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THREE-DIMENSIONAL FLOW OF A VISCOELASTIC FLUID ON AN EXPONENTIALLY STRETCHING SURFACE

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Abstract: An analysis of a three-dimensional viscoelastic fluid flow over an exponentially stretching surface is carried out in the presence of heat transfer. Constitutive equations of a second-grade fluid are employed. The governing boundary layer equations are reduced by appropriate transformations to ordinary differential equations. Series solutions of these equations are found, and their convergence is discussed. The influence of the prominent parameters involved in the heat transfer process is analyzed. It is found that the effects of the Prandtl number, viscoelastic parameter, velocity ratio parameter, and temperature exponent on the Nusselt number are qualitatively similar.

Keywords: viscoelastic fluid, exponentially stretching surface, heat transfer.

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INTRODUCTION

The recent interest of researchers in the investigation of non-Newtonian fluids is caused by a large number of practical applications of such fluids in industry and technology (e.g., manufacturing of foods and paper, polymer extrusion, petroleum drilling, oil recovery, etc.). Additional rheological parameters in constitutive relationships of such fluids increase the order of the resulting equations and make them much more complicated than the Navier–Stokes equations. Non-Newtonian fluids are divided into three classes depending on the strain rate law. The second-grade fluid model describes the normal stress effects on heat transfer. Such fluid flows were studied by many researchers (see [1–10] and references therein).

The boundary layer flow induced by a continuously moving sheet occurs widely in cooling of metallic sheets, drawing of polymer sheets, crystal growing, aerodynamic extrusion of plastic sheets, liquid film condensation, etc. Sakiadis [11] analyzed the flow of a viscous fluid over a continuously moving surface. Crane [12] extended the work of Sakiadis [11] for a linearly stretching surface and obtained closed-form solutions. Some recent studies can be found in [13–18]. Heat transfer on a linearly stretching surface was analyzed in [19–25].

Elbashbeshy [26] analyzed the effects of heat transfer in a steady viscous fluid flow induced by an exponentially stretching porous surface. Heat and mass transfer in the boundary layer flow of a viscoelastic fluid over an exponentially stretching surface were numerically examined by Sanjaynand and Khan [27]. Sajid and Hayat [28] analytically discussed the radiative effects in the flow of a viscous fluid over an exponentially stretching sheet. Bidin and Nazar [29] obtained a numerical solution of the problem considered in [28]. Sahoo and Poncet [30] carried out a study to analyze partial slip effects in the boundary layer flow of a third-grade fluid with heat transfer. A two-dimensional boundary layer flow of an incompressible second-grade fluid over an exponentially stretching porous

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surface with viscous dissipation was numerically discussed by Singh and Agarwal [31]. Liu et al. [32] presented a study for a three-dimensional flow of a viscous fluid over an exponentially stretching surface with heat transfer.

According to the authors' knowledge, no attempt has been presented to analyze a three-dimensional flow of a second-grade fluid over an exponentially stretching surface. The basic challenge of the present work is to model a three-dimensional flow of a second-grade fluid due to an exponentially stretching sheet. The problem is solved by the homotopy analysis method (HAM) [33–38].

GOVERNING EQUATIONS

We consider an incompressible three-dimensional boundary layer flow of a viscoelastic fluid over an exponentially stretching surface. In addition, heat transfer effects are taken into account. Viscous dissipation effects are neglected. In this case, the governing boundary layer equations have the following form [16, 20]:

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0; \tag{1}$$

$$u\,\frac{\partial u}{\partial x}+v\,\frac{\partial u}{\partial y}+w\,\frac{\partial u}{\partial z}=\nu\,\frac{\partial^2 u}{\partial z^2}+k_0\Big(u\,\frac{\partial^3 u}{\partial x\,\partial z^2}+w\,\frac{\partial^3 u}{\partial z^3}$$

$$-\Big(\frac{\partial u}{\partial x}\,\frac{\partial^2 u}{\partial z^2}+\frac{\partial u}{\partial z}\,\frac{\partial^2 w}{\partial z^2}+2\,\frac{\partial u}{\partial z}\,\frac{\partial^2 u}{\partial x\,\partial z}+2\,\frac{\partial w}{\partial z}\,\frac{\partial^2 u}{\partial z^2}\Big)\Big),$$

