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STOCHASTIC WAVE FINITE ELEMENT METHOD IN UNCERTAIN ELASTIC MEDIA THROUGH THE SECOND ORDER PERTURBATION

F. Bouchoucha[a], **M. N. Ichchou**[b], **and M. Haddar**[a]

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Abstract: In this work, the stochastic wave finite element (SWFE) method for uncertain media through the second-order perturbation is formulated. A parametric approach for uncertainties is considered and combined to the finite element technique. The stochastic state space formulation is detailed in this work. The originality of this paper is the study of the second-order perturbation. The sensitivity and the precision of the SWFE approach are treated through the second-order perturbation introduced in the structural parameters. The question of the statistics of the propagation constants and the wave modes is considered. Comparisons with analytical results and Monte Carlo simulations are performed.

Keywords: stochastic wave finite element method, dispersion, uncertainty, second-order perturbation.

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INTRODUCTION

Many researchers proposed some structural health monitoring (SHM) techniques in order to carry out the monitoring and the diagnosis of the risks [1, 2]. SHM is among the fields of application of guided wave propagation. Guided waves are still a subject of intensive research. One of the primary properties of guided waves is the dispersion curve, which yields the velocity-frequency relationship for all the modes that may propagate in the studied structure. A wave finite element (WFE) method provides an effective way to calculate the dispersion curves of complex guided structures and investigate their properties [3]. The WFE method regards the waveguide structure as a periodic system assembled from identical substructures [4, 5].

In the literature, however, most of numerical issues of wave propagation simulations are mainly limited to deterministic media. Numerical guided wave techniques in spatially homogeneous random media are investigated in this paper. To deal with uncertainties in structural dynamics, extensive research has been performed (see, e.g., [6, 7]). Uncertainties are often present in geometric properties, material characteristics, and boundary conditions of the model. These variables are taken into account in models according to both parametric [8] and non-parametric [9, 10] approaches. Ichchou et al. [11] considered the wave propagation features in random guided

[a]Unit of Dynamics of the Mechanical Systems (UDSM), National School of Engineers of Sfax. BP. W3038 Sfax-Tunisia; fakersbouchoucha@yahoo.fr, Mohamed.Ichchou@ec-lyon.fr, mohamed.haddar@enis.rnu.tn. [b]Laboratory of Tribology and Dynamics of Systems (LTDS), Ecole Centrale de Lyon. 36 Avenue Guy de Collongues 69130 Ecully. Translated from Prikladnaya Mekhanika i Tekhnicheskaya Fizika, Vol. 58, No. 2, pp. 202–211, March–April, 2017. Original article submitted February 5, 2015; revision submitted August 12, 2015.

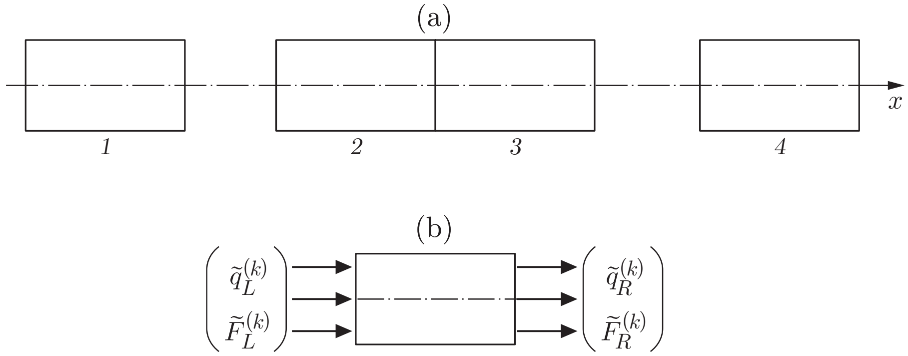

Fig. 1. Periodic waveguide (a) and degrees of freedom of the subsystem k on the left and right boundaries (b): subsystem 1 (*1*), subsystem k (*2*), subsystem $k+1$ (*3*), and subsystem N (*4*).

elastic media and formulated the stochastic wave finite element (SWFE) approach. This formulation allows definition of wave characteristics by means of a stochastic finite element model. The case of spatially homogeneous random properties is dealt with, through a parametric probabilistic technique. Bouchoucha et al. [12] presented a numerical approach to study guided elastic wave propagation in uncertain elastic media. The SWFE method formulation with consideration for spatial variability of the material and geometrical properties was developed for the probabilistic analysis of structures. This work was extended in [13] to the diffusion matrix for uncertain media through the SWFE method. The stochastic diffusion relationship derived in that work allows one to evaluate the statistics of reflection and transmission coefficients under structural uncertainty.

