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DETERMINATION OF THE INHOMOGENEOUS PRELIMINARY STRESS–STRAIN STATE IN A PIEZOELECTRIC DISK

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Abstract: The problem of steady radial vibrations of a thin electroelastic hollow disk in the presence of a preliminary inhomogeneous plane stress–strain state is solved. Vibrations are induced by applying a potential difference across the electrodes placed on the end surfaces of the disk. Equations of the vibrations and boundary conditions are formulated. The preliminary stress state corresponding to the solution of the Lamé problem was investigated. The direct problem of determining the displacement function is solved numerically by the shooting method. The inverse problem of determining a pre-stress parameter from the change in the natural frequency of the disk is formulated and solved. The accuracy of determining the prestressed state for initial data specified with an error is analyzed.

Keywords: piezoelectric disk, non-uniform prestresses, vibrations, acoustic method, inverse coefficient problem, natural frequencies.

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INTRODUCTION

Modern piezomaterials are widely used in machine building (pressure and vibration sensors, ultrasonic converters, accelerometers, etc.), acoustics (piezoelectric transducers, microphones, etc.), optics (lens positioning systems), robotics, medicine (ultrasonic diagnostics of tissues and organs), in the development of space vehicles etc. As a rule, piezoelements in the form of a plate, disk, ring, rod or cylinder are used.

Investigation of resonant frequencies and other mechanical characteristics of piezoceramic bodies started in the second half of the 20th century. Adelman and Stavsky [1] obtained resonant and antiresonant frequencies for a thin disk and an infinite cylinder made of PZT-4 ceramic and steel. These results are used in calculations for composite transducers used as band-pass filters.

Problems of vibrations of piezoceramic bodies have also been studied by laser methods. In particular, Parali et al. [2] have investigated forced bending vibrations of a piezoceramic (PZT material) disk with the electroded end surfaces located on a metal plate. Test results show that the measurement error of resonance frequencies in the range from 0 to 40 kHz obtained using a directional laser is 1 Hz.

Recently, considerable attention has been paid to cylindrical piezoelements in the presence of prestresses (PSs). Arnold and Muhlen [3] have studied the influence of longitudinal PS of a cylindrical piezoceramic piezoelectric transducer on resonant and antiresonant frequencies. It is noted that the results of this study may be useful for developing new electroacoustic transducers.

For an adequate description of the dynamic behavior of prestressed bodies, the theory of small perturbations imposed on finite deformations is used [4]. Some studies use the hypothesis of homogeneity of PSs, but they are substantially inhomogeneous, as a rule, as they can arise as a result of latent loads or in a production process.

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There are several models describing the behavior of elastic and electroelastic bodies in the presence of PSs [5–7]. Investigation of the dynamic characteristics of a prestressed electroelastic body depends on the choice of the model. In the first stage, theoretical calculations of the main characteristics of the body are made and the possibility of their experimental measurement is assessed taking into account the characteristics of the equipment used. In the second stage, relevant experiments and comparison of the results obtained are performed.

One of the most effective methods of investigating the properties of inhomogeneous electroelastic bodies is the acoustic method [8]. Bogachev et al. [8], using the theory of integral Fredholm equations of the first and second kind, developed an iterative process for determining variable elastic and piezoelectric moduli from the chosen initial approximation.

In this paper, the general linearized formulation of the problem of the motion of an electroelastic body in the presence of a preliminary stress–strain state (PSSS) is presented in tensor form. Using this formulation, the direct problem of steady radial vibrations of a pre-stressed piezoelectric cylindrical disk of small thickness is formulated. The inverse coefficient problem of determining a parameter characterizing the PSSS using an additional information on the first natural frequency is formulated and solved.

1. FORMULATION OF THE DIRECT PROBLEM

On the basis of the results described in [7], we can formulate a linearized boundary-value problem of vibrations of an electroelastic body with allowance for preliminary stresses and strains in the metric of the configuration of the perturbed state:

