw_y - w^2 = \nu\,\Delta w + p_2; \tag{2}$$

the function $W(t,x,y)$ is determined from the linear equation

$$W_t + uW_x + vW_y - wW = \nu\,\Delta W - p_1 \tag{3}$$

(the subscripts t, x, and y denote the partial derivatives of corresponding variables, and Δ is a two-dimensional Laplace operator).

System (2) is simpler than the original three-dimensional Navier–Stokes equations, but it is difficult for solution (1) to be regarded as a class of exact solutions of fluid dynamics equations in the above-mentioned meaning. As the first couple of equations (2) coincides in form with equations of two-dimensional motion of viscous fluid, the reduction can be continued in two directions: either consider quasipotential flows or apply the known classes of exact solutions of Navier–Stokes equations to expressions (2).

2. QUASIPOTENTIAL SOLUTIONS

Quasipotential solutions are solutions similar to (1) and such that the vector $(V_x, V_y, 0)$ has a potential: $u = \Phi_x$, $v = \Phi_y$. The first two equations (2) are reduced to an integral of the type of a Lagrange–Cauchy integral, whose presence allows obtaining explicit expressions for p_0, and an equation for the potential $\Phi(t,x,y)$ is derived from the remaining equations. As a result,

$$\Delta\Phi_t + \Phi_x\,\Delta\Phi_x + \Phi_y\,\Delta\Phi_y - \Delta\Phi\,\Delta\Phi = \nu\Delta\Delta\Phi + p_2,$$
$$w = \Delta\Phi, \qquad p_0 = p_{00} + \nu\,\Delta\Phi - (\Phi_x\Phi_x + \Phi_y\Phi_y)/2 - \Phi_t. \tag{4}$$

Thus, system (2) is integrated once [$p_{00}(t)$ is a function resulting from integration], and the description of three-dimensional quasipotential flows similar to (1) is reduced to finding the solutions of Eq. (4). For example, the solution of Eq. (4) satisfies the family of solutions

$$\Phi = Ax + By + Cx^2/2 + Dy^2/2 + Exy + F\sin(\lambda x + \mu y + \theta),$$

where the nine time functions A, B, C, D, E, F, λ, μ, and θ are bound by the equations

$$(C+D)_t - (C+D)^2 - (\lambda^2+\mu^2)^2 F^2 = p_2,$$

$$[(\lambda^2+\mu^2)F]_t + [\nu(\lambda^2+\mu^2) - 2(C+D)](\lambda^2+\mu^2)F = 0,$$

$$(\theta_t + \lambda A + \mu B)(\lambda^2+\mu^2)F = 0, \qquad \lambda_t + C\lambda + E\mu = 0, \qquad \mu_t + D\mu + E\lambda = 0.$$

3. GENERALIZED CLASS OF HIEMENZ SOLUTIONS

Two-dimensional Navier–Stokes equations allow for a class of exact Hiemenz solutions [11], which makes it possible to seek for the solution of the first two equations (2) in the form

$$u = u(t,x), \qquad v = V(t,x) + yg(t,x), \qquad p_0 = p(t,x) + yp_3(t) + y^2 p_4(t)/2.$$

As a result, system (2) reduces to a closed system of one-dimensional equations

$$w = u_x + g, \qquad g_t + ug_x + g^2 = \nu g_{xx} - p_4, \qquad w_t + uw_x - w^2 = \nu w_{xx} + p_2,$$
$$V_t + uV_x + gV = \nu V_{xx} - p_3, \qquad p = \nu u_x - \frac{u^2}{2} - \int u_t\, dx. \tag{5}$$

As in the case considered above, the function W is regarded as dependent on time t and two spatial variables and satisfying the equation that follows from expression (3):

$$W_t + uW_x + (V + yg)W_y - wW = \nu\, \Delta W - p_1.$$

The partial solutions of this equation can be sought in the form of polynomials with respect to the powers of the variable y with coefficients satisfying the chains of linear differential equations:

$$W = \sum_{n=0}^{N} y^n W_n(t,x),$$
$$W_{nt} + uW_{nx} + (ng - w)W_n + (n+1)(1-\delta_{n,N})VW_{n+1}$$
$$= \nu[W_{nxx} + (n+1)(1-\delta_{n,N-1})(1-\delta_{n,N})W_{n+2}] - p_1\delta_{n,0} \tag{6}$$

($\delta_{i,j}$ are Kronecker symbols).

System (6) should be solved sequentially, starting with the number N.