The current study extends the WFE technique to stochastic media through the second-order perturbation. A parametric approach for treating uncertainties is considered and combined to the WFE technique. The SWFE method through the state space formulations is proposed. The originality of this paper is the study of the second-order perturbation. The sensitivity and the precision of the SWFE approach is treated through the second-order uncertainty introduced in the structural parameters. The second-order perturbation of the propagation constants and the wave modes are investigated. The numerical accuracy and the computational efficiency of the method are demonstrated through comparisons with analytical results.

1. FORMULATION OF THE SWFE APPROACH THROUGH THE STATE SPACE FORMULATION

Introduction of uncertainties and specific developments leads to the state space formulation for studying stochastic phenomena. An uncertain media is considered. The system is assumed to be a set of identical subsystems connected along the principal direction, say, the x axis. The length of each subsystem in the x direction is denoted as d. The formulation is based on the finite element model of a typical subsystem (Fig. 1). The left (index L) and right (index R) boundaries of a given subsystem are assumed to contain the same number of degrees of freedom n. The stochastic equation of motion for each sample is

$$\tilde{D}\tilde{q} = \tilde{F},$$

where

$$\tilde{D} = \begin{pmatrix} \tilde{D}_{LL} & \tilde{D}_{LR} \\ \tilde{D}_{RL} & \tilde{D}_{RR} \end{pmatrix} = \tilde{K} - \omega^2 \tilde{M},$$

ω, $\tilde{D}_{LL}$, $\tilde{D}_{LR}$, $\tilde{D}_{RL}$, and $\tilde{D}_{RR}$ are the elements of the dynamic matrix of size $n \times n$, $\tilde{K}$ is the stiffness matrix of size $2n \times 2n$, $\tilde{M}$ is the mass matrix of size $2n \times 2n$, ω is the wave number, $\tilde{q} = \begin{pmatrix} \tilde{q}_L \\ \tilde{q}_R \end{pmatrix}$ and $\tilde{F} = \begin{pmatrix} \tilde{F}_L \\ \tilde{F}_R \end{pmatrix}$ are the displacement and force vectors of size $2n \times 1$, and $\tilde{q}_L$, $\tilde{q}_R$, $\tilde{F}_L$, and $\tilde{F}_R$ are the displacement and force vectors of size $n \times 1$ on the left and right boundaries, respectively. The kinematic stochastic variables $\tilde{q}$ and $\tilde{F}$ can be represented through the following stochastic state vectors:

$$\tilde{u}_L^{(k)} = \begin{pmatrix} \tilde{q}_L^{(k)} \\ -\tilde{F}_L^{(k)} \end{pmatrix}, \qquad \tilde{u}_R^{(k)} = \begin{pmatrix} \tilde{q}_R^{(k)} \\ \tilde{F}_R^{(k)} \end{pmatrix}.$$

In this way, it can be shown that the state vectors $\tilde{u}_L^{(k)}$ and $\tilde{u}_R^{(k)}$ are related by the stochastic transfer matrix $\tilde{S}$:

$$\tilde{u}_R = \tilde{S}\tilde{u}_L.$$

The uncertainties are assumed to be mostly in the material properties. Such uncertainties are assumed to be spatially homogeneous. This guarantees that the assumed periodicity will be respected both in the stochastic case and in the deterministic situation. Random variables are modelled by the Gaussian variables through a second-order perturbation $\tilde{\nu} = \bar{\nu} + \nu_1\varepsilon + \nu_2\varepsilon^2$, where $\tilde{\nu}$ is the random variable, $\bar{\nu}$ is its mean, ν_1 is the first-order perturbation, ν_2 is the second-order perturbation, and ε is the standard normal distribution. The polynomial chaos $(1, \varepsilon, \varepsilon^2)$ is used as a supplementary dimension of the problem. The random variables $\tilde{u}_L$, $\tilde{u}_R$, and $\tilde{S}$ are modelled by the Gaussian variables through a second-order perturbation as