$$\begin{gathered} \nabla\cdot T=\rho\ddot{\boldsymbol{u}},\qquad \nabla\cdot\boldsymbol{D}=0,\\ T=\sigma+\sigma\cdot\nabla\boldsymbol{u}^0+\sigma^0\cdot\nabla\boldsymbol{u},\qquad \sigma=C\odot\varepsilon-e\odot\boldsymbol{E},\qquad \boldsymbol{D}=e\odot\varepsilon+\hat{\varepsilon}\odot\boldsymbol{E},\\ \varepsilon=(\nabla\boldsymbol{u}+\nabla\boldsymbol{u}^{\mathrm{t}}+\nabla\boldsymbol{u}^0\cdot\nabla\boldsymbol{u}^{\mathrm{t}}+\nabla\boldsymbol{u}\cdot\nabla\boldsymbol{u}^{0\mathrm{t}})/2,\\ \boldsymbol{n}\cdot T\Big|_{S_\sigma}=\boldsymbol{P},\qquad \boldsymbol{u}\Big|_{S_u}=0,\qquad \boldsymbol{n}\cdot D\Big|_{S_D}=0,\qquad \varphi\Big|_{S_\pm}=\pm V_0. \end{gathered} \tag{1}$$

Here ∇ is the nabla-operator in the metric of the natural undeformed configuration in Cartesian coordinates, T is the asymmetric additional tensor of Piola stresses, $\boldsymbol{E}=-\nabla\varphi$ is the electric field strength vector, $\boldsymbol{D}$ is the electric induction vector, σ is the additional tensor of objective stresses, σ^0 is the PS tensor, C is the tensor of elastic constants, e is the tensor of piezoelectric constants, ε is the additional strain tensor, $\hat{\varepsilon}$ is the permittivity tensor, $\boldsymbol{u}$ is the additional vector of small displacements, $\boldsymbol{u}^0$ is the vector of preliminary displacements (PD), ρ is the density of the body material, $\boldsymbol{n}$ is the unit normal vector to the boundary of the body S, $\boldsymbol{P}$ is the vector of the active mechanical load applied to the body, and $\Delta\varphi=\varphi\Big|_{S_+}-\varphi\Big|_{S_-}=2V_0$ is the potential difference applied to the electrodes. The model formulated above does not take into account the mass forces and the density of free electrical charges in the current configuration and the nonlinear and electrostriction coefficients in the constitutive relations.

Using problem (1), we can investigate the behavior of a prestressed piezoelectric body with inhomogeneous properties by specifying the laws of variation of the elastic and electrical moduli.

We consider the problem of steady radial vibrations of a prestressed electroelastic disk of small thickness h in a cylindrical coordinate system (Fig. 1). The end surfaces of the disk $z=\pm h/2$ are electroded. Vibrations are induced by applying a potential difference $\pm V_0\,\mathrm{e}^{i\omega t}$.

We consider a plane inhomogeneous PSSS described by two components of the tensor σ^0 [$\sigma^0_{rr}=\sigma^0_{rr}(r)\neq 0$ and $\sigma^0_{\varphi\varphi}=\sigma^0_{\varphi\varphi}(r)\neq 0$], which satisfy the equilibrium equation, and one component of the vector $\boldsymbol{u}^0$ [$u^0_r=u^0_r(r)$]. We will assume that due to axial symmetry, the radial displacement component is a function of r: $u_r=u_r(r)$. The disk is made of PZT-4 piezoceramic with 6 mm hexagonal symmetry. In this case, the matrices of the components of the tensors C and e have the form [9]

$$C=\begin{pmatrix} c_{11} & c_{12} & c_{13} & 0 & 0 & 0\\ c_{12} & c_{11} & c_{13} & 0 & 0 & 0\\ c_{13} & c_{13} & c_{33} & 0 & 0 & 0\\ 0 & 0 & 0 & c_{44} & 0 & 0\\ 0 & 0 & 0 & 0 & c_{44} & 0\\ 0 & 0 & 0 & 0 & 0 & c_{66} \end{pmatrix},\qquad e=\begin{pmatrix} 0 & 0 & 0 & 0 & e_{15} & 0\\ 0 & 0 & 0 & e_{15} & 0 & 0\\ e_{31} & e_{31} & e_{33} & 0 & 0 & 0 \end{pmatrix}.$$

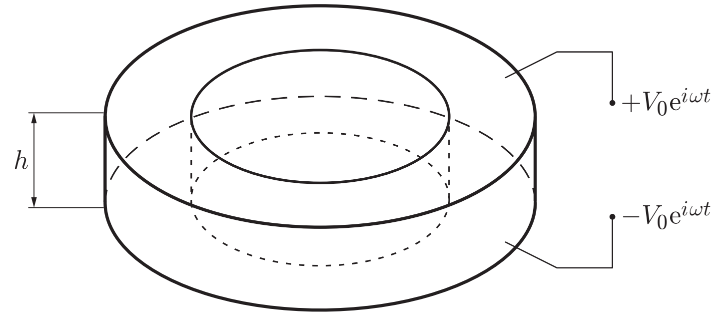

Fig. 1. Schematic of a thin electroelastic disk with electroded end surfaces.