Thus, the hydrodynamics equations of the viscous fluid allow for the solutions in which one of the velocity vector components can be represented in the form of a polynomial of an arbitrary power N with respect to one of the Cartesian coordinates:

$$V_x = u(t,x), \qquad V_y = V(t,x) + yg(t,x), \qquad V_z = \sum_{n=0}^{N} y^n W_n(t,x) - zw(t,x),$$
$$P = p(t,x) + zp_1(t) + z^2 p_2(t)/2 + yp_3(t) + y^2 p_4(t)/2.$$

New unknowns satisfy Eqs. (5) and (6). For $N = 1$, the obtained result is a partial case of the solutions with two linearly separated variables [3].

4. ADDITIONS FOR THE CLASS OF CONICAL FLOWS

Let solutions (1) be considered in a polar coordinate system:

$$V_r = \frac{u(t,r,\varphi)}{r}, \quad V_\varphi = \frac{v(t,r,\varphi)}{r}, \quad V_z = W(t,r,\varphi) - \frac{z}{r^2}\, w(t,r,\varphi), \qquad P = \frac{p_0(t,r,\varphi)}{r^2} + zp_1(t) + \frac{z^2}{2}\, p_2(t). \tag{7}$$

Again, there is a system of nonlinear equations, closed with respect to u, v, w, and p_0, while the equations for the velocity vector components V_r and V_φ are similar in form to the equations of the two-dimensional Navier–Stokes equations (no incompressibility condition), which makes it possible to use known two-dimensional classes to construct three-dimensional solutions of hydrodynamic equations. The function W satisfies the linear equation in partial derivatives with variable coefficients. Below, there are two cases of reduction of the system for u, v, w, and p_0 to ordinary differential equations and one-dimensional equations in partial derivatives, and the function W is sought in the form of a function with separated variables.

Following [12, 13], the next case under consideration is the one where the unknowns u, v, w, and p_0 are independent of a self-simulated variable $\xi = m \ln r + n\varphi$. The partial solutions of the equation with respect to W

are sought in the form $W(r,\xi) = r^{-k}W_k(\xi)$ (n is an integer, and m and k are real numbers). Thus, there is a class of exact solutions of the Navier–Stokes equations, such that the z component of the velocity can be represented in the form of a series or its section in the powers of a radial variable:

$$V_r = \frac{u(\xi)}{r}, \quad V_\varphi = \frac{v(\xi)}{r}, \quad V_z = \sum_k r^{-k}W_k(\xi) - \frac{z}{r^2}\,w(\xi), \quad P = \frac{p_0(\xi)}{r^2} + zp_1(t). \tag{8}$$

The new unknowns satisfy the system

$$(mu+nv)u' - u^2 - v^2 = 2p_0 - mp_0' + \nu[(m^2+n^2)u'' - 2(mu+nv)'],$$

$$(mu+nv)v' = -np_0' + \nu[(m^2+n^2)v'' + 2(nu-mv)'], \qquad w = (mu+nv)',$$

$$(mu+nv)w' - w^2 - 2uw = \nu[(m^2+n^2)w'' - 4mw' + 4w],$$

$$(mu+nv)W_k' - (w+ku)W_k = \nu[(m^2+n^2)W_k'' + k^2W_k] - p_1\delta_{k,2},$$

where the prime denotes the differentiation with respect to ξ. With $W_k = 0$, the class (8) coincides with the class of solutions determined in [6]; with $m = 0$ (see [14]), it generalizes some subclass of three-dimensional conical flows [2].

Based on system (7), it is assumed that, in any plane perpendicular to the z axis, the flow is asymmetric, and the function $W(t,r,\varphi)$ is sought in the form of a trigonometric series or its section with respect to an azimuthal angle φ:

$$V_r = \frac{u(t,r)}{r}, \qquad V_\varphi = \frac{v(t,r)}{r}, \qquad P = \frac{p_0(t,r)}{r^2} + zp_1(t) + \frac{z^2}{2}\,p_2(t),$$

$$V_z = \sum_n\,[W_{1n}(t,r)\sin(n\varphi) + W_{2n}(t,r)\cos(n\varphi)] - \frac{z}{r^2}\,w(t,r,\varphi).$$

The new unknowns satisfy the system

$$r^2u_t + ru_ru - u^2 - v^2 = 2p_0 - rp_{0r} + \nu[r(ru_r)_r - 2ru_r], \qquad r^2v_t + rv_ru = \nu[r(rv_r)_r - 2rv_r], \qquad w = ru_r,$$

$$r^2w_t + rw_ru - w^2 - 2uw = \nu[r(rw_r)_r - 4rw_r + 4w] + p_2r^4,$$

$$r^2W_{1nt} + rW_{1nr}u - nvW_{2n} - wW_{1n} = \nu[r(rW_{1nr})_r - n^2W_{1n}],$$

$$r^2W_{2nt} + rW_{2nr}u + nvW_{1n} - wW_{2n} = \nu[r(rW_{2nr})_r - n^2W_{2n}] - p_1r^2\delta_{n,0},$$

where the subscripts t and r denote differentiation of the corresponding variable, and $n = 0,1,\dots$. The hydrodynamic fields of the form (8) generalize the class of solutions with a velocity field that linear on the axial coordinate z [5].

5. EXAMPLE OF AN OVERDETERMINED SYSTEM

Kármán solution, describing the flow of a viscous fluid created by the rotation of a disk [11], belongs to the class of hydrodynamic solutions with the linear dependence of some velocity vector components on two coordinates [3, 15]. Two examples of constructing new solutions of the Navier–Stokes equations with separated variables on the basis of Kármán class are under consideration. A cylindrical coordinate system is used to represent hydrodynamic fields in the form

$$V_r = \frac{U(t,z)}{r} + ru_z, \qquad V_\varphi = \frac{V(t,z)}{r} + rv(t,z), \qquad V_z = -2u(t,z),$$
$$P = 2\Big(\int u_t\,dz - u^2 - \nu u_z\Big) + p_1(t)\ln r - \frac{r^2}{2}\,p_2(t) - \frac{r^{-2}}{2}\,p_3^2(t) \tag{9}$$

(the subscripts t and z denote differentiations with respect to corresponding variables). The new unknowns satisfy the overdetermined system of equations

$$u_{tz} + u_z u_z - 2uu_{zz} - v^2 = \nu u_{zzz} + p_2, \qquad v_t + 2vu_z - 2uv_z = \nu v_{zz}; \tag{10}$$

$$U_t - 2uU_z - 2vV = \nu U_{zz} - p_1, \qquad V_t - 2uV_z + 2vU = \nu V_{zz}, \qquad U^2 + V^2 = p_3^2. \tag{11}$$

The isolated subsystem (10) is a closed system, the passive subsystem (11) is overdetermined, and its trivial solution $U = 0$, $V = 0$, $p_1 = 0$, $p_3 = 0$ transforms solution (9) into Kármán class. Two examples of closured are given below without conducting a general compatibility analysis.

Let $v = 0$, $U(t)$, $V = \text{const}$, $p_1 = -U_t$, and $p_3^2 = U^2 + V^2$, then system (10) and (11) is compatible and solution (9) describes some axisymmetric viscous flow in the background of a nonstationary potential vortex source.

Another example of compatibility of system (10) and (11) is less obvious. With $U = S_1 u_{zz}$, $V = S_1 v_z$, $p_1 = 0$, and $S_1 = \text{const}$, the first two equations (11) coincide with the differentiated system (10). The analysis of compatibility of the remaining equations yields a family of solutions $u(t,z)$ and $v(t,z)$ containing four time functions A, B, p_2, and p_3 and constants S_1 and S_2, bound by two equations

$$u = \frac{B_t}{2A} - \frac{S_2}{A^2} + \frac{A_t}{2A}\,z - \frac{p_3}{A^2 S_1}\sin(Az+B), \qquad v = \frac{S_2}{A} + \frac{p_3}{AS_1}\sin(Az+B),$$

$$p_{3t} = -\nu p_3 A^2, \qquad p_2 = \frac{A_{tt}}{2A} - \Big(\frac{A_t}{2A}\Big)^2 - \Big(\frac{S_2}{A}\Big)^2 + \Big(\frac{p_3}{AS_1}\Big)^2.$$

CONCLUSIONS

It is shown that the Navier–Stokes equations allow for exact solutions representing the superposition of the main flow, which is described within the framework of one of the classes of exact solutions with a linear or inversely proportional dependence between the velocity and some spatial variables, and the secondary flow with the nonlinear dependence of the velocity on one of the coordinates, which evolves in the background of the primer flow. In all the cases under consideration, except for the latter, the secondary flow is described by the linear equations with coefficients calculated in the description of the main flow. The example of a perturbation that satisfies the nonlinear overdetermined system is described. The solutions obtained supplement the known classes of exact solutions of hydrodynamic equations.

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