$$\tilde{u}_L = \begin{pmatrix} q_{0L} + q_{1L}\varepsilon + q_{2L}\varepsilon^2 \\ -F_{0L} - F_{1L}\varepsilon - F_{2L}\varepsilon^2 \end{pmatrix}, \quad \tilde{u}_R = \begin{pmatrix} q_{0R} + q_{1R}\varepsilon + q_{2R}\varepsilon^2 \\ F_{0R} + F_{1R}\varepsilon + F_{2R}\varepsilon^2 \end{pmatrix}, \quad \tilde{S} = S_0 + S_1\varepsilon + S_2\varepsilon^2,$$

where S_0 is the deterministic transfer matrix, and S_1 and S_2 are the first-order and second-order transfer matrices, respectively. Using the determination of S_0 and S_1 [11] and applying some transformations, we obtain

$$S_0 = \begin{pmatrix} -D_{0LR}^{-1}D_{0LL} & D_{0LR}^{-1} \\ D_{0RR}D_{0LR}^{-1}D_{0RR} - D_{0RL} & D_{0RR}D_{0LR}^{-1} \end{pmatrix},$$

$$S_1 = \begin{pmatrix} S_{1LL} & S_{1LR} \\ S_{1RL} & S_{1RR} \end{pmatrix}, \qquad S_2 = \begin{pmatrix} S_{2LL} & S_{2LR} \\ S_{2RL} & S_{2RR} \end{pmatrix},$$

where

$$S_{1LL} = D_{0LR}^{-1}D_{1LL} - D_{0LR}^{-1}D_{1LR}D_{0LR}^{-1}D_{0LL}, \qquad S_{1LR} = -D_{0LR}^{-1}D_{1LR}D_{0LR}^{-1},$$

$$S_{1RL} = D_{0RR}D_{0LR}^{-1}D_{1LL} - D_{0RR}D_{0LR}^{-1}D_{1LR}D_{0LR}^{-1}D_{0LL} - D_{1RL} + D_{1RR}D_{0LR}^{-1}D_{0LL},$$

$$S_{1RR} = -D_{0RR}D_{0LR}^{-1}D_{1LR}D_{0LR}^{-1} + D_{0LR}^{-1},$$

$$S_{2LL} = -D_{0LR}^{-1}D_{2LL} - D_{0LR}^{-1}D_{1LR}D_{1LL} - D_{0LR}^{-1}D_{2LR}S_{0LL},$$

$$S_{2LR} = -D_{0LR}^{-1}D_{1LR}S_{1LR} - D_{0LR}^{-1}D_{2LR}S_{0LL},$$

$$S_{2RL} = D_{0RR}D_{0LR}^{-1}(D_{2LL} + D_{1LR}S_{1LL} + D_{2LR}S_{0LL}) + D_{2RL} + D_{1RR}S_{1LL} + D_{2RR}S_{0LL},$$

$$S_{2RR} = -D_{0RR}D_{0LR}^{-1}(D_{1LR}S_{1LR} + D_{2LR}S_{0LR}) + D_{1RR}S_{1LR} + D_{2RR}S_{0LR}.$$

Then we pose the stochastic eigenvalue problem

$$\tilde{S}\tilde{\varphi} = \tilde{\mu}\tilde{\varphi},$$

where $\tilde{\mu} = \mu_0 + \mu_1\varepsilon + \mu_2\varepsilon^2$ are the eigenvalues associated to the eigenvector $\tilde{\varphi} = \varphi_0 + \varphi_1\varepsilon + \varphi_2\varepsilon^2$.

The polynomial chaos projection of the eigenvalue problem yields the equation for identifying the statistical characteristics

$$(S_0 + S_1\varepsilon + S_2\varepsilon^2)(\varphi_0 + \varphi_1\varepsilon + \varphi_2\varepsilon^2) = (\mu_0 + \mu_1\varepsilon + \mu_2\varepsilon^2)(\varphi_0 + \varphi_1\varepsilon + \varphi_2\varepsilon^2).$$

The deterministic and first-order eigenvalue problems were treated by Ichchou et al. [11]. Solving the deterministic equation leads to a problem of calculating the mean of the eigenvalue μ_0 and eigenvector φ_0. In turn, solving the first-order equation, one can calculate the standard deviation of the eigenvalue μ_1 and eigenvector φ_1.