Using problem (1), we write the equations and the boundary conditions of the problem for the disk after separating the time multiplier:

$$T_{rr,r} + (T_{rr} - T_{\varphi\varphi})/r + T_{zr,z} + \rho\omega^2 u_r = 0; \tag{2}$$

$$T_{rz,r} + T_{rz}/r + T_{zz,z} + \rho\omega^2 u_z = 0; \tag{3}$$

$$D_{r,r} + D_r/r + D_{z,z} = 0,$$

$$T_{rr} = \sigma_{rr}(1 + u^0_{r,r}) + u_{r,r}\sigma^0_{rr}, \qquad T_{\varphi\varphi} = \sigma_{\varphi\varphi}(1 + u^0_r/r) + u_r\sigma^0_{\varphi\varphi}/r,$$

$$T_{rz} = \sigma_{rz} + u_{z,r}\sigma^0_{rr}, \qquad T_{zr} = \sigma_{rz}(1 + u^0_{r,r}), \qquad T_{zz} = \sigma_{zz},$$

$$\sigma_{rr} = c_{11}(1 + u^0_{r,r})u_{r,r} + c_{12}(1 + u^0_r/r)u_r/r + c_{13}u_{z,z} + e_{31}\varphi_{,z},$$

$$\sigma_{\varphi\varphi} = c_{12}(1 + u^0_{r,r})u_{r,r} + c_{11}(1 + u^0_r/r)u_r/r + c_{13}u_{z,z} + e_{31}\varphi_{,z},$$

$$\sigma_{rz} = \sigma_{zr} = c_{44}(u_{r,z} + u_{z,r}) + e_{15}\varphi_{,r} = (c_{44}u_z + e_{15}\varphi)_{,r}, \tag{4}$$

$$\sigma_{zz} = c_{13}(1 + u^0_{r,r})u_{r,r} + c_{13}(1 + u^0_r/r)u_r/r + c_{33}u_{z,z} + e_{33}\varphi_{,z},$$

$$D_r = -\hat{\varepsilon}_{11}\varphi_{,r} + e_{15}(u_{r,z} + u_{z,r}),$$

$$D_z = -\hat{\varepsilon}_{33}\varphi_{,z} + e_{31}((1 + u^0_{r,r})u_{r,r} + (1 + u^0_r/r)u_r/r) + e_{33}u_{z,z},$$

$$T_{rr}\Big|_{r=r_1,r_2} = 0, \qquad \varphi\Big|_{z=\pm h/2} = \pm V_0, \qquad D_r\Big|_{r=r_1,r_2} = 0.$$

Since, for the disk considered here, the generalized plane stress conditions are satisfied, we set $T_{zz} = \sigma_{zz} \approx 0$. Then, from (4) we express the displacement vector component as

$$u_z = -\frac{c_{13}}{c_{33}}\, z\Big((1 + u^0_{r,r})u_{r,r} + \Big(1 + \frac{u^0_r}{r}\Big)\frac{u_r}{r}\Big) - \frac{e_{33}}{c_{33}}\,\varphi.$$

Taking this representation into account, we write the components of the tensor σ and the vector $\boldsymbol{D}$ as

$$\sigma_{rr} = c_{11}^*(1 + u^0_{r,r})u_{r,r} + c_{12}^*(1 + u^0_r/r)u_r/r + e_{31}^*\varphi_{,z},$$

$$\sigma_{\varphi\varphi} = c_{12}^*(1 + u^0_{r,r})u_{r,r} + c_{11}^*(1 + u^0_r/r)u_r/r + e_{31}^*\varphi_{,z},$$

$$\sigma_{rz} = \sigma_{zr} = -z\,\frac{c_{44}c_{13}}{c_{33}}\Big[(1 + u^0_{r,r})u_{r,r} + \Big(1 + \frac{u^0_r}{r}\Big)\frac{u_r}{r}\Big]_{,r} + e_{15}^*\varphi_{,r},$$

$$D_r = -\hat{\varepsilon}_{11}^*\varphi_{,r} - z\,\frac{e_{15}c_{13}}{c_{33}}\Big[(1 + u^0_{r,r})u_{r,r} + \Big(1 + \frac{u^0_r}{r}\Big)\frac{u_r}{r}\Big]_{,r},$$