$$u\,\frac{\partial v}{\partial x}+v\,\frac{\partial v}{\partial y}+w\,\frac{\partial v}{\partial z}=\nu\,\frac{\partial^2 v}{\partial z^2}+k_0\Big(v\,\frac{\partial^3 v}{\partial y\,\partial z^2}+w\,\frac{\partial^3 v}{\partial z^3} \tag{2}$$

$$-\Big(\frac{\partial v}{\partial y}\,\frac{\partial^2 v}{\partial z^2}+\frac{\partial v}{\partial z}\,\frac{\partial^2 w}{\partial z^2}+2\,\frac{\partial v}{\partial z}\,\frac{\partial^2 v}{\partial y\,\partial z}+2\,\frac{\partial w}{\partial z}\,\frac{\partial^2 v}{\partial z^2}\Big)\Big),$$

$$u\,\frac{\partial T}{\partial x}+v\,\frac{\partial T}{\partial y}+w\,\frac{\partial T}{\partial z}=\frac{k}{\rho c_p}\,\frac{\partial^2 T}{\partial z^2}.$$

Here u, v, and w are the velocity components in the x, y, and z directions, k_0 is the material parameter of the fluid, T is the fluid temperature, $\nu=\mu/\rho$ is the kinematic viscosity, μ is the dynamic viscosity, ρ is the fluid density, c_p is the specific heat at constant pressure, and k is the thermal conductivity of the fluid.

The boundary conditions for the flow under consideration are written as

$$z=0:\qquad u=U_w,\quad v=V_w,\quad w=0,\quad T=T_w,$$

$$z\to\infty:\qquad u\to 0,\quad v\to 0,\quad T\to T_\infty, \tag{3}$$

where the subscript w indicates the wall condition.

The velocities and temperature at the wall are

$$U_w=U_0\,\mathrm{e}^{(x+y)/L},\quad V_w=V_0\,\mathrm{e}^{(x+y)/L},\quad T_w=T_\infty+T_0\,\mathrm{e}^{A(x+y)/(2L)}, \tag{4}$$

where U_0, V_0, and T_0 are constants, L is the reference length, T_∞ is the ambient temperature, and A is the temperature exponent.

We introduce the following similarity transformations [32]:

$$u=U_0\,\mathrm{e}^{(x+y)/L}\,f'(\eta),\quad v=V_0\,\mathrm{e}^{(x+y)/L}\,g'(\eta),\quad w=-\Big(\frac{\nu U_0}{2L}\Big)^{1/2}(f+\eta f'+g+\eta g'),$$

$$T=T_\infty+T_0\,\mathrm{e}^{A(x+y)/(2L)}\,\theta(\eta),\qquad \eta=\Big(\frac{U_0}{2\nu L}\Big)^{1/2}\mathrm{e}^{(x+y)/(2L)}\,z. \tag{5}$$

Equation (1) is satisfied automatically, and Eqs. (2)–(5) yield

$$f''' + (f+g)f'' - 2(f'+g')f'$$

$$+ K_1(6f'''f' + (3g'' - 3f'' + \eta g''')f'' + (4g' + 2\eta g'')f''' - (f + g + \eta g')f'''') = 0,$$

$$g''' + (f+g)g'' - 2(f'+g')g'$$

$$+ K_1(6g'''g' + (3f'' - 3g'' + \eta f''')g'' + (4f' + 2\eta f'')g''' - (f + g + \eta f')g'''') = 0,$$

$$\theta'' + \Pr\,(f+g)\theta' - \Pr A(f'+g')\theta = 0,$$

$$\eta = 0: \qquad f = 0, \quad g = 0, \quad f' = 1, \quad g' = \alpha, \quad \theta = 1,$$

$$\eta \to \infty: \qquad f' \to 0, \quad g' \to 0, \quad \theta \to 0.$$

Here $K_1 = k_0 U w/(2\nu L)$ is the viscoelastic parameter, $\alpha = V_0/U_0$ is the velocity ratio parameter ($\alpha = 0$ corresponds to the two-dimensional case), and $\Pr = \mu c_p/k$ is the Prandtl number.

The local Nusselt number is expressed by the formula

$$\mathrm{Nu} = \frac{-k}{k(T_w - T_\infty)/x}\,\frac{\partial T}{\partial z} = -\frac{x}{L}\Big(\frac{\mathrm{Re}_x}{2}\Big)^{1/2} \mathrm{e}^{(x+y)/(2L)}\,\theta'(0)$$

($\mathrm{Re}_x = U_0 L/\nu$ is the local Reynolds number).

SERIES SOLUTIONS

The initial approximations and auxiliary linear operators for analytical solutions are chosen as

$$f_0(\eta) = (1 - \mathrm{e}^{-\eta}), \qquad g_0(\eta) = \alpha(1 - \mathrm{e}^{-\eta}), \qquad \theta_0(\eta) = \mathrm{e}^{-\eta},$$

$$L_f = f''' - f', \qquad L_g = g''' - g', \qquad L_\theta = \theta'' - \theta.$$

The auxiliary linear operators satisfy the relations

$$L_f(C_1 + C_2\,\mathrm{e}^{\eta} + C_3\,\mathrm{e}^{-\eta}) = 0, \quad L_g(C_4 + C_5\,\mathrm{e}^{\eta} + C_6\,\mathrm{e}^{-\eta}) = 0, \quad L_\theta(C_7\,\mathrm{e}^{\eta} + C_8\,\mathrm{e}^{-\eta}) = 0,$$

where C_i ($i = 1, \ldots, 8$) are arbitrary constants.

The associated zeroth-order deformation problems can be written as

$$(1-p)L_f[\hat f(\eta;p) - f_0(\eta)] = p h_f N_f[\hat f(\eta;p), \hat g(\eta;p)],$$

$$(1-p)L_g[\hat g(\eta;p) - g_0(\eta)] = p h_g N_g[\hat f(\eta;p), \hat g(\eta;p)],$$

$$(1-p)L_\theta[\hat\theta(\eta;p) - \theta_0(\eta)] = p h_\theta N_\theta[\hat f(\eta;p), \hat g(\eta;p), \hat\theta(\eta,p)],$$

$$\hat f(0;p) = 0, \quad \hat f'(0;p) = 1, \quad \hat f'(\infty;p) = 0, \quad \hat g(0;p) = 0,$$

$$\hat g'(0;p) = \alpha, \quad \hat g'(\infty;p) = 0, \quad \hat\theta(0,p) = 1, \quad \hat\theta(\infty,p) = 0,$$

$$N_f[\hat f(\eta,p),\hat g(\eta,p)] = \frac{\partial^3 \hat f(\eta,p)}{\partial \eta^3} - 2\Big(\frac{\partial \hat f(\eta,p)}{\partial \eta} + \frac{\partial \hat g(\eta,p)}{\partial \eta}\Big)\frac{\partial \hat f(\eta,p)}{\partial \eta}$$

$$+ (\hat f(\eta,p) + \hat g(\eta,p))\,\frac{\partial^2 \hat f(\eta,p)}{\partial \eta^2} + K_1\Big(6\,\frac{\partial \hat f(\eta,p)}{\partial \eta}\,\frac{\partial^3 \hat f(\eta,p)}{\partial \eta^3} + \Big(3\,\frac{\partial^2 \hat g(\eta,p)}{\partial \eta^2} - 3\,\frac{\partial^2 \hat f(\eta,p)}{\partial \eta^2} + \eta\,\frac{\partial^3 \hat g(\eta,p)}{\partial \eta^3}\Big)\frac{\partial^2 \hat f(\eta,p)}{\partial \eta^2}$$

$$+ \Big(4\,\frac{\partial \hat g(\eta,p)}{\partial \eta} + 2\eta\,\frac{\partial^2 \hat g(\eta,p)}{\partial \eta^2}\Big)\,\frac{\partial^3 \hat f(\eta,p)}{\partial \eta^3} - \Big(\hat f(\eta,p) + \hat g(\eta,p) + \eta\,\frac{\partial \hat g(\eta,p)}{\partial \eta}\Big)\,\frac{\partial^4 \hat f(\eta,p)}{\partial \eta^4}\Big),$$

$$N_g[\hat g(\eta,p),\hat f(\eta,p)] = \frac{\partial^3 \hat g(\eta,p)}{\partial \eta^3} - 2\Big(\frac{\partial \hat f(\eta,p)}{\partial \eta} + \frac{\partial \hat g(\eta,p)}{\partial \eta}\Big)\,\frac{\partial \hat g(\eta,p)}{\partial \eta}$$

$$+ (\hat f(\eta,p) + \hat g(\eta,p))\,\frac{\partial^2 \hat g(\eta,p)}{\partial \eta^2} + K_1\Big(6\,\frac{\partial \hat g(\eta,p)}{\partial \eta}\,\frac{\partial^3 \hat g(\eta,p)}{\partial \eta^3} + \Big(3\,\frac{\partial^2 \hat f(\eta,p)}{\partial \eta^2} - 3\,\frac{\partial^2 \hat g(\eta,p)}{\partial \eta^2} + \eta\,\frac{\partial^3 \hat f(\eta,p)}{\partial \eta^3}\Big)\frac{\partial^2 \hat g(\eta,p)}{\partial \eta^2}$$

$$+ \Big(4\,\frac{\partial \hat f(\eta,p)}{\partial \eta} + 2\eta\,\frac{\partial^2 \hat f(\eta,p)}{\partial \eta^2}\Big)\,\frac{\partial^3 \hat g(\eta,p)}{\partial \eta^3} - \Big(\hat f(\eta,p) + \hat g(\eta,p) + \eta\,\frac{\partial \hat f(\eta,p)}{\partial \eta}\Big)\,\frac{\partial^4 \hat g(\eta,p)}{\partial \eta^4}\Big),$$

$$N_\theta[\hat\theta(\eta,p),\hat f(\eta,p),\hat g(\eta,p)] = \frac{\partial^2 \hat\theta(\eta,p)}{\partial \eta^2} + \Pr\,(\hat f(\eta,p) + \hat g(\eta,p))\,\frac{\partial \hat\theta(\eta,p)}{\partial \eta} - \Pr A\Big(\frac{\partial \hat f(\eta,p)}{\partial \eta} + \frac{\partial \hat g(\eta,p)}{\partial \eta}\Big)\hat\theta(\eta,p).$$

Here p is the embedding parameter, h_f, h_g, and h_θ are nonzero auxiliary parameters, and N_f, N_g, and N_θ are nonlinear operators.

For $p = 0$ and $p = 1$, we have

$$\hat f(\eta;0) = f_0(\eta), \qquad \hat g(\eta;0) = g_0(\eta), \qquad \hat\theta(\eta,0) = \theta_0(\eta), \qquad \hat f(\eta;1) = f(\eta),$$

$$\hat g(\eta;1) = g(\eta), \qquad \hat\theta(\eta,1) = \theta(\eta).$$

For $0 < p < 1$, the values of the functions $f(\eta,p)$, $g(\eta,p)$, and $\theta(\eta,p)$ change from $f_0(\eta)$, $g_0(\eta)$, and $\theta_0(\eta)$ to $f(\eta)$, $g(\eta)$, and $\theta(\eta)$. Using the Taylor expansion, we obtain the series solutions

$$f(\eta,p) = f_0(\eta) + \sum_{m=1}^{\infty} f_m(\eta)p^m, \qquad f_m(\eta) = \frac{1}{m!}\,\frac{\partial^m f(\eta;p)}{\partial \eta^m}\Big|_{p=0},$$

$$g(\eta,p) = g_0(\eta) + \sum_{m=1}^{\infty} g_m(\eta)p^m, \qquad g_m(\eta) = \frac{1}{m!}\,\frac{\partial^m g(\eta;p)}{\partial \eta^m}\Big|_{p=0}, \tag{6}$$

$$\theta(\eta,p) = \theta_0(\eta)\sum_{m=1}^{\infty} \theta_m(\eta)p^m, \qquad \theta_m(\eta) = \frac{1}{m!}\,\frac{\partial^m \theta(\eta;p)}{\partial \eta^m}\Big|_{p=0},$$

whose convergence strongly depends on h_f, h_g, and h_θ. The auxiliary parameters h_f, h_g, and h_θ were chosen properly, so that series (6) converge at $p = 1$. Therefore, we have

$$f(\eta) = f_0(\eta) + \sum_{m=1}^{\infty} f_m(\eta), \quad g(\eta) = g_0(\eta) + \sum_{m=1}^{\infty} g_m(\eta), \quad \theta(\eta) = \theta_0(\eta) + \sum_{m=1}^{\infty} \theta_m(\eta). \tag{7}$$

The mth-order deformation problems are written as