The deterministic equation is given as

$$S_0\varphi_0 = \mu_0\varphi_0;$$

the first-order equation is

$$S_0\varphi_2 + S_1\varphi_1 + S_2\varphi_0 = \mu_0\varphi_2 + \mu_1\varphi_1 + \mu_2\varphi_0;$$

finally, the second-order equation can be extracted in the following form:

$$S_0\varphi_2 + S_1\varphi_1 + S_2\varphi_0 = \mu_0\varphi_2 + \mu_1\varphi_1 + \mu_2\varphi_0.$$

In order to extract the second-order perturbation of the eigenvalue μ_2 and eigenvector φ_2, we use the left propagation constants. The vector $\tilde{\varphi}_i^{\mathrm{T}} J_n$ can be chosen as the left eigenvector $\tilde{S}$ associated to the eigenvalue $\tilde{\mu}_i^{-1}$, where $J_n = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$ [14]. The left stochastic eigenvalue problem can be established as follows:

$$(\tilde{\varphi}_i^{\mathrm{T}} J_n)\tilde{S} = \tilde{\mu}_i^{-1}(\tilde{\varphi}_i^{\mathrm{T}} J_n).$$

The polynomial chaos projection of the eigenvalue problem allows their statistics to be identified. In particular, the left second-order equation is given as

$$\varphi_2^{\mathrm{T}} J_n S_0 + \varphi_1^{\mathrm{T}} J_n S_1 + \varphi_0^{\mathrm{T}} J_n S_2 = \mu_0^{-1}\varphi_2^{\mathrm{T}} J_n + \mu_1^*\varphi_1^{\mathrm{T}} J_n + \mu_2^*\varphi_0^{\mathrm{T}} J_n,$$

where $\mu_1^* = -\mu_1/\mu_0^2$ and $\mu_2^* = -\mu_2/\mu_0^2 + \mu_1^2/\mu_0^3$.

After some analytical treatments, the statistics of the wave characteristics can be expressed via the second-order perturbation of the propagation constants

$$\mu_2 = -[(S_0 - \mu_0 I)(S_0^{\mathrm{T}} J_n - \mu_0^{-1} J_n) J_n \varphi_0 \mu_0^{-2} - \varphi_0]^+ \times$$
$$\times [(S_0 - \mu_0 I)(S_0^{\mathrm{T}} J_n - \mu_0^{-1} J_n)(-J_n\varphi_0\mu_1^2\mu_0^{-3} + S_1^{\mathrm{T}} J_n\varphi_1 + S_2^{\mathrm{T}} J_n\varphi_0 + \mu_1\mu_0^{-2} J_n\varphi_1) -$$
$$- S_1\varphi_1 - S_2\varphi_0 + \mu_1\varphi_1]$$

and the second-order perturbation of the eigenvector

$$\varphi_2 = [S_0^{\mathrm{T}} J_n - \mu_0^{-1} J_n]^+[\mu_2^{-1} J_n\varphi_0 - S_1^{\mathrm{T}} J_n\varphi_1 - S_2^{\mathrm{T}} J_n\varphi_0 - \mu_1\mu_0^{-2} J_n\varphi_1]$$

(the plus sign corresponds to the pseudo-inverse operator).

Let us consider the wave numbers statistics:

$$\tilde{k} = (i/d)\log\,\tilde{\mu},$$

which is of interest in numerous applications. Indeed, knowing the zeroth-, first-, and second-order perturbations of the stochastic eigenvalue $\tilde{\mu}$ and considering the wave number to be written as $\tilde{k} = k_0 + k_1\varepsilon + k_2\varepsilon^2$, we can express the statistics of $\tilde{k}$. The mean of the wave number is given in [11] as $k_0 = (i/d)\log\mu_0$, and its first-order term is $k_1 = (i/d)\mu_0^{-1}\mu_1$. After some mathematical treatments, we can extract the second-order term of the wave number:

$$k_2 = (i/d)(\mu_0^{-2}\mu_1^2 + \mu_0^{-1}\mu_2).$$

2. NUMERICAL RESULTS AND DISCUSSION

In this Section, the main issue is the validation of the second-order SWFE finding by using the analytical results and Monte Carlo (MC) simulations (5000 samples). The compression (longitudinal) and flexural waves are considered.

2.1. Longitudinal Wave Case Study

We study the longitudinal wave case in order to validate the SWFE formulation for the traction–compression mode. The waveguide is assumed to be a beam element with two nodes and one degree of freedom per node.

The material is steel ($\rho = 7800$ kg/m^3, $\nu = 0.3$, and $E = 2 \cdot 10^{11}$ Pa). In numerical simulations, the level of uncertainty is chosen approximately at 0.5% for the second-order perturbation. The mass and stiffness matrices for the traction–compression mode are

$$M_{tc} = \frac{\rho s d}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}, \qquad K_{tc} = \frac{Es}{d} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix},$$

where E is Young's modulus, s is the cross-sectional area, ρ is the mass density, and d is the length of the considered element.

Let us consider now a random medium. It is then feasible in this basic case to compare the statistics of the wave numbers and the propagation constants for an uncertain waveguide. Young's modulus and mass density are assumed to be random.

The random Young's modulus $\tilde{E}$ is modelled by the Gaussian variable: $\tilde{E} = E_0 + E_1\varepsilon + E_2\varepsilon^2$ ($E_0 = 2 \cdot 10^{11}$ Pa, $E_1 = 5\ \% \cdot E_0 = 10^{10}$ Pa, and $E_2 = 0.5\ \% \cdot E_0 = 10^9$ Pa). The validation of the SWFE results is performed through the analytical wave number expression derived for the longitudinal mode:

$$\tilde{k}_{tc} = \omega\sqrt{\rho/E}\,.$$

The polynomial chaos projection of the analytical wave number for the traction–compression mode is given as $\tilde{k}_{tc} = k_{0tc} + k_{1tc}\varepsilon + k_{2tc}\varepsilon^2$, where $k_{0tc} = \omega\sqrt{\rho/E}$ is the deterministic wave number and $k_{1tc} = -(1/2)\omega\sqrt{\rho}\,E_1E_0^{-3/2}$ is its first-order perturbation. The second-order perturbation can be presented in the following form (Fig. 2a):

$$k_{2tc} = \frac{1}{2}\,\omega\sqrt{\rho}\,\Big(\frac{3}{4}\,E_1^2E_0^{-5/2} - E_2E_0^{-3/2}\Big).$$

In the next paragraph, the random mass density $\tilde{\rho}$ is modeled by the Gaussian variables through a second-order perturbation as

$$\tilde{\rho} = \rho_0 + \rho_1\varepsilon + \rho_2\varepsilon^2$$

($\rho_0 = 7800$ kg/m^3, $\rho_1 = 5\ \% \cdot \rho_0 = 390$ kg/m^3, and $\rho_2 = 0.5\ \% \cdot \rho_0 = 39$ kg/m^3). The expression for the first-order term of the analytical wavenumber is

$$k_{1tc} = \frac{\omega}{2\sqrt{E}}\,\rho_0^{-1/2}\rho_1,$$

and the second-order term (Fig. 2b) is

$$k_{2tc} = \frac{\omega}{2\sqrt{E}}\Big(\rho_2\rho_0^{-1/2} - \frac{1}{4}\,\rho_1^2\rho_0^{-3/2}\Big).$$

2.2. Flexural Wave Case Study

Let us now consider a simple dispersive media. A two-node beam element is analyzed. The mass and stiffness matrices of the beam are given simply as

$$M_{fl} = \frac{\rho s d}{420} \begin{bmatrix} 156 & 22d & 54 & -13d \\ 22d & 4d^2 & 13d & -3d \\ 54 & 13d & 156 & -22d \\ -13d & -3d & -22d & 4d^2 \end{bmatrix}, \qquad K_{fl} = \frac{EI}{d^3} \begin{bmatrix} 12 & 6d & -12 & 6d \\ 6d & 4d^2 & -6d & 2d^2 \\ -12 & -6d & 12 & -6d \\ 6d & 2d^2 & -6d & 4d^2 \end{bmatrix},$$

where EI is the bending stiffness.

The analytical wave number in the flexural mode is

$$\tilde{k}_{fl} = (\rho s\omega^2/(EI))^{1/4}.$$

The polynomial chaos projection of the analytical wave number for the flexural mode is written as $\tilde{k}_{fl} = k_{0fl} + k_{1fl}\varepsilon + k_{2fl}\varepsilon^2$, where $k_{0fl} = (\rho s\omega^2/(E_0I))^{1/4}$ is the analytical deterministic wave number for the flexural mode.

In the case of random media, if Young's modulus is considered as a random parameter, the first-order term of the analytical wave number for the flexural mode is

$$k_{1fl} = -\Big(\frac{\rho s\omega^2}{I}\Big)^{1/4}\frac{1}{4}\,E_1E_0^{-5/4}.$$

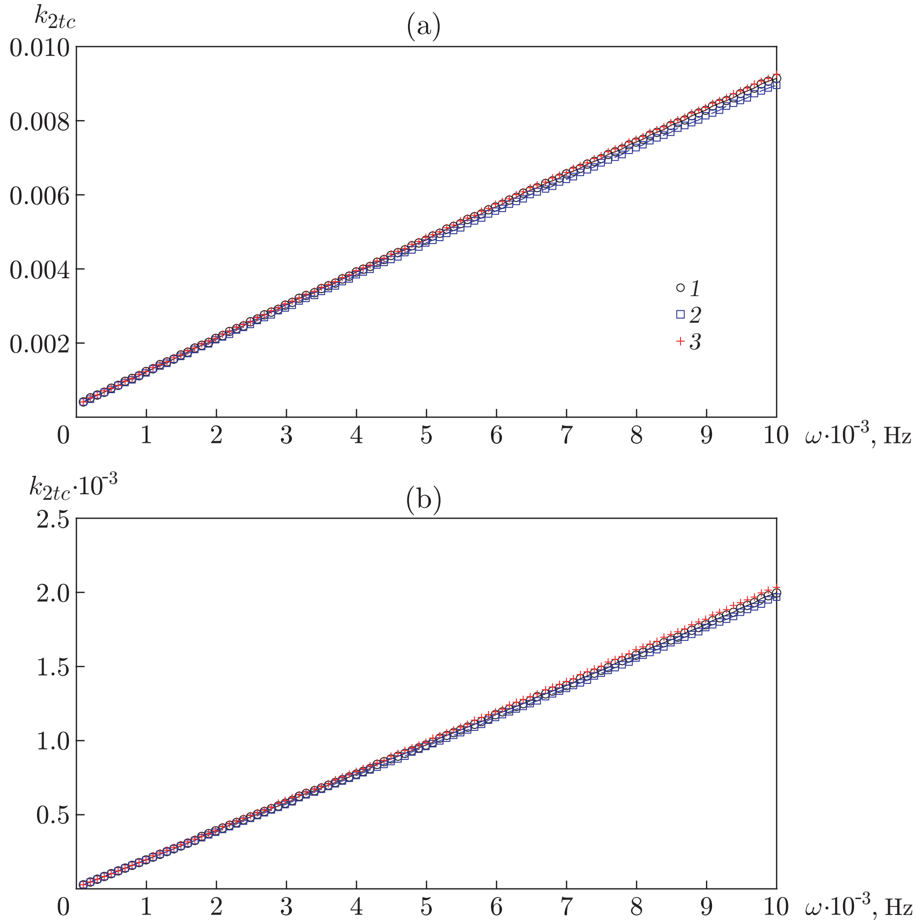

Fig. 2. Second-order perturbation of the wave number versus frequency for the longitudinal mode obtained by different methods: (a) E is the stochastic variable; (b) ρ is the stochastic variable; SWFE results (*1*), analytical solution (*2*), and MC results (*3*).

The second-order term is written as follows (Fig. 3a):

$$k_{2fl} = \left(\frac{\rho s\omega^2}{I}\right)^{1/4}\left(\frac{5}{32}\,E_1^2E_0^{-9/4} - \frac{1}{4}\,E_2E_0^{-5/4}\right).$$

The analytical expression for the wave number obtained for the flexural mode is

$$k_{1fl} = \left(\frac{s\omega^2}{EI}\right)^{1/4}\frac{1}{4}\,\rho_0^{-3/4}\rho_1.$$

After some analytical treatments, we can express the second-order term of the analytical wave number as follows (Fig. 3b):

$$k_{2fl} = \left(\frac{s\omega^2}{EI}\right)^{1/4}\left(-\frac{3}{32}\,\rho_0^{-7/4}\rho_1 + \frac{1}{4}\,\rho_0^{-3/4}\rho_2\right).$$

2.3. Results of Simulations with Allowance for the First-Order and Second-Order Perturbations

Let the length d be a stochastic parameter:

$$\tilde{d} = d_0 + d_1\varepsilon + d_2\varepsilon^2.$$

The polynomial chaos projection of the mass matrix for the longitudinal mode has the form

$$\tilde{M} = M_0 + M_1\varepsilon + M_2\varepsilon^2,$$

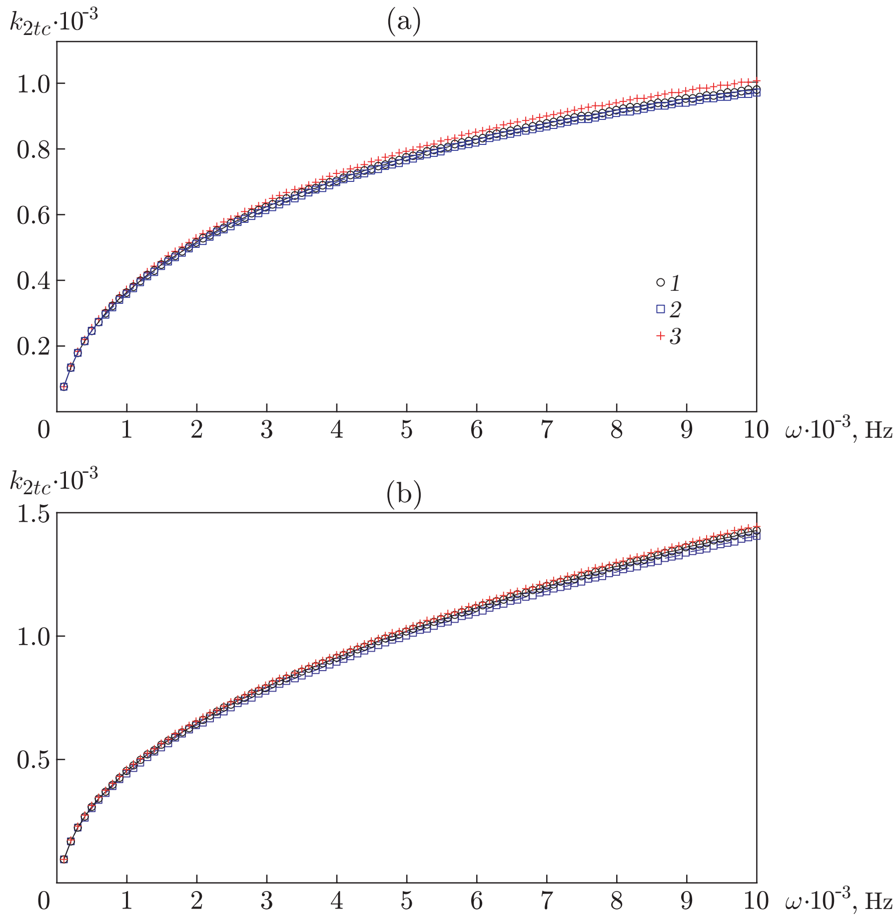

Fig. 3. Second-order perturbation of the wave number versus frequency for the flexural mode obtained by different methods: (a) E is the stochastic variable; (b) ρ is the stochastic variable (notation the same as in Fig. 2).

where $M_0 = \dfrac{\rho s d_0}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ is the deterministic mass matrix and $M_1 = \dfrac{\rho s d_1}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ is its first-order perturbation. The second-order perturbation of the mass matrix can be given as

$$M_2 = \frac{\rho s d_2}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}.$$

The polynomial chaos projection of the stiffness matrix for the longitudinal mode is

$$\tilde{K} = K_0 + K_1\varepsilon + K_2\varepsilon^2,$$

where $K_0 = \dfrac{Es}{d_0}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$ is the deterministic stiffness matrix and $K_1 = -\dfrac{Esd_1}{d_0^2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$ is its first-order perturbation. Then the second-order perturbation of the stiffness matrix can be given as

$$K_2 = Es\Big(\frac{d_1^2}{d_0^3} - \frac{d_2}{d_0^2}\Big)\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}.$$

In numerical simulations, the level of uncertainty is chosen to be approximately 10% for the first-order perturbation and 0.5% for the second-order perturbation for studying the efficiency of the SWFE method in the case of high perturbations around the mean of the random variable. The numerical simulations are performed for

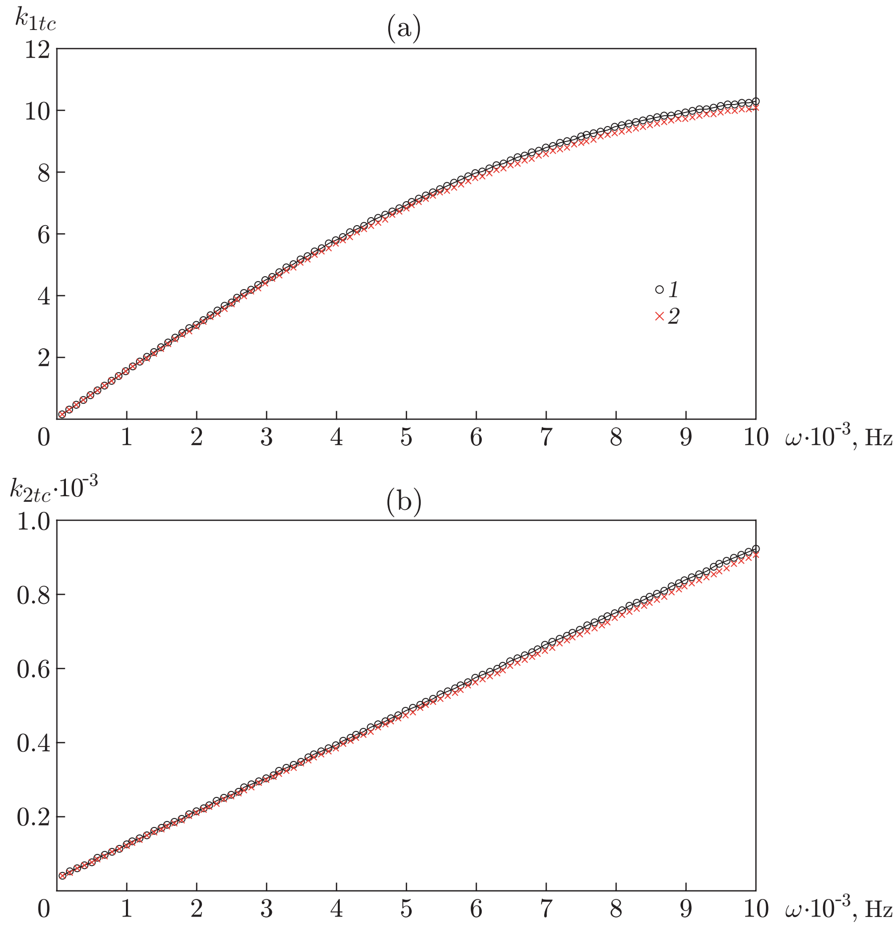

Fig. 4. First-order (a) and second-order perturbations of the wave number versus frequency for the longitudinal mode (d is the stochastic variable) obtained by different methods: SWFE results (*1*) and MC results (*2*).

the longitudinal mode under the assumption that the stochastic element length is $d_0 = 0.02$ m, $d_1 = 10\%\, d_0 = 0.002$ m, and $d_2 = 0.5\% d_0 = 0.0001$ m.

It is seen in Fig. 4 that the first-order perturbation is insufficient in the case of high uncertainty. It is remarkable that the SWFE and MC results are close to each other. In the case of high uncertainties, the second-order perturbations should be taken into account.

CONCLUSIONS

In this paper, the stochastic wave finite element method is generalized for the analysis of uncertain periodic waveguides with considering the second-order perturbation in the stochastic variables. Both the spectral and state space formulations of the method are presented. The novelty and practical importance of the research consists in the calculation of the second-order perturbation of eigenvalues and eigenvectors of the problem for the introduced uncertainties in the physical and geometrical characteristics of the waveguide. The validation of method is based on comparisons of numerical and analytical solutions for the longitudinal and flexural modes of wave propagation through an elastic beam with random Young's modulus and mass density. It is demonstrated that the second-order perturbation should be taken into account at high uncertainties.

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