$$D_z = -\hat{\varepsilon}_{33}^*\varphi_{,z} + e_{31}^*\Big[(1 + u^0_{r,r})u_{r,r} + \Big(1 + \frac{u^0_r}{r}\Big)\frac{u_r}{r}\Big],$$

where

$$c_{11}^* = c_{11}(1 - c_{13}^2/(c_{11}c_{33})), \qquad c_{12}^* = c_{12}(1 - c_{13}^2/(c_{12}c_{33})),$$

$$e_{31}^* = e_{31}(1 - c_{13}e_{33}/(c_{33}e_{31})), \qquad e_{15}^* = e_{15}(1 - c_{44}e_{33}/(c_{33}e_{15})),$$

$$\hat{\varepsilon}_{11}^* = \hat{\varepsilon}_{11}(1 + e_{33}e_{15}/(c_{33}\hat{\varepsilon}_{11})), \qquad \hat{\varepsilon}_{33}^* = \hat{\varepsilon}_{33}(1 + e_{33}^2/(c_{33}\hat{\varepsilon}_{33})).$$

In accordance with the theory of the generalized stress state, we perform averaging over the thickness of the disk according to the rule

$$\langle Q\rangle = \frac{1}{h}\int_{-h/2}^{h/2} Q\,dz.$$

The equation of motion (2) for the averaged components of the tensor T takes the form

$$\langle T_{rr,r}\rangle + \frac{1}{r}\,(\langle T_{rr}\rangle - \langle T_{\varphi\varphi}\rangle) + \rho\omega^2 u_r = 0,$$

where

$$\langle T_{rr}\rangle = u_{r,r}(c_{11}^*(1 + u^0_{r,r})^2 + \sigma^0_{rr}) + c_{12}^*\Big(1 + \frac{u^0_r}{r}\Big)(1 + u^0_{r,r})\,\frac{u_r}{r} + \frac{2V_0e_{31}^*}{h}\,(1 + u^0_{r,r}),$$

$$\langle T_{\varphi\varphi}\rangle = u_{r,r}c_{12}^*\Big(1 + \frac{u^0_r}{r}\Big)(1 + u^0_{r,r}) + \frac{u_r}{r}\Big(c_{11}^*\Big(1 + \frac{u^0_r}{r}\Big)^2 + \sigma^0_{\varphi\varphi}\Big) + \frac{2V_0e_{31}^*}{h}\Big(1 + \frac{u^0_r}{r}\Big).$$

Here we take into account the boundary condition for the potential $\varphi\Big|_{z=\pm h/2} = \pm V_0$. Since the functions u_z and φ are odd with respect to the coordinate z, Eq. (3) is satisfied identically after averaging.

Thus, for the thin disk under consideration, the equations and boundary conditions for the radial displacement function and potential take the form

$$u_{r,rr}P_2 + u_{r,r}P_1 + u_r(\rho\omega^2 + P_0) + \frac{2V_0e_{31}^*}{h}\,F_0 = 0,$$

$$u_{r,r}P_2 + \Big[u_r\,\frac{c_{12}^*}{r}\Big(1 + \frac{u^0_r}{r}\Big) + \frac{2V_0e_{31}^*}{h}\Big](1 + u^0_{r,r})\Big|_{r=r_1,r_2} = 0; \tag{5}$$

$$\varphi_{,rr} + \frac{1}{r}\,\varphi_{,r} + \nu^2\varphi_{,zz} = -\frac{z\mu}{r}\,(rF_1)_{,r},$$

$$\varphi\Big|_{z=\pm h/2} = \pm V_0, \qquad -\hat{\varepsilon}_{11}^*\varphi_{,r} - z\,\frac{e_{15}c_{13}}{c_{33}}\,F_1\Big|_{r=r_1,r_2} = 0, \tag{6}$$

where

$$P_2 = c_{11}^*(1+u_{r,r}^0)^2 + \sigma_{rr}^0, \qquad P_1 = c_{11}^*(1+u_{r,r}^0)\Big(2u_{r,rr}^0 + \frac{1+u_{r,r}^0}{r}\Big) + \sigma_{rr,r}^0 + \frac{\sigma_{rr}^0}{r},$$

$$P_0 = \frac{c_{12}^*}{r}\Big(u_{r,rr}^0\Big(1+\frac{u_r^0}{r}\Big) + (1+u_{r,r}^0)\Big(\frac{u_{r,r}^0}{r} - \frac{u_r^0}{r^2}\Big)\Big) - \frac{c_{11}^*}{r^2}\Big(1+\frac{u_r^0}{r}\Big)^2 - \frac{\sigma_{\varphi\varphi}^0}{r^2},$$

$$F_0 = u_{r,rr}^0 + \frac{u_{r,r}^0}{r} - \frac{u_r^0}{r^2}, \quad F_1 = \Big((1+u_{r,r}^0)u_{r,r} + \Big(1+\frac{u_r^0}{r}\Big)\frac{u_r}{r}\Big)_{,r}, \quad \nu^2 = \frac{\hat{\varepsilon}_{33}^*}{\hat{\varepsilon}_{11}^*}, \quad \mu = \frac{c_{13}e_{15}}{c_{33}\hat{\varepsilon}_{11}^*}.$$

The solution of problem (5), (6) can be obtained in two stages. In the first stage, problem (5) for the function $u_r(r)$ is considered, and in the second stage, the distribution of the potential in solution (6) is determined.

We introduce the following dimensionless parameters and functions for the elastic field:

$$u_r(r) = U(\xi)r_2, \quad \xi = \frac{r}{r_2}, \quad \xi_0 = \frac{r_1}{r_2}, \quad u_r^0(r) = \tau\psi(\xi)r_2, \quad c = \frac{c_{12}^*}{c_{11}^*}, \quad k = -\frac{2e_{31}^*V_0}{hc_{11}^*},$$

$$\varkappa^2 = \frac{\rho\omega^2 r_2^2}{c_{11}^*}, \quad \frac{\sigma_{rr}^0(r)}{c_{11}^*} = \tau\phi(\xi), \quad \frac{\sigma_{\varphi\varphi}^0(r)}{c_{11}^*} = \tau\phi'(\xi)\xi + \tau\phi(\xi), \quad \tau = \frac{P}{c_{11}^*}.$$

Neglecting second-order terms, we write the problem for the function $U(\xi)$ as

$$U''(1+2\tau\psi'+\tau\phi) + U'\Big((1+2\tau\psi'+\tau\phi)' + \frac{1+2\tau\psi'+\tau\phi}{\xi}\Big)$$

$$+ U\Big[c\Big(1+\frac{\tau\psi}{\xi}+\tau\psi'\Big)'\frac{1}{\xi} - \Big(1+\frac{2\tau\psi}{\xi}+\xi\tau\phi'+\tau\phi\Big)\frac{1}{\xi^2} + \varkappa^2\Big] = k\Big(\tau\psi'' + \frac{\tau\psi'}{\xi} - \frac{\tau\psi}{\xi^2}\Big), \tag{7}$$

$$(1+2\tau\psi'+\tau\phi)U' + c\Big(1+\tau\psi'+\frac{\tau\psi}{\xi}\Big)\frac{U}{\xi} - k(1+\tau\psi')\Big|_{\xi=\xi_0,1} = 0.$$

Problem (7) reduces to a system of two first-order differential equations for the functions U and $Y = (1+2\tau\psi'+\tau\phi)U' + c(1+\tau\psi'+\tau\psi/\xi)U/\xi - k(1+\tau\psi')$. The solution of this system for the given functions ψ and ϕ and parameter τ is obtained numerically by the shooting method using the Maple package.

2. INVERSE PROBLEM OF DETERMINING THE PRE-STRESS STATE FROM DATA ON THE CHANGE IN NATURAL FREQUENCIES

Since piezoelements are widely used in various decices, it is necessary to know their electroelastic properties and stress–strain state. The resolution of modern measuring instruments is constantly increasing, making it possible to evaluate even minor changes in acoustic characteristics [10, 11].

In this paper, we consider the inverse coefficient problem of determining a parameter that characterizes the PSSS in an electroelastic disk from data on its natural vibration frequencies.

On the basis of problem (7), we consider free vibrations of the disk using short-circuited electrodes ($V_0 = 0$ and $k = 0$).

To linearize the problem, we introduce a formal parameter ε:

$$\psi = \varepsilon\psi_1, \quad \phi = \varepsilon\phi_1, \quad U = U_0 + \varepsilon U_1, \quad \varkappa^2 = \varkappa_0^2 + \varepsilon\,\Delta\varkappa.$$

The boundary-value problems corresponding to the zero and first powers of the parameter ε can be written as:

(1) for ε^0,

$$\Big(U_0' + \frac{U_0}{\xi}\Big)' + \varkappa_0^2 U_0 = 0; \tag{8}$$

$$U_0' + c\,\frac{U_0}{\xi}\Big|_{\xi=\xi_0,1} = 0; \tag{9}$$

(2) for ε^1,

$$\Big(U_1' + \frac{U_1}{\xi}\Big)' + \varkappa_0^2 U_1 + (GU_0')' + G\,\frac{U_0'}{\xi} + \Big(\frac{cM'}{\xi} - \frac{B}{\xi^2} + \Delta\varkappa\Big)U_0 = 0; \tag{10}$$

$$GU_0' + cM\,\frac{U_0}{\xi} + U_1' + c\,\frac{U_1}{\xi}\Big|_{\xi=\xi_0,1} = 0. \tag{11}$$

Here $G(\xi) = 2\tau\psi_1'(\xi) + \tau\phi_1(\xi)$, $M(\xi) = \tau\psi_1'(\xi) + \tau\psi_1(\xi)/\xi$, and $B(\xi) = 2\tau\psi_1(\xi)/\xi + \xi\tau\phi_1'(\xi) + \tau\phi_1(\xi)$.

The homogeneous boundary-value problem (8), (9) describes free vibrations of a piezoceramic disk in the absence of PSSS, and problem (10), (11) describes them in the presence of the disk. We note that the solution of problem (8), (9) is known and can be expressed analytically in terms of Bessel functions:

$$U_0(\xi) = C_1 J_1(\varkappa_0\xi) + C_2 Y_1(\varkappa_0\xi).$$

Here C_1 and C_2 are constants, $J_1(\varkappa_0\xi)$ is a Bessel function of the first kind of the first order, and $Y_1(\varkappa_0\xi)$ is a Bessel function of the second kind of the first order (Neumann function). Using boundary conditions (9), we construct a frequency equation for determining the value of the parameter $\varkappa_0$ proportional to the natural vibration frequency:

$$\Delta(\varkappa_0) = 0. \tag{12}$$

Here

$$\Delta = \Big(\varkappa_0 J_0(\varkappa_0\xi_0) + \frac{c-1}{\xi_0}\,J_1(\varkappa_0\xi_0)\Big)(\varkappa_0 Y_0(\varkappa_0) + (c-1)Y_1(\varkappa_0))$$

$$-\,(\varkappa_0 J_0(\varkappa_0) + (c-1)J_1(\varkappa_0))\Big(\varkappa_0 Y_0(\varkappa_0\xi_0) + \frac{c-1}{\xi_0}\,Y_1(\varkappa_0\xi_0)\Big).$$

The constants C_1 and C_2 are related by the formula

$$C_2 = -C_1\,\frac{\varkappa_0 J_0(\varkappa_0) + (c-1)J_1(\varkappa_0)}{\varkappa_0 Y_0(\varkappa_0) + (c-1)Y_1(\varkappa_0)}.$$

Thus, the vibration eigenmode $U_0(\xi)$ is determined up to the constant multiplier C_1 and has the form

$$U_0(\xi) = C_1\Big(J_1(\varkappa_0\xi) - \frac{\varkappa_0 J_0(\varkappa_0) + (c-1)J_1(\varkappa_0)}{\varkappa_0 Y_0(\varkappa_0) + (c-1)Y_1(\varkappa_0)}\,Y_1(\varkappa_0\xi)\Big).$$

To determine the correction $\Delta\varkappa$, we multiply the equations of motion (8) and (10) by the quantities $\xi U_1(\xi)$ and $\xi U_0(\xi)$, respectively, and subtract and integrate the resulting expression in the interval $[\xi_0, 1]$. After a series of transformations, we have the formula

$$\Delta\varkappa = \varkappa^2 - \varkappa_0^2 = \Big[cMU_0^2\Big|_{\xi_0}^{1} + \int_{\xi_0}^{1} GU_0^{2\prime}\xi\,d\xi - \int_{\xi_0}^{1}\Big(cM'U_0^2 - \frac{B}{\xi}\,U_0^2\Big)\,d\xi\Big] \Big/ \int_{\xi_0}^{1}\xi U_0^2\,d\xi. \tag{13}$$

This relation links the change in the natural frequency of radial vibrations of the piezoceramic disk with the functions M, G, and B characterizing the change in the PS and PD, and the shape of the free vibrations U_0.

Formula (13) can be used to estimate the change in the natural frequency as a function of the type and parameter of PSSS.

3. RESULTS OF NUMERICAL CALCULATIONS

Consider a disk of thickness $h = 0.001$ m with an inner radius $r_1 = 0.03$ m and an outer radius $r_2 = 0.05$ m ($\xi_0 = 0.6$). The density of the disk material is $\rho = 5250$ kg/m^3. The voltage across the electrodes is $V_0 = 250$ V. The material constants of PZT-4 piezoceramic have the following values [9]: $c_{11} = 13.9\cdot10^{10}$ N/m^2, $c_{12} = 7.78\cdot10^{10}$ N/m^2, $c_{13} = 7.43\cdot10^{10}$ N/m^2, $c_{33} = 11.5\cdot10^{10}$ N/m^2, $c_{44} = 2.56\cdot10^{10}$, $e_{15} = 12.7$ C/m^2, $e_{31} = -5.2$ C/m^2, $e_{33} = 15.1$ C/m^2, $\hat{\varepsilon}_{11} = 6.45\cdot10^{-9}$ F/m, and $\hat{\varepsilon}_{33} = 5.62\cdot10^{-9}$ F/m.

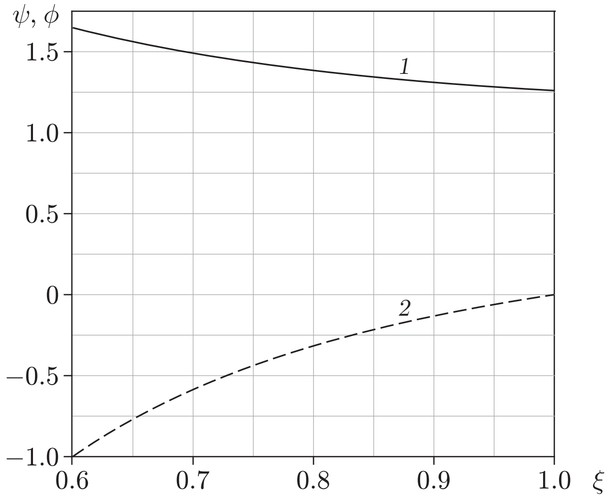

Fig. 2.

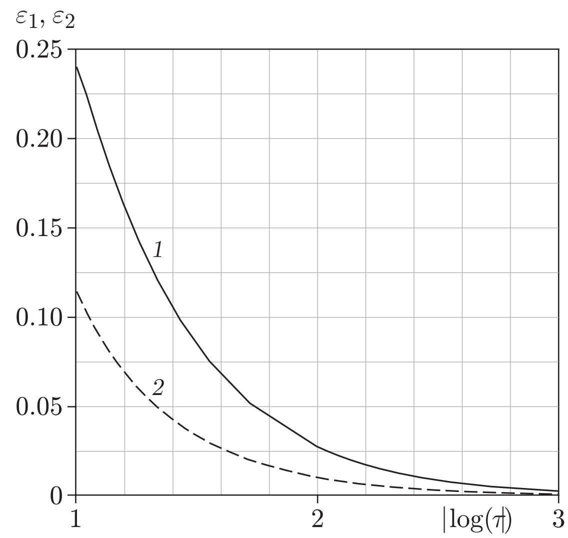

Fig. 3.

Fig. 2. Dependences of $\psi(\xi)$ (*1*) and $\phi(\xi)$ (*2*).

Fig. 3. Dependences of $\varepsilon_1(\tau)$ (*1*) and $\varepsilon_2(\tau)$ (*2*).

Table 1. Reconstructed values of τ_R

τ	τ_R	δ_τ, %	$\varkappa_1$	$\delta_\varkappa$, %
10^{-1}	0.09600	4.00067	1.50462	24.39782
10^{-2}	0.00997	0.24070	1.24288	2.75730
10^{-3}	0.00099	0.13589	1.21291	0.27948
10^{-4}	0.00009	0.15898	1.20986	0.02793
10^{-5}	0.00001	0.06334	1.20956	0.00283

Figure 2 shows graphs of the functions $\psi(\xi)$ and $\phi(\xi)$ characterizing the distribution of PD and PS in the electroelastic disk:

$$\psi(\xi) = \frac{(1-c)\xi + (1+c)/\xi}{(1-c^2)(\xi_0^{-2}-1)}, \qquad \phi(\xi) = \frac{1-\xi^{-2}}{\xi_0^{-2}-1}.$$

To study the relative variation of the first two resonance frequencies, we consider the characteristics $\varepsilon_j(\tau)$ defined by formula

$$\varepsilon_j(\tau) = \frac{\varkappa_j(\tau) - \varkappa_{j0}}{\varkappa_{j0}},$$

where $\varkappa_{j0}$ $(j = 1, 2)$ are the first two dimensionless resonant frequencies, which are the roots of the frequency equation (12); $\varkappa_j(\tau)$ $(j = 1, 2)$ are the first two dimensionless resonant frequencies obtained by numerical solution of the direct problem.

Figure 3 shows graphs of the functions $\varepsilon_1(\tau)$ and $\varepsilon_2(\tau)$. It can be seen that in the presence of the PSSS, the first resonant frequency changes more significantly than the second. Therefore, in investigating the inverse problem, we should use the values of the first resonant frequency. The curve of $\varepsilon_3(\tau)$ is similar to the curve of $\varepsilon_2(\tau)$.

Table 1 shows the values of the PSSS parameter τ_R obtained from formula (13) [τ is the exact value of the PSSS parameter, $\varkappa_1$ is the corresponding first natural frequency obtained by solving the direct problem, δ_τ is the relative error of determining τ_R, $\delta_\varkappa$ is the deviation of the value $\varkappa_1$ from the value $\varkappa_{10}$, and $\varkappa_{10} = 1.209\,52$ is the first natural frequency obtained by solving the frequency equation (12)]. From Table 1, it follows that the accuracy of determining the values of τ_R by formula (13) increases with decreasing parameter τ. At the same time, for small values of τ, the value of $\delta_\varkappa$, characterizing the difference of the first natural frequencies of disk vibrations in the presence and absence of PSSS decreases, which should be taken into account in studies.

Table 2. Exact τ and reconstructed τ_R values of the PSSS parameter for different noise levels

τ	τ_R	ε_0	$\varkappa_1^e$	$\varkappa_1^n$	δ_τ, %
10^{-1}	0.097600	10^{-4}	1.50462359	1.50458146	2.32
	0.098000	10^{-3}		1.50558655	1.95
	0.102500	10^{-2}		1.51771382	2.51
	0.102600	10^{-1}		1.51816521	2.68
10^{-2}	0.009970	10^{-5}	1.24287593	1.24287556	0.24
	0.009960	10^{-4}		1.24283616	0.36
	0.010060	10^{-3}		1.24317422	0.66
	0.011100	10^{-2}		1.24660456	11.08
10^{-3}	0.000998	10^{-6}	1.21290654	1.21290597	0.13
	0.000997	10^{-5}		1.21290242	0.24
	0.001029	10^{-4}		1.21301085	2.96
	0.000873	10^{-3}		1.21248202	12.67

To simulate a full-scale experiment, we performed a reconstruction of the PSSS parameter τ for noisy values of the first natural frequency $\varkappa_1$. Noising was applied according to the formula

$$\varkappa_1^n = \varkappa_1^e\Big(1 + \frac{d\varepsilon_0}{100}\Big),$$

where $\varkappa_1^e$ is the exact value of the first resonant frequency obtained by solving the direct problem for given laws of variation of the functions $\psi(\xi)$ and $\phi(\xi)$, $\varkappa_1^n$ is the noisy value of the resonant frequency, ε_0 is a parameter characterizing the noise level, and $d \in [-100, 100]$ is a random variable with a uniform distribution law.

Table 2 shows the exact and reconstructed values of the parameter τ for different noise levels.

Thus, the accuracy of measuring the vibration frequencies must be the higher, the smaller the PSSS parameter τ. For values of the parameter $\tau < 10^{-4}$, the measurement accuracy should be very high.

CONCLUSIONS

A general formulation of the problem of the motion of an electroelastic body in the presence of an inhomogeneous preliminary stress–strain state was presented. Based on this formulation, the problem of steady radial vibrations of a thin hollow piezoelectric cylindrical disk was formulated. The solution of the direct dimensionless problem of determining the radial displacement function was obtained numerically using the shooting method. The influence of the level and structure of the preliminary stress–strain state on the values of the first two resonant frequencies was analyzed.

The free vibrations of an electroelastic disk were analyzed to solve the inverse problem of determining the level of pre-stresses from data on the change in the natural frequency. A formula for the change in the natural frequency of the disk depending on the laws of distribution of the pre-stresses and residual displacements was obtained. The results of the numerical experiments showed that the proposed solution using this formula is applicable in practice. The accuracy of determining the PSSS for small values of the parameter τ does not exceed 5%. It is recommended that the reconstruction procedure should be carried out for the first natural frequency.

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