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GREEN'S FUNCTION FOR A PRESTRESSED THERMOELASTIC HALF-SPACE WITH AN INHOMOGENEOUS COATING

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Abstract: A mathematical model is developed for an inhomogeneous thermoelastic prestressed half-space consisting of a stack of homogeneous or functionally graded layers rigidly attached to a homogeneous base. Each component of the inhomogeneous medium is subjected to initial mechanical stresses and temperature. Successive linearization of the constitutive relations of the nonlinear mechanics of a thermoelastic medium is performed using the theory of superposition of small deformations on finite deformations with the inhomogeneity of the medium taken into account. Integral formulas are derived to explore dynamic processes in inhomogeneous prestressed thermoelastic media.

Keywords: thermoelasticity, functionally graded material, prestressed thermoelastic medium with a coating, initial stresses, preheating, three-dimensional Green's function.

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INTRODUCTION

The linear theory of thermoelasticity for prestressed bodies in the isothermal and non-isothermal cases has been developed in the works of Green [1] and Iesan [2]. Constitutive equations and equations of motion for thermoelastic bodies at large initial strains and initial temperature have been derived by Wang and Slattery [3]. Singh and Renu [4, 5] have studied the effect of initial stresses on the wave field on the surface of a homogeneous transversely isotropic thermoelastic half-space was in a linear approximation. The effect of preheating and initial mechanical loads on the dynamics of a homogeneous medium has been investigated using linearized theory [6–8]. Kalinchuk et al. [6] have constructed a three-dimensional Green's function for a homogeneous thermoelastic layer and analyzed the influence of initial stresses on its dispersion properties. Belyankova et al. [7] have studied a mixed problem of vibrations of a homogeneous thermoelastic layer. Kalinchuk and Levi [8] have studied the problem of a homogeneous thermoelastic half-space under a thermal load that simulates the action of a frequency-modulated laser beam and obtained heat flux distributions in the contact area depending on the nature, type, and magnitude of initial loads. Sheidakov et al. [9] linearized the constitutive relations and equations of motion of the nonlinear mechanics of a thermoelastic homogeneous medium under the assumption that the initial stress state is also homogeneous. During linearization, fourth-order terms in strain and second-order terms in temperature were retained in the expansion of the thermodynamic potential. This approach is possible in studies of the dynamics of homogeneous thermoelastic materials, but in studies of the behavior of composite structures or media with an inhomogeneous coating, it greatly complicates the problem. In the present study of inhomogeneous thermoelastic materials, simpler and more convenient linearized constitutive equations and equations of motion of the medium were obtained that take into account the effect of the inhomogeneity of the initial stress state and properties of the material on its thermoelastic properties. Green's function matrices for a prestressed thermoelastic half-space

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with an inhomogeneous coating are constructed using a hybrid numerical analytical method [10–12], which is a combination of analytical methods and numerical schemes for constructing the solution and a matrix approach to satisfying the boundary conditions.

1. FORMULATION OF BOUNDARY-VALUE DYNAMIC PROBLEMS FOR THERMOELASTIC BODIES

Consider the problem of vibrations a thermoelastic medium that occupies a certain volume V bounded by the surface $o = o_1 + o_2 = o_3 + o_4$. We introduce an orthonormal vector basis of Cartesian coordinates $\boldsymbol{i}_1, \boldsymbol{i}_2, \boldsymbol{i}_3$ in space. Denote the Lagrangian and Eulerian Cartesian coordinates by x_1, x_2, x_3 and X_1, X_2, X_3, respectively. The nabla operators, the vectors of the direct and reciprocal bases are given by the formulas

$$\nabla_0 = \boldsymbol{i}_m \frac{\partial}{\partial x_m}, \quad \nabla = \boldsymbol{i}_m \frac{\partial}{\partial X_m}, \quad \boldsymbol{r} = x_k \boldsymbol{i}_k, \quad \boldsymbol{R} = X_k \boldsymbol{i}_k. \tag{1.1}$$

Suppose that v is the reference configuration associated with the natural state, V is the actual configuration associated with the initial deformed state, and $\boldsymbol{R}$ and $\boldsymbol{r}$ are the radius vectors of a point in the initial deformed and natural states, respectively. The stress–strain state of the nonlinear thermoelastic material in the natural state is defined by the Piola stress tensor

$$\Pi = P \cdot C, \qquad P = \chi_S, \tag{1.2}$$

the Cauchy–Green strain tensor

$$S = (G - I)/2, \qquad C = \nabla_0 \boldsymbol{R}, \quad G = C \cdot C^{\mathrm{t}} \tag{1.3}$$

and the specific (per unit volume) entropy

$$\eta = -\chi_\theta. \tag{1.4}$$

Here C is the strain gradient, G is Cauchy–Green strain measure, and θ is the temperature. The thermal characteristics of the material are described using the following parameters defined in the metric of the natural state: the temperature gradient

$$\boldsymbol{g} = \nabla_0 \theta \tag{1.5}$$

and the heat flux vector

$$\boldsymbol{h} = \boldsymbol{h}(C, \theta, \boldsymbol{g}), \tag{1.6}$$

which is generally a nonlinear function [13, 14]. In this paper, we use its well-known representation

$$\boldsymbol{h} = -\lambda \boldsymbol{g} \tag{1.7}$$

[$\lambda = \lambda(C, \theta, \boldsymbol{g})$ is the thermal conductivity tensor]. For materials with hexagonal symmetry (class 6 mm), we have $\lambda = \|\lambda_{kk}\|_{k=1}^3$ and $\lambda_{11} = \lambda_{22} \neq \lambda_{33}$.

The tensor χ_S and the scalar χ_θ used in representations (1.2) and (1.4) are derivatives of the thermodynamic potential $\chi = \chi(S, \theta)$ [13, 14]. The boundary-value problem of vibrations of a prestressed thermoelastic medium in Lagrangian coordinates is described by the equations

$$\nabla_0 \cdot \Pi = \rho_0 \frac{\partial^2 \boldsymbol{R}}{\partial t^2}; \tag{1.8}$$

$$\nabla_0 \cdot \boldsymbol{h} + \theta \frac{\partial \eta}{\partial t} = 0 \tag{1.9}$$

with the boundary conditions

$$\boldsymbol{n} \cdot \Pi \Big|_{o_1} = \boldsymbol{t}_{\boldsymbol{n}}, \quad \boldsymbol{R}\Big|_{o_2} = \boldsymbol{R}^*, \quad \theta\Big|_{o_3} = T^*, \quad \boldsymbol{n} \cdot \boldsymbol{h}\Big|_{o_4} = -h^*, \tag{1.10}$$

where ρ_0 is the density of the undeformed body and $\boldsymbol{n}$ is the normal to the surface.

Suppose that there is a state of equilibrium of the thermoelastic body:

$$\boldsymbol{R} = \boldsymbol{R}_1(\boldsymbol{r}), \qquad \theta = T_1(\boldsymbol{r}). \tag{1.11}$$

According to (1.8)–(1.10), the equations of statics within the volume and on the surface can be represented as

$$\nabla_0 \cdot \Pi_1 = 0, \qquad \nabla_0 \cdot \boldsymbol{h}_1 = 0; \tag{1.12}$$

$$\boldsymbol{n} \cdot \Pi_1 \Big|_{o_1} = \boldsymbol{t}^1_{\boldsymbol{n}}, \quad \boldsymbol{R}_1 \Big|_{o_2} = \boldsymbol{R}_1, \quad \theta \Big|_{o_3} = T_1, \quad \boldsymbol{n} \cdot \boldsymbol{h}_1 \Big|_{o_4} = h_1.$$

Consider a small perturbation of the equilibrium configuration (1.11) [9, 13, 14]

$$\boldsymbol{R}^{\times} = \boldsymbol{R}_1 + \varepsilon \boldsymbol{u}, \qquad \theta^{\times} = T_1 + \varepsilon T, \tag{1.13}$$

where $\boldsymbol{u}$ and T are the incremental displacement vector and temperature and ε is a small parameter. The Piola stress tensor, the specific entropy, and the heat flux vector can be represented as

$$\Pi^{\times} = \Pi_1 + \varepsilon \Pi^{\bullet} + o(\varepsilon^2), \quad \eta^{\times} = \eta_1 + \varepsilon \eta^{\bullet} + o(\varepsilon^2), \quad \boldsymbol{h}^{\times} = \boldsymbol{h}_1 + \varepsilon \boldsymbol{h}^{\bullet} + o(\varepsilon^2), \tag{1.14}$$

where the bold dot marks convective derivatives

$$f^{\bullet} = \frac{d}{d\varepsilon} f(\boldsymbol{R} + \varepsilon \boldsymbol{u}, T_1 + \varepsilon T) \Big|_{\varepsilon = 0}.$$

The parameters defining the perturbed state of a thermoelastic body should satisfy the equations of motion (1.8) and (1.9)

$$\nabla_0 \cdot \Pi^{\times} + \rho_0 \boldsymbol{b}^{\times} = \rho_0 \frac{\partial^2 \boldsymbol{R}^{\times}}{\partial t^2}, \qquad \nabla_0 \cdot \boldsymbol{h}^{\times} + \theta^{\times} \frac{\partial \eta^{\times}}{\partial t} = 0. \tag{1.15}$$

Substituting relations (1.13) and (1.14) into (1.15) and taking into account that the stress state (1.12) is in equilibrium, with accuracy up to $o(\varepsilon^2)$ we obtain the equations of motion and heat conduction linearized in the neighborhood of the initial state (1.11):

$$\nabla_0 \cdot \Theta = \nabla_0 \cdot \Pi^{\bullet} = \rho_0 \frac{\partial^2 \boldsymbol{u}}{\partial t^2}; \tag{1.16}$$

$$\nabla_0 \cdot \boldsymbol{h}^{\bullet} + T_1 \frac{\partial \eta^{\bullet}}{\partial t} = 0. \tag{1.17}$$

Here

$$\Theta = \Pi^{\bullet} = P^{\bullet} \cdot C + P \cdot \nabla_0 \boldsymbol{u}; \tag{1.18}$$

$$\boldsymbol{h}^{\bullet} = \frac{\partial \boldsymbol{h}}{\partial S} \circ S^{\bullet} + \frac{\partial \boldsymbol{h}}{\partial \theta} \theta^{\bullet} + \frac{\partial \boldsymbol{h}}{\partial \boldsymbol{g}} \cdot \boldsymbol{g}^{\bullet}; \tag{1.19}$$

$$\eta^{\bullet} = \frac{\partial \eta}{\partial S} \circ S^{\bullet} + \frac{\partial \eta}{\partial \theta} \theta^{\bullet}; \tag{1.20}$$

$$P^{\bullet} = \frac{\partial P}{\partial S} \circ S^{\bullet} + \frac{\partial P}{\partial \theta} \theta^{\bullet}. \tag{1.21}$$

The symbol $\circ$ denotes the complete multiplication operation. In constructing the constitutive relations, we assume that the state

$$S = 0, \qquad \theta = \theta_0 \tag{1.22}$$

is the state with minimum free energy. Retaining quadratic terms in strain and temperature deviation in the expansion of the function $\chi = \chi(S, \theta)$ [13, 14] in the neighborhood of state (1.22), we obtain

$$\chi = \frac{1}{2} {}^4C^W \cdot\cdot S \cdot\cdot S - \frac{1}{2} C_\varepsilon \rho_0 \frac{(\theta - \theta_0)^2}{\theta_0} - {}^2Q \cdot\cdot S \, (\theta - \theta_0). \tag{1.23}$$

Here ${}^4C^W$ is the fourth-rank tensor of second-order elastic constants, C_ε is the specific heat, ρ_0 is the density of the material, and ${}^2Q$ is the thermoelasticity tensor. For materials with hexagonal 6 mm symmetry, we have ${}^2Q = \|q_{ii}\|_{i=1}^3$, and $q_{11} = q_{22} \neq q_{33}$.

The tensor constants of the thermodynamic potential and heat conductivity are given by the formulas

$$ {}^4C = C_{ijkl}\boldsymbol{i}_i\boldsymbol{i}_j\boldsymbol{i}_k\boldsymbol{i}_l, \qquad Q = Q_{ij}\boldsymbol{i}_i\boldsymbol{i}_j, \qquad \lambda = \lambda_{ij}\boldsymbol{i}_i\boldsymbol{i}_j. \tag{1.24} $$

Adding expression (1.23) to formulas (1.2) and (1.4), we obtain

$$ P = {}^4C^W \cdot\cdot\, S - {}^2Q(\theta - \theta_0), \qquad \eta = \frac{C_\varepsilon \rho_0}{\theta_0}\,(\theta - \theta_0) + {}^2Q \cdot\cdot\, S; \tag{1.25} $$

$$ \frac{\partial P}{\partial S} = {}^4C^W, \qquad \frac{\partial P}{\partial \theta} = -{}^2Q, \qquad \frac{\partial \eta}{\partial \theta} = \frac{C_\varepsilon \rho_0}{\theta_0}. \tag{1.26} $$

Substituting formulas (1.25) and (1.26) into (1.18)–(1.20), considering (1.21) and (1.2)–(1.7), and using the relations

$$ \frac{\partial \eta}{\partial S} = \frac{\partial}{\partial S}\Big(-\frac{\partial \chi}{\partial \theta}\Big) = -\frac{\partial P}{\partial \theta}, \qquad S^\bullet = \frac{1}{2}\Big(\nabla_0 \boldsymbol{u}\cdot C^{\rm t} + C\cdot \nabla_0 \boldsymbol{u}^{\rm t}\Big), $$
$$ \theta^\bullet = T, \qquad \boldsymbol{g}^\bullet = \nabla_0 T, $$

we have

$$ \Pi^\bullet = ({}^4C^W \circ S^\bullet - {}^2Q\,T)\cdot C + P\cdot \nabla_0 \boldsymbol{u}; \tag{1.27} $$

$$ \boldsymbol{h}^\bullet = -\lambda \cdot \nabla_0 T; \tag{1.28} $$

$$ \eta^\bullet = {}^2Q \circ S^\bullet + \frac{C_\varepsilon \rho_0}{\theta_0}\,T. \tag{1.29} $$

We will assume that the initial strain state in the thermoelastic material is given by the conditions

$$ \boldsymbol{R} = \Lambda\cdot \boldsymbol{r}, \qquad \Lambda = \nu_k \boldsymbol{i}_k\boldsymbol{i}_k, \qquad \theta = T_1, \qquad \nu_k = \text{const}, \qquad T_1 = \text{const}, \tag{1.30} $$

where $\nu_k = 1 + \delta_k$; δ_k $(k = 1, 2, 3)$ are the relative elongations of fiber along the coordinate axes whose direction in the natural configuration coincides with the Cartesian coordinates; T_1 is the body temperature in the initial strained state.

In view of formulas (1.1), (1.23), (1.24), and (1.30), the stress tensor, the heat flux vector, and the specific entropy (1.27)–(1.29) can be represented in component form

$$ \Theta = \Theta_{ij}\boldsymbol{i}_i\boldsymbol{i}_j = \Pi^*_{ij}\boldsymbol{i}_i\boldsymbol{i}_j, \qquad \boldsymbol{h}^\bullet = h^*_i \boldsymbol{i}_i, $$
$$ \Theta_{ij} = \Big(C_{ijkl}\nu_k \frac{\partial u_k}{\partial x_l} - q_{ij}T\Big)\nu_j + P_{ik}\frac{\partial u_j}{\partial x_k} = C^*_{ijkl}\frac{\partial u_k}{\partial x_l} - q^*_{ij}T; \tag{1.31} $$

$$ h^*_i = -\lambda_{ii}\frac{\partial T}{\partial x_i}; \tag{1.32} $$

$$ \eta^\bullet = q^*_{ii}\frac{\partial u_i}{\partial x_i} + \frac{C_\varepsilon \rho_0}{\theta_0}\,T. \tag{1.33} $$

The linearized equations of motion are written in the component form

$$ \frac{\partial}{\partial x_i}\Big(C^*_{ijkl}\frac{\partial u_k}{\partial x_l} - q^*_{ij}T\Big) = \rho_0 \frac{\partial^2 u_j}{\partial t^2}, \qquad j = 1, 2, 3, $$
$$ \frac{\partial}{\partial x_i}\Big(\lambda_{ii}\frac{\partial T}{\partial x_i}\Big) - T_1 q^*_{ii}\frac{\partial}{\partial t}\Big(\frac{\partial u_i}{\partial x_i}\Big) - T_1\frac{C_\varepsilon \rho_0}{\theta_0}\frac{\partial T}{\partial t} = 0. $$

In the case of the homogeneous stress–strain state (1.30), we have

$$ C = \nu_k \boldsymbol{i}_k\boldsymbol{i}_k, \qquad S = S_{kk}\boldsymbol{i}_k\boldsymbol{i}_k, \qquad S_{kk} = \frac{1}{2}\,(\nu_k^2 - 1), $$
$$ P = P_{kk}\boldsymbol{i}_k\boldsymbol{i}_k, \qquad P_{kk} = \frac{1}{2}\,C_{kkkk}(\nu_k^2 - 1) - (T_1 - \theta_0)q_{kk}. \tag{1.34} $$

The coefficients C^*_{ijkl} and q^*_{ij} are defined as follows:

$$ C^*_{ijkl} = C_{ijkl}\nu_k\nu_j + P_{il}\delta_{jk}, \qquad q^*_{ij} = q_{ij}\nu_j. \tag{1.35} $$

From (1.35) it follows that for any form of the stress–strain state ($P_{ii} = P^0_i$, $i = 1, 2, 3$), the symmetry of the starting material is broken. Thus, the Voigt notation cannot be used to represent the coupling matrix in the stress–strain state.

2. FORMULATION OF DYNAMIC PROBLEMS FOR A PRESTRESSED THERMOELASTIC HALF-SPACE WITH A COATING

We consider harmonic vibrations of an inhomogeneous prestressed thermoelastic medium consisting of a stack of $M-1$ homogeneous or functionally graded thermoelastic layers $0 \leqslant x_3 \leqslant H$, $H = h_1 \geqslant h_2 \geqslant \ldots \geqslant h_M = 0$, $|x_1, x_2| \leqslant \infty$, in a homogeneous half-space $x_3 \leqslant 0$, $|x_1, x_2| \leqslant \infty$. We assume that the coating materials are thermoelastic materials which in the natural state have hexagonal 6 mm symmetry. The initial strain state of each component of the medium is homogeneous (1.30) and is due to the action of the initial stress and temperature. System (1.16), (1.17) can be represented as

$$\nabla_0 \cdot \Theta^{(n)} = \rho^{(n)} \ddot{\boldsymbol{u}}^{(n)}; \tag{2.1}$$

$$\nabla_0 \cdot \boldsymbol{h}^{(n)} + T_1^{(n)} \frac{\partial \eta^{(n)}}{\partial t} = 0. \tag{2.2}$$

The mechanical boundary conditions on the surface $o = o_1 + o_2$ are written as

$$\boldsymbol{n} \cdot \Theta^{(1)} = \boldsymbol{f}^* \Big|_{o_1}; \tag{2.3}$$

$$\boldsymbol{u}^{(1)} = \boldsymbol{u}^* \Big|_{o_2}, \tag{2.4}$$

and the thermal boundary conditions on the surface of $o = o_3 + o_4$ are represented as

$$\boldsymbol{n} \cdot \boldsymbol{h}^{(1)} = -h^* \Big|_{o_3}; \tag{2.5}$$

$$T^{(1)} = T^* \Big|_{o_4}. \tag{2.6}$$

In (2.1)–(2.6), $\boldsymbol{u}^*$, $\boldsymbol{f}^*$, and $\boldsymbol{n}$ are the displacement, stress, and outward normal vectors to the surface of the medium defined in the natural state, respectively (the quantities specified in the corresponding region are marked with an asterisk), $\rho^{(n)}$ is the density of the material, $T^{(n)}$ is the dynamic temperature, and h^* and T^* are the heat flux and temperature, respectively; the superscript in parentheses denotes the parameters of the nth layer of the coating ($n = 1, \ldots, M-1$), and the subscript M the parameters of the half-space. For functionally graded components of the coating, the dependence of the elastic and temperature parameters of the coordinate x_3 has the form

$$\rho^{(n)} = \rho_0^{(n)} f_\rho^{(n)}(x_3), \qquad C_{lksm}^{(n)} = C_{lksm}^{0(k)} f_c^{(k)}(x_3), \qquad q_{lk}^{(n)} = q_{lk}^{0(n)} f_q^{(n)}(x_3),$$

$$\lambda_{lk}^{(n)} = \lambda_{lk}^{0(n)} f_\lambda^{(n)}(x_3), \qquad C_\varepsilon^{(n)} = C_\varepsilon^{0(n)} f_{c_\varepsilon}^{(n)}(x_3)$$

(the subscript and superscript "0" marks the constants of the corresponding material of the base).

In view of the assumptions made and expressions (1.31)–(1.33), the components of the tensor $\Theta^{(n)}$, vector $\boldsymbol{h}^{(n)}$, and function $\eta^{(n)}$ are given by the formulas

$$\Theta_{lk}^{(n)} = C_{lksm}^{*(n)} u_{s,m}^{(n)} - q_{lk}^{*(n)} T^{(n)}, \quad h_i^{(n)} = -\lambda_{ii}^{(n)} \frac{\partial T^{(n)}}{\partial x_i}, \quad \eta^{(n)} = q_{sp}^{*(n)} u_{s,p}^{(n)} + \frac{\rho^{(n)} C_\varepsilon^{(n)}}{T_0} T^{(n)}, \tag{2.7}$$

where, in view of the representation (1.35),

$$C_{lksp}^{*(n)} = P_{lp}^{(n)} \delta_{ks} + \nu_k^{(n)} \nu_s^{(n)} C_{lksp}^{(n)}, \qquad q_{lk}^{*(n)} = \nu_k^{(n)} q_{lk}^{(n)} \tag{2.8}$$

(the subscript after the comma denotes differentiation).

We introduce the extended displacement vector $\boldsymbol{u}_\tau^{(n)} = \{u_1^{(n)}, u_2^{(n)}, u_3^{(n)}, u_4^{(n)} = T^{(n)}\}$, load vector $\boldsymbol{f}_\tau = \{f_1, f_2, f_3, f_4 = -h^*\}$, and notation

$$\theta_{lksp}^{(n)} = C_{lksp}^{*(n)}, \quad \theta_{lk44}^{(n)} = -q_{lk}^{*(n)}, \quad \theta_{4444}^{(n)} = -C_\varepsilon^{(n)} \rho^{(n)} (T_0)^{-1}, \quad k, l, s, p = 1, 2, 3. \tag{2.9}$$

In view of formulas (2.7), (2.8), and (1.34) and the properties of the material [15], the coupling matrix Hin notation (2.9) takes the form

$$
H = \left(\begin{array}{c|ccccccccc|c}
 & u_{1,1}^{(n)} & u_{2,2}^{(n)} & u_{3,3}^{(n)} & u_{2,3}^{(n)} & u_{3,2}^{(n)} & u_{1,3}^{(n)} & u_{3,1}^{(n)} & u_{1,2}^{(n)} & u_{2,1}^{(n)} & u_{4}^{(n)} \\
\hline
\Theta_{11}^{(n)} & \theta_{1111}^{(n)} & \theta_{1122}^{(n)} & \theta_{1133}^{(n)} & 0 & 0 & 0 & 0 & 0 & 0 & \theta_{1144}^{(n)} \\
\Theta_{22}^{(n)} & \theta_{1122}^{(n)} & \theta_{2222}^{(n)} & \theta_{2233}^{(n)} & 0 & 0 & 0 & 0 & 0 & 0 & \theta_{2244}^{(n)} \\
\Theta_{33}^{(n)} & \theta_{1133}^{(n)} & \theta_{2233}^{(n)} & \theta_{3333}^{(n)} & 0 & 0 & 0 & 0 & 0 & 0 & \theta_{3344}^{(n)} \\
\Theta_{23}^{(n)} & 0 & 0 & 0 & \theta_{2323}^{(n)} & \theta_{2332}^{(n)} & 0 & 0 & 0 & 0 & 0 \\
\Theta_{32}^{(n)} & 0 & 0 & 0 & \theta_{3223}^{(n)} & \theta_{2323}^{(n)} & 0 & 0 & 0 & 0 & 0 \\
\Theta_{13}^{(n)} & 0 & 0 & 0 & 0 & 0 & \theta_{1313}^{(n)} & \theta_{1331}^{(n)} & 0 & 0 & 0 \\
\Theta_{31}^{(n)} & 0 & 0 & 0 & 0 & 0 & \theta_{3113}^{(n)} & \theta_{1313}^{(n)} & 0 & 0 & 0 \\
\Theta_{12}^{(n)} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \theta_{1212}^{(n)} & \theta_{1221}^{(n)} & 0 \\
\Theta_{21}^{(n)} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \theta_{2112}^{(n)} & \theta_{1212}^{(n)} & 0 \\
\hline
-\eta^{(n)} & \theta_{1144}^{(n)} & \theta_{2244}^{(n)} & \theta_{3344}^{(n)} & 0 & 0 & 0 & 0 & 0 & 0 & \theta_{4444}^{(n)}
\end{array}\right). \tag{2.10}
$$

Following [4–7], we convert to the dimensionless variables

$$
\begin{gathered}
x_i' = \frac{\omega^* x_i}{V_p^{(M)}}, \qquad u_i^{(n)\prime} = \frac{\rho^{(M)}\omega^* V_p^{(M)}}{q_{11}^{(M)} T_0}\, u_i^{(n)}, \qquad T^{(n)\prime} = \frac{T^{(n)}}{T_0}, \\
\omega' = \frac{\omega}{\omega^*}, \qquad \omega^* = \frac{C_\varepsilon^{(M)} C_{11}^{(M)}}{\lambda_{11}^{(M)}}, \qquad h_i^{(n)\prime} = \frac{V_p^{(M)}}{\omega^* T_0 \lambda_{11}^{(M)}}\, h_i^{(n)}, \\
\Theta_{ij}^{(n)\prime} = \frac{\Theta_{ij}^{(n)}}{q_{11}^{(M)} T_0}, \qquad \theta_{ijkl}^{(n)\prime} = \frac{\theta_{ijkl}^{(n)}}{C_{11}^{(M)}}, \qquad \theta_{kk44}^{(n)\prime} = \frac{\theta_{kk44}^{(n)}}{q_{11}^{(M)}}, \qquad \theta_{4444}^{(n)\prime} = \frac{\rho^{(n)} C_\varepsilon^{(n)}}{T_0 \rho^{(M)} C_\varepsilon^{(M)}}, \\
\lambda_{ij}^{(n)\prime} = \frac{\lambda_{ij}^{(n)}}{\lambda_{11}^{(M)}} \quad (i,j,k,l = 1,2,3), \qquad E = \frac{T_0 (q_{11}^{(M)})^2}{\rho^{(M)} C_\varepsilon^{(M)} C_{11}^{(M)}}, \qquad E_T^{(n)} = E T_1^{(n)\prime}
\end{gathered} \tag{2.11}
$$

(E and E_T are the dimensionless normalizing factors).

After substitution of expressions (2.7) into representations (2.1)–(2.6) in view of formulas (2.9)–(2.11), the boundary-value problem for the prestressed thermoelastic half-space with an inhomogeneous coating takes the following form in dimensionless parameters:

$$
\begin{gathered}
L_{11}^*[u_1^{(n)}] + \theta_1^{(n)} u_{2,12}^{(n)} + \theta_2^{(n)} u_{3,13}^{(n)} + \theta_{1144}^{(n)} u_{4,1}^{(n)} = 0, \\
\theta_1^{(n)} u_{1,12}^{(n)} + L_{22}^*[u_2^{(n)}] + \theta_3^{(n)} u_{3,23}^{(n)} + \theta_{2244}^{(n)} u_{4,2}^{(n)} = 0, \\
\theta_2^{(n)} u_{1,13}^{(n)} + \theta_3^{(n)} u_{2,23}^{(n)} + L_{33}^*[u_3^{(n)}] + \theta_{3344}^{(n)} u_{4,3}^{(n)} = 0, \\
i\omega E_T^{(n)} [\theta_{1144}^{(n)} u_{1,1}^{(n)} + \theta_{2244}^{(n)} u_{2,2}^{(n)} + \theta_{3344}^{(n)} u_{3,3}^{(n)}] - L_{44}^*[u_4^{(n)}] = 0
\end{gathered} \tag{2.12}
$$

for the homogeneous components of the medium and

$$
\begin{gathered}
L_{11}^{*f}[u_1^{(n)}] + \theta_1^{(n)} u_{2,12}^{(n)} + L_{13}^{*f}[u_3^{(n)}] + \theta_{1144}^{(n)} u_{4,1}^{(n)} = 0, \\
\theta_1^{(n)} u_{1,12}^{(n)} + L_{22}^{*f}[u_2^{(n)}] + L_{23}^{*f}[u_3^{(n)}] + \theta_{2244}^{(n)} u_{4,2}^{(n)} = 0, \\
L_{31}^{*f}[u_1^{(n)}] + L_{32}^{*f}[u_2^{(n)}] + L_{33}^{*f}[u_3^{(n)}] + L_{34}^{*f}[u_4^{(n)}] = 0, \\
i\omega E_T^{(n)} [\theta_{1144}^{(n)} u_{1,1}^{(n)} + \theta_{2244}^{(n)} u_{2,2}^{(n)} + \theta_{3344}^{(n)} u_{3,3}^{(n)}] - L_{44}^{*f}[u_4^{(n)}] = 0
\end{gathered} \tag{2.13}
$$

for the functionally graded coating components.

The linearized boundary conditions are written as

$$\Sigma_k^{(1)}\Big|_{x_3=h,\ (x_1,x_2)\in o_1} = f_k(x_1,x_2), \quad u_k^{(1)}\Big|_{x_3=h,\ (x_1,x_2)\in o_2} = u_k^*(x_1,x_2), \quad k=1,2,3; \tag{2.14}$$

$$\lambda_{33}^{(1)} u_{4,3}^{(1)}\Big|_{x_3=h,\ (x_1,x_2)\in o_3} = f_4(x_1,x_2), \qquad u_4^{(1)}\Big|_{x_3=h,\ (x_1,x_2)\in o_4} = T^*(x_1,x_2); \tag{2.15}$$

$$\Sigma_k^{(n)}\Big|_{x_3=h_k} = \Sigma_k^{(n+1)}\Big|_{x_3=h_k}, \qquad u_k^{(n)}\Big|_{x_3=h_k} = u_k^{(n+1)}\Big|_{x_3=h_k},$$
$$k=1,\dots,4, \quad n=2,\dots,M-1; \tag{2.16}$$

$$u_k^{(M)}\Big|_{x_3\to-\infty} \to 0. \tag{2.17}$$

In formulas (2.12)–(2.17), we have

$$\theta_1^{(n)} = \theta_{1122}^{(n)} + \theta_{1212}^{(n)}, \qquad \theta_2^{(n)} = \theta_{1133}^{(n)} + \theta_{1313}^{(n)}, \qquad \theta_3^{(n)} = \theta_{2233}^{(n)} + \theta_{2323}^{(n)},$$

$$L_{kk}^* = \theta_{ikki}^{(n)} \frac{\partial^2}{\partial x_i^2} + \rho^{(n)}\omega^2 \quad (k,i=1,2,3), \qquad L_{44}^* = \lambda_{ii}^{(n)} \frac{\partial^2}{\partial x_i^2} - i\omega T_1^{(n)}\theta_{4444}^{(n)}, \tag{2.18}$$

$$L_{kk}^{*f} = L_{kk}^* + \frac{\partial\theta_{3kk3}^{(n)}}{\partial x_3}\frac{\partial}{\partial x_3} \quad (k=1,2,3), \qquad L_{44}^{*f} = L_{44}^* + \frac{\partial\lambda_{33}^{(n)}}{\partial x_3}\frac{\partial}{\partial x_3};$$

$$L_{s3}^{*f} = (\theta_{ss33}^{(n)} + \theta_{s3s3}^{(n)})\frac{\partial^2}{\partial x_s\,\partial x_3} + \frac{\partial\theta_{s3s3}^{(n)}}{\partial x_3}\frac{\partial}{\partial x_s},$$

$$L_{3s}^{*f} = (\theta_{ss33}^{(n)} + \theta_{s3s3}^{(n)})\frac{\partial^2}{\partial x_s\,\partial x_3} + \frac{\partial\theta_{ss33}^{(n)}}{\partial x_3}\frac{\partial}{\partial x_s}, \qquad s=1,2, \tag{2.19}$$

$$L_{34}^{*f} = \theta_{3344}^{(n)}\frac{\partial}{\partial x_3} + \frac{\partial\theta_{3344}^{(n)}}{\partial x_3};$$

$$\Sigma_\tau^{(n)} = \{\Sigma_k^{(n)}\}_{k=1}^4, \quad \Sigma_p^{(n)} = \Theta_{3p}^{(n)}, \quad \Sigma_4^{(n)} = -h_3^{(n)} = \lambda_{33}^{(n)}u_{4,3}^{(n)}, \quad p=1,2,3. \tag{2.20}$$

The components of the extended vector $\Sigma_\tau^{(n)}$ (2.20) are given by the formulas

$$\Sigma_1^{(n)} = \Theta_{31}^{(n)} = \theta_{3113}^{(n)}u_{1,3}^{(n)} + \theta_{1313}^{(n)}u_{3,1}^{(n)},$$

$$\Sigma_2^{(n)} = \Theta_{32}^{(n)} = \theta_{3223}^{(n)}u_{2,3}^{(n)} + \theta_{2323}^{(n)}u_{3,2}^{(n)}, \qquad \Sigma_4^{(n)} = -h_3^{(n)} = \lambda_{33}^{(n)}u_{4,3}^{(n)},$$

$$\Sigma_3^{(n)} = \Theta_{33}^{(n)} = \theta_{1133}^{(n)}u_{1,1}^{(n)} + \theta_{2233}^{(n)}u_{2,2}^{(n)} + \theta_{3333}^{(n)}u_{3,3}^{(n)} + \theta_{3344}^{(n)}u_4^{(n)}.$$

3. GREEN'S FUNCTION FOR AN INHOMOGENEOUS PRESTRESSED THERMOELASTIC HALF-SPACE

We subject problem (2.12), (2.13) with boundary conditions (2.14)–(2.17) to a Fourier transform over the coordinates x_1 and x_2 (α_1 and α_2 are the transform parameters). In the space of images, systems (2.12), (2.13) take the form

$$L_{11}^\Lambda[U_1^{(n)}] - \alpha_1\alpha_2\theta_1^{(n)}U_2^{(n)} - i\alpha_1\theta_2^{(n)}U_3^{(n)\prime} - i\alpha_1\theta_{1144}^{(n)}U_4^{(n)} = 0,$$

$$-\alpha_1\alpha_2\theta_1^{(n)}U_1^{(n)} + L_{22}^\Lambda[U_2^{(n)}] - i\alpha_2\theta_3^{(n)}U_3^{(n)\prime} - i\alpha_2\theta_{2244}^{(n)}U_4^{(n)} = 0,$$

$$-i\alpha_1\theta_2^{(n)}U_1^{(n)\prime} - i\alpha_2\theta_3^{(n)}U_2^{(n)\prime} + L_{33}^\Lambda[U_3^{(n)}] + \theta_{3344}^{(n)}U_4^{(n)\prime} = 0, \tag{3.1}$$

$$\omega E_T^{(n)}(\alpha_1\theta_{1144}^{(n)}U_1^{(n)} + \alpha_2\theta_{2244}^{(n)}U_2^{(n)} + i\theta_{3344}^{(n)}U_3^{(n)\prime}) - L_{44}^{\Lambda}[U_4^{(n)}] = 0;$$

$$\begin{gathered}
L_{11}^{\Lambda f}[U_1^{(n)}] - \alpha_1\alpha_2\theta_1^{(n)}U_2^{(n)} + L_{13}^{\Lambda f}[U_3^{(n)}] - i\alpha_1\theta_{1144}^{(n)}U_4^{(n)} = 0,\\
-\alpha_1\alpha_2\theta_1^{(n)}U_1^{(n)} + L_{22}^{\Lambda f}[U_2^{(n)}] + L_{23}^{\Lambda f}[U_3^{(n)}] - i\alpha_2\theta_{2244}^{(n)}U_4^{(n)} = 0,\\
L_{31}^{\Lambda f}[U_1^{(n)}] + L_{32}^{\Lambda f}[U_2^{(n)}] + L_{33}^{\Lambda f}[U_3^{(n)}] + L_{34}^{\Lambda f}[U_4^{(n)}] = 0,\\
\omega E_T^{(n)}(\alpha_1\theta_{1144}^{(n)}U_1^{(n)} + \alpha_2\theta_{2244}^{(n)}U_2^{(n)} + i\theta_{3344}^{(n)}U_3^{(n)\prime}) - L_{44}^{\Lambda f}[U_4^{(n)}] = 0.
\end{gathered} \tag{3.2}$$

In (3.1) and (3.2), we have

$$\begin{gathered}
L_{kk}^{\Lambda} = \theta_{3kk3}^{(n)}\frac{\partial^2}{\partial x_3^2} - \alpha_s^2\theta_{skks}^{(n)} + \rho^{(n)}\omega^2, \qquad k = 1,2,3,\\
L_{44}^{\Lambda} = \lambda_{33}^{(n)}\frac{\partial^2}{\partial x_3^2} - \alpha_s^2\lambda_{ss}^{(n)} - i\omega T_1^{(n)}\theta_{4444}^{(n)}, \qquad s = 1,2,
\end{gathered} \tag{3.3}$$

$$\begin{gathered}
L_{kk}^{\Lambda f} = L_{kk}^{\Lambda} + \theta_{3kk3}^{(n)\prime}\frac{\partial}{\partial x_3}, \qquad L_{44}^{\Lambda f} = L_{44}^{\Lambda} + \lambda_{33}^{(n)\prime}\frac{\partial}{\partial x_3}, \qquad k,i = 1,2,3;\\
L_{s3}^{\Lambda f} = -i\alpha_s(\theta_{ss33}^{(n)} + \theta_{s3s3}^{(n)})\frac{\partial}{\partial x_3} - i\alpha_s\theta_{s3s3}^{(n)\prime},\\
L_{3s}^{\Lambda f} = -i\alpha_s(\theta_{ss33}^{(n)} + \theta_{s3s3}^{(n)})\frac{\partial}{\partial x_3} - i\alpha_s\theta_{ss33}^{(n)\prime}, \qquad s = 1,2,\\
L_{34}^{\Lambda f} = \theta_{3344}^{(n)}\frac{\partial}{\partial x_3} + \theta_{3344}^{(n)\prime}.
\end{gathered} \tag{3.4}$$

For the boundary conditions, we have

$$\Sigma_k^{\Lambda(1)} = F_k\Big|_{x_3=h}, \qquad k = 1,\dots,4; \tag{3.5}$$

$$\Sigma_k^{\Lambda(n)} = \Sigma_k^{\Lambda(n+1)}\Big|_{x_3=h_n}, \quad U_k^{(n)} = U_k^{(n+1)}\Big|_{x_3=h_n}, \quad k = 1,\dots,4, \quad n = 2,\dots,M-1; \tag{3.6}$$

$$U_k^{(M)}\Big|_{x_3\to-\infty} \to 0, \tag{3.7}$$

where $U_k^{(n)}$, $\Sigma_k^{\Lambda(n)}$, and F_k ($k = 1,\dots,4$) are transforms of the Fourier components $\boldsymbol{u}_\tau^{(n)}$, $\Sigma_\tau^{(n)}$, and $\boldsymbol{f}_\tau$, respectively; the prime denotes differentiation with respect to x_3.

Thus, the boundary-value problem of harmonic vibrations of a prestressed thermoelastic medium consisting of a stack of homogeneous or functionally graded layers lying on homogeneous base, depending on the type of source, the nature of its effect, and the structure of the medium is described by system (2.12), (2.13) with boundary conditions (2.14)–(2.17) in notation (2.18)–(2.20). Using methods of operational calculus, we reduce the problem to solving the boundary-value problem (3.1), (3.2), (3.5)–(3.7) in notation (3.3), (3.4). The solution of system (3.1) has the following form:

$$\begin{gathered}
U_p^{(M)}(\alpha_1,\alpha_2,x_3) = -i\alpha_p\sum_{k=1}^{4} f_{pk}^{(M)}c_{k+g}\,\mathrm{e}^{\sigma_k^{(M)}x_3}, \qquad p = 1,2, \quad g = 8(M-1),\\
U_p^{(M)}(\alpha_1,\alpha_2,x_3) = \sum_{k=1}^{4} f_{pk}^{(M)}c_{k+g}\,\mathrm{e}^{\sigma_k^{(M)}x_3} \qquad (p = 3,4)
\end{gathered} \tag{3.8}$$

for the half-space or

$$U_p^{(n)}(\alpha_1,\alpha_2,x_3) = -i\alpha_p \sum_{k=1}^{4} f_{pk}^{(n)}\Big(c_{k+g}\sinh\,\sigma_k^{(n)}x_3 + c_{k+4+g}\cosh\,\sigma_k^{(n)}x_3\Big), \quad p=1,2,\ g=8(n-1),$$

$$U_3^{(n)}(\alpha_1,\alpha_2,x_3) = \sum_{k=1}^{4} f_{3k}^{(n)}\Big(c_{k+g}\cosh\,\sigma_k^{(n)}x_3 + c_{k+4+g}\sinh\,\sigma_k^{(n)}x_3\Big), \tag{3.9}$$

$$U_4^{(n)}(\alpha_1,\alpha_2,x_3) = \sum_{k=1}^{4} f_{4k}^{(n)}\Big(c_{k+g}\sinh\,\sigma_k^{(n)}x_3 + c_{k+4+g}\cosh\,\sigma_k^{(n)}x_3\Big)$$

for the homogeneous coating layers. Here $\sigma_k^{(n)}$ are the roots of the characteristic equation $\det M_\sigma^{(n)} = 0$,

$$M_\sigma^{(n)} = \begin{pmatrix} A_{11}^{(n)} & -\alpha_2^2\theta_1^{(n)} & \sigma_k^{(n)}\theta_2^{(n)} & \theta_{1144}^{(n)} \\ -\alpha_1^2\theta_1^{(n)} & A_{22}^{(n)} & \sigma_k^{(n)}\theta_3^{(n)} & \theta_{2244}^{(n)} \\ -\alpha_1^2\sigma_k^{(n)}\theta_2^{(n)} & -\alpha_2^2\sigma_k^{(n)}\theta_3^{(n)} & A_{33}^{(n)} & \sigma_k^{(n)}\theta_{3344}^{(n)} \\ -\alpha_1^2 i\omega\, E_T^{(n)}\theta_{1144}^{(n)} & -\alpha_2^2 i\omega\, E_T^{(n)}\theta_{2244}^{(n)} & \sigma_k^{(n)} i\omega\, E_T^{(n)}\theta_{3344}^{(n)} & -A_{44}^{(n)} \end{pmatrix}, \tag{3.10}$$

$$A_{ll}^{(n)} = \theta_{3ll3}^{(n)}(\sigma_k^{(n)})^2 - \alpha_s^2\theta_{slls}^{(n)} + \rho^{(n)}\omega^2, \qquad A_{44}^{(n)} = \lambda_{33}^{(n)}(\sigma_k^{(n)})^2 - \alpha_s^2\lambda_{ss}^{(n)} - i\omega T_1^{(n)}\theta_{4444}^{(n)},$$

$$s=1,2, \qquad l=1,2,3.$$

The coefficients of $f_{pk}^{(n)}$ $(p,k = 1,\ldots,4)$ satisfy the homogeneous system of equations with the matrix $M_\sigma^{(n)}(\sigma_k)$ (3.10).

In accordance with [10, 11], we introduce the variables

$$Y^{(n)} = \begin{pmatrix} Y_\Sigma^n \\ Y_u^n \end{pmatrix}, \qquad Y_\Sigma^n = \|\Sigma_k^{\Lambda(n)}\|_{k=1}^4, \qquad Y_u^n = \|U_k^{(n)}\|_{k=5}^8,$$

in which system (3.2) is represented as

$$Y^{(n)\prime} = M^{(n)}(\alpha_1,\alpha_2,x_3)Y^{(n)}, \tag{3.11}$$

$$M^{(n)} = \begin{pmatrix} 0 & 0 & m_{13}^{(n)} & 0 & m_{15}^{(n)} & m_{16}^{(n)} & 0 & m_{18}^{(n)} \\ 0 & 0 & m_{23}^{(n)} & 0 & m_{25}^{(n)} & m_{26}^{(n)} & 0 & m_{28}^{(n)} \\ m_{31}^{(n)} & m_{32}^{(n)} & 0 & 0 & 0 & 0 & m_{37}^{(n)} & 0 \\ 0 & 0 & m_{43}^{(n)} & 0 & m_{45}^{(n)} & m_{46}^{(n)} & 0 & m_{48}^{(n)} \\ m_{51}^{(n)} & 0 & 0 & 0 & 0 & 0 & m_{57}^{(n)} & 0 \\ 0 & m_{62}^{(n)} & 0 & 0 & 0 & 0 & m_{67}^{(n)} & 0 \\ 0 & 0 & m_{73}^{(n)} & 0 & m_{75}^{(n)} & m_{76}^{(n)} & 0 & m_{78}^{(n)} \\ 0 & 0 & 0 & m_{84}^{(n)} & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$m_{13}^{(n)} = \frac{i\alpha_1\theta_{1133}^{(n)}}{\theta_{3333}^{(n)}}, \qquad m_{15}^{(n)} = -\frac{(\theta_{1133}^{(n)})^2\alpha_1^2}{\theta_{3333}^{(n)}} + P_1^{(n)},$$

$$m_{16}^{(n)} = (\theta_1^{(n)}\theta_{3333}^{(n)} - \theta_{2233}^{(n)}\theta_{1133}^{(n)})\,\frac{\alpha_1\alpha_2}{\theta_{3333}^{(n)}}, \qquad m_{s8}^{(n)} = \frac{i\alpha_s}{\theta_{3333}^{(n)}}\,(\theta_{ss44}^{(n)}\theta_{3333}^{(n)} - \theta_{3344}^{(n)}\theta_{ss33}^{(n)}), \qquad s=1,2,$$

$$m_{23}^{(n)} = \frac{i\alpha_2\theta_{2233}^{(n)}}{\theta_{3333}^{(n)}}, \qquad m_{25}^{(n)} = m_{16}^{(n)}, \qquad m_{26}^{(n)} = -\frac{(\theta_{2233}^{(n)})^2\alpha_2^2}{\theta_{3333}^{(n)}} + P_2^{(n)},$$

$$m_{3s}^{(n)} = \frac{i\alpha_s\theta_{s3s3}^{(n)}}{\theta_{3ss3}^{(n)}}, \quad s=1,2, \qquad m_{37}^{(n)} = -\frac{\alpha_k^2(\theta_{k3k3}^{(n)})^2}{\theta_{3kk3}^{(n)}} + P_3^{(n)}, \quad k=1,2,$$

$$m_{43}^{(n)} = -\frac{i\omega E^* \theta_{3344}^{(n)}}{\theta_{3333}^{(n)}}, \qquad m_{45}^{(n)} = \frac{\omega E^* \alpha_1}{\theta_{3333}^{(n)}}(\theta_{3344}^{(n)}\theta_{1133}^{(n)} - \theta_{1144}^{(n)}\theta_{3333}^{(n)}),$$

$$m_{46}^{(n)} = \frac{\omega E^* \alpha_2}{\theta_{3333}^{(n)}}(\theta_{3344}^{(n)}\theta_{2233}^{(n)} - \theta_{2244}^{(n)}\theta_{3333}^{(n)}), \qquad m_{48}^{(n)} = \frac{i\omega E^* (\theta_{3344}^{(n)})^2}{\theta_{3333}^{(n)}} + P_4^{(n)},$$

$$m_{51}^{(n)} = (\theta_{3113}^{(n)})^{-1}, \qquad m_{57}^{(n)} = m_{31}^{(n)}, \qquad m_{62}^{(n)} = (\theta_{3223}^{(n)})^{-1}, \qquad m_{67}^{(n)} = m_{32}^{(n)},$$

$$m_{73}^{(n)} = (\theta_{3333}^{(n)})^{-1}, \quad m_{75}^{(n)} = m_{13}^{(n)}, \quad m_{76}^{(n)} = m_{23}^{(n)}, \quad m_{78}^{(n)} = -\frac{\theta_{3344}^{(n)}}{\theta_{3333}^{(n)}}, \quad m_{84}^{(n)} = (\lambda_{33}^{(n)})^{-1},$$

$$P_k^{(n)} = \alpha_i^2 \theta_{ikki}^{(n)} - \rho^{(n)}\omega^2 \quad (k = 1, 2, 3), \qquad P_4^{(n)} = \alpha_i^2 \lambda_{ii}^{(n)} + i\omega T_1^{(n)} \theta_{4444}^{(n)} \quad (i = 1, 2).$$

System (3.11) is a system of first-order ordinary differential equations with variable coefficients, which can be solved using numerical methods, in particular, the Runge–Kutta method. We represent $Y_k^{(n)}$ in the form of the expansion

$$Y_k^{(n)} = \sum_{p=1}^{8} c_{p+g}(\alpha_1, \alpha_2) y_{kp}^{(n)}(\alpha_1, \alpha_2, x_3), \qquad k = 1, 2, \ldots, 8, \quad g = 8(n-1), \tag{3.12}$$

where $y_{kp}^{(n)}(\alpha_1, \alpha_2, x_3)$ are linear independent solutions of the Cauchy problem for system (3.11) with the initial conditions $y_{kp}^{(n)}(\alpha_1, \alpha_2, x_{30}) = \delta_{kp}$.

The solution of the boundary-value problem (3.1)–(3.7) is the set of solutions for the homogeneous (3.9) and inhomogeneous (3.12) components of the coating and half-space (3.8). The unknowns c_k are determined by substituting the solutions into the boundary conditions

$$AC = \boldsymbol{F}. \tag{3.13}$$

Here $C^{\mathrm{t}} = \{c_p\}_{p=1}^{8(M-1)+4}$ is the vector of unknowns; $\boldsymbol{F}^{\mathrm{t}} = \{\boldsymbol{F}_\tau, \boldsymbol{F}_0\}$ ($\boldsymbol{F}_\tau$ is the Fourier transform of a given load vector and $\boldsymbol{F}_0$ is the zero vector, whose dimension is determined by the geometry of the problem), and

$$A = \begin{pmatrix} B^{(1)}(h_1) & 0 \\ A^{(1)}(h_{2,\ldots,M}) & B^{(M)}(h_M) \end{pmatrix}. \tag{3.14}$$

Here $B^{(1)}(h_1)$ and $B^{(M)}(h_M)$ are 4×8 and 8×4 rectangular matrices; A and $A^{(1)}(h_{2,\ldots,M})$ are $[4(2M-1)]$ and $[8(M-1)]$ square matrices, respectively. The elements of the matrix (3.14) have the following form:
—for the homogeneous upper layer,

$$B^{(1)}(h_1) = \begin{pmatrix} l_{11}^{(1)} c_{11}^1 & l_{12}^{(1)} c_{21}^1 & l_{13}^{(1)} c_{31}^1 & l_{14}^{(1)} c_{41}^1 & l_{11}^{(1)2} s_{11}^{01} & l_{12}^{(1)2} s_{21}^{01} & l_{13}^{(1)2} s_{31}^{01} & l_{14}^{(1)2} s_{41}^{01} \\ l_{21}^{(1)} c_{11}^1 & l_{22}^{(1)} c_{21}^1 & l_{23}^{(1)} c_{31}^1 & l_{24}^{(1)} c_{41}^1 & l_{21}^{(1)2} s_{11}^{01} & l_{22}^{(1)2} s_{21}^{01} & l_{23}^{(1)2} s_{21}^{01} & l_{24}^{(1)2} s_{41}^{01} \\ l_{31}^{(1)0} s_{11}^{01} & l_{32}^{(1)0} s_{21}^{01} & l_{33}^{(1)0} s_{31}^{01} & l_{34}^{(1)0} s_{41}^{01} & l_{31}^{(1)0} c_{11}^1 & l_{32}^{(1)0} c_{21}^1 & l_{33}^{(1)0} c_{31}^1 & l_{34}^{(1)0} c_{41}^1 \\ l_{41}^{(1)} c_{11}^1 & l_{42}^{(1)} c_{21}^1 & l_{43}^{(1)} c_{31}^1 & l_{44}^{(1)} c_{41}^1 & l_{41}^{(1)2} s_{11}^{01} & l_{42}^{(1)2} s_{21}^{01} & l_{43}^{(1)2} s_{31}^{01} & l_{44}^{(1)2} s_{41}^{01} \end{pmatrix}, \tag{3.15}$$

where

$$l_{pk}^{(n)} c_{ki}^n = l_{pk}^{(n)} \cosh \sigma_k^{(n)} h_i, \qquad l_{pk}^{(n)} s_{ki}^{0n} = l_{pk}^{(n)} \sinh^0 \sigma_k^{(n)} h_i, \qquad l_{pk}^{(n)2} s_{ki}^{0n} = l_{pk}^{(n)} (\sigma_k^{(n)})^2 \sinh^0 \sigma_k^{(n)} h_i,$$

$$\sinh^0 \sigma_k^{(n)} h_i = (\sigma_k^{(n)})^{-1} \sinh \sigma_k^{(n)} h_i, \qquad p, k = 1, \ldots, 4, \quad n = 1, \ldots, M, \quad i = 1, \ldots, M-1,$$

$$l_{sk}^{(n)} = \theta_{3ss3}^{(n)} f_{sk}^{(n)0} + \theta_{s3s3}^{(n)} f_{3k}^{(n)} + E_p \theta_{3s4s}^{(n)} f_{4k}^{(n)}, \quad s = 1, 2, \qquad l_{3k}^{(n)} = (\sigma_k^{(n)})^{-1} l_{3k}^{(n)0}, \tag{3.16}$$

$$l_{3k}^{(n)0} = -\alpha_1^2 \theta_{1133}^{(n)} f_{1k}^{(n)0} - \alpha_2^2 \theta_{2233}^{(n)} f_{2k}^{(n)0} + (\sigma_k^{(n)})^2 \theta_{3333}^{(n)} f_{3k}^{(n)} + \theta_{3344}^{(n)} f_{4k}^{(n)0},$$

$$l_{4k}^{(n)} = f_{4k}^{(n)0}, \qquad f_{sk}^{(n)} = (\sigma_k^{(n)})^{-1} f_{sk}^{(n)0}, \qquad s = 1, 2, 4;$$

—for the functionally graded upper layer,

$$B^{(1)}(h_1) = \|y_{kp}^{(1)}(\alpha_1, \alpha_2, h_1)\|_{k=1,\dots,4}^{p=1,\dots,8}; \tag{3.17}$$

$$B^{(M)}(h_M) = \begin{pmatrix} l^{(M)} \\ f^{(M)} \end{pmatrix}, \qquad l^{(M)} = \| - l_{ij}^{(M)}\|_{i,j=1}^4, \qquad f^{(M)} = \| - f_{ij}^{(M)}\|_{i,j=1}^4; \tag{3.18}$$

$$A^{(1)}(h_{2,\dots,M}) = \begin{pmatrix} B^{(1)}(h_2) & P^{(2)}(h_2) & 0 & 0 & \vdots & 0 & 0 \\ 0 & B^{(2)}(h_3) & P^{(3)}(h_3) & 0 & \vdots & 0 & 0 \\ 0 & 0 & B^{(3)}(h_4) & P^{(4)}(h_4) & \vdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \vdots & B^{(M-2)}(h_{M-1}) & P^{(M-1)}(h_{M-1}) \\ 0 & 0 & 0 & 0 & \vdots & 0 & B^{(M-1)}(h_M) \end{pmatrix}. \tag{3.19}$$

Here $B^{(n)}(h_k)$ and $P^{(k)}(h_l) = -B^{(k)}(h_l)$ are 8×8 matrices; the superscript corresponds to the layer number, and the argument to the interface between layers. In the general form, taking into account (3.9), (3.12), and (3.16), the matrices $B^{(n)}(h_k)$ are defined as follows:

$$B^{(n)}(h_k) = \|y_{lp}^{(n)}(\alpha_1, \alpha_2, h_k)\|_{l,p=1}^8 \tag{3.20}$$

for the functionally graded nth layer and

$$B^{(n)}(h_k) = \begin{pmatrix}
l_{11}^{(n)} c_{1k}^n & l_{12}^{(n)} c_{2k}^n & l_{13}^{(n)} c_{3k}^n & l_{14}^{(n)} c_{4k}^n & l_{11}^{(n)2} s_{1k}^{0n} & l_{12}^{(n)2} s_{2k}^{0n} & l_{13}^{(n)2} s_{3k}^{0n} & l_{14}^{(n)2} s_{4k}^{0n} \\
l_{21}^{(n)} c_{1k}^n & l_{22}^{(n)} c_{2k}^n & l_{23}^{(n)} c_{3k}^n & l_{24}^{(n)} c_{4k}^n & l_{21}^{(n)2} s_{1k}^{0n} & l_{22}^{(n)2} s_{2k}^{0n} & l_{23}^{(n)2} s_{3k}^{0n} & l_{24}^{(n)2} s_{4k}^{0n} \\
l_{31}^{(n)0} s_{1k}^{0n} & l_{32}^{(n)0} s_{2k}^{0n} & l_{33}^{(n)0} s_{3k}^{0n} & l_{34}^{(n)0} s_{4k}^{0n} & l_{31}^{(n)0} c_{1k}^n & l_{32}^{(n)0} c_{2k}^n & l_{33}^{(n)0} c_{3k}^n & l_{34}^{(n)0} c_{4k}^n \\
l_{41}^{(n)} c_{1k}^n & l_{42}^{(n)} c_{2k}^n & l_{43}^{(n)} c_{3k}^n & l_{44}^{(n)} c_{4k}^n & l_{41}^{(n)2} s_{1k}^{0n} & l_{42}^{(n)2} s_{2k}^{0n} & l_{43}^{(n)2} s_{3k}^{0n} & l_{44}^{(n)2} s_{4k}^{0n} \\
f_{11}^{(n)0} s_{1k}^{0n} & f_{12}^{(n)0} s_{2k}^{0n} & f_{13}^{(n)0} s_{3k}^{0n} & f_{14}^{(n)0} s_{4k}^{0n} & f_{11}^{(n)0} c_{1k}^n & f_{12}^{(n)0} c_{2k}^n & f_{13}^{(n)0} c_{3k}^n & f_{14}^{(n)0} c_{4k}^n \\
f_{21}^{(n)0} s_{1k}^{0n} & f_{22}^{(n)0} s_{2k}^{0n} & f_{23}^{(n)0} s_{3k}^{0n} & f_{24}^{(n)0} s_{4k}^{0n} & f_{21}^{(n)0} c_{1k}^n & f_{22}^{(n)0} c_{2k}^n & f_{23}^{(n)0} c_{3k}^n & f_{24}^{(n)0} c_{4k}^n \\
f_{31}^{(n)} c_{1k}^n & f_{32}^{(n)} c_{2k}^n & f_{33}^{(n)} c_{3k}^n & f_{34}^{(n)} c_{4k}^n & f_{31}^{(n)2} s_{1k}^{0n} & f_{32}^{(n)2} s_{2k}^{0n} & f_{33}^{(n)2} s_{3k}^{0n} & f_{34}^{(n)2} s_{4k}^{0n} \\
f_{41}^{(n)0} s_{1k}^{0n} & f_{42}^{(n)0} s_{2k}^{0n} & f_{43}^{(n)0} s_{3k}^{0n} & f_{44}^{(n)0} s_{4k}^{0n} & f_{41}^{(n)0} c_{1k}^n & f_{42}^{(n)0} c_{2k}^n & f_{43}^{(n)0} c_{3k}^n & f_{44}^{(n)0} c_{4k}^n
\end{pmatrix} \tag{3.21}$$

for the homogeneous layer.

If the last layer of the coating is homogeneous, the matrix $B^{(M-1)}(h_M)$ has the form

$$B^{(M-1)}(h_M) = \begin{pmatrix}
l_{11}^{(M-1)} & l_{12}^{(M-1)} & l_{13}^{(M-1)} & l_{14}^{(M-1)} & 0 & 0 & 0 & 0 \\
l_{21}^{(M-1)} & l_{22}^{(M-1)} & l_{23}^{(M-1)} & l_{24}^{(M-1)} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & l_{31}^{(M-1)\,0} & l_{32}^{(M-1)\,0} & l_{33}^{(M-1)\,0} & l_{34}^{(M-1)\,0} \\
l_{41}^{(M-1)} & l_{42}^{(M-1)} & l_{43}^{(M-1)} & l_{44}^{(M-1)} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & f_{11}^{(M-1)0} & f_{12}^{(M-1)0} & f_{13}^{(M-1)0} & f_{14}^{(M-1)0} \\
0 & 0 & 0 & 0 & f_{21}^{(M-1)0} & f_{22}^{(M-1)\,0} & f_{23}^{(M-1)0} & f_{24}^{(M-1)\,0} \\
f_{31}^{(M-1)} & f_{32}^{(M-1)} & f_{33}^{(M-1)} & f_{34}^{(M-1)} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & f_{41}^{(M-1)0} & f_{42}^{(M-1)0} & f_{43}^{(M-1)0} & f_{44}^{(M-1)0}
\end{pmatrix}. \tag{3.22}$$

If the last coating layer is functionally graded, the elements of the matrix $B^{(M-1)}(h_M)$ are defined by formulas (3.20) where $n = M - 1$ and $k = M$.

Substituting solution (3.13) into representations (3.8), (3.12), and (3.9) and applying an inverse Fourier transform, we obtain

$$\boldsymbol{u}_\tau^{(n)}(x_1, x_2, x_3) = \frac{1}{4\pi^2} \iint\limits_\Omega k_\tau^{(n)}(x_1 - \xi, x_2 - \eta, x_3)\, f_\tau(\xi, \eta)\, d\xi\, d\eta; \tag{3.23}$$

$$k_\tau^{(n)}(s, t, x_3) = \int\limits_{\Gamma_1} \int\limits_{\Gamma_2} K_\tau^{(n)}(\alpha_1, \alpha_2, x_3)\, \mathrm{e}^{-i(\alpha_1 s + \alpha_2 t)}\, d\alpha_1\, d\alpha_2; \tag{3.24}$$

$$K_\tau^{(n)}(\alpha_1, \alpha_2, x_3) = \|K_{lj}^{(n)}\|_{l,j=1}^4. \tag{3.25}$$

The components of the matrix $K_\tau^{(n)}$ have the following form:

$$K_{lj}^{(n)} = \frac{-i\alpha_l}{\Delta_0} \sum_{k=1}^{4} f_{lk}^{(n)} \Big(\Delta_{j,\,k+p} \sinh\, \sigma_k^{(n)} x_3 + \Delta_{j,k+4+p} \sigma_k^{(n)} \cosh\, \sigma_k^{(n)} x_3\Big), \qquad l = 1, 2,$$

$$K_{3j}^{(n)} = \frac{1}{\Delta_0} \sum_{k=1}^{4} f_{3k}^{(n)} \Big(\Delta_{j,\,k+p} \cosh\, \sigma_k^{(n)} x_3 + \Delta_{j,k+4+p} \sigma_k^{(n)} \sinh\, \sigma_k^{(n)} x_3\Big), \tag{3.26}$$

$$K_{4j}^{(n)} = \frac{1}{\Delta_0} \sum_{k=1}^{4} f_{4k}^{(n)} \Big(\Delta_{j,\,k+p} \sinh\, \sigma_k^{(n)} x_3 + \Delta_{j,k+4+p} \sigma_k^{(n)} \cosh\, \sigma_k^{(n)} x_3\Big)$$

for the homogeneous components ($p = 8(n-1)$, $n = 1, 2, \ldots, M-1$);

$$K_{lj}^{(n)} = \frac{1}{\Delta_0} \sum_{k=1}^{8} \Delta_{j,k+p} y_{l+4,k}^{(n)}(\alpha_1, \alpha_2, x_3) \qquad (l = 1, \ldots, 4) \tag{3.27}$$

for the functionally graded components, and

$$K_{lj}^{(M)} = \frac{\beta_l}{\Delta_0} \sum_{k=1}^{4} f_{lk}^{(M)} \Delta_{j,k+8(M-1)}\, \mathrm{e}^{\sigma_k^{(M)} x_3} \quad (l = 1, \ldots, 4, \qquad \boldsymbol{\beta} = \{-i\alpha_1, -i\alpha_2, 1, 1\}). \tag{3.28}$$

for the half-space. Here Δ_0 and Δ_{ns} are the determinant and algebraic cofactor of the corresponding element of the matrix A (3.14) with elements (3.15)–(3.22).

The integral representation (3.23), (3.24) and the Green's function (3.25)–(3.28) define the displacement of an arbitrary point of the medium under the action of the specified load on its surface. The contours Γ_1 and Γ_2 in representation (3.24) are found in the region of analyticity of the integrand and are selected in accordance with the rules [16].

CONCLUSIONS

A mathematical model was developed for an inhomogeneous prestressed thermoelastic half-space which is a stack of homogeneous or functionally graded layers rigidly attached to a homogeneous base. Each inhomogeneous component of the medium can be subjected to an inhomogeneous initial stress and temperature. Consistent linearization of the constitutive relations of the nonlinear mechanics of a thermoelastic medium is performed using the theory of superposition of small deformations on finite deformations with the inhomogeneity of the medium taken into account. A method was proposed and integral formulas were derived that take into account the inhomogeneity of the coating elements, different laws of change in the properties of these elements, the inhomogeneity of their stress state, and different conditions at the interface between the layers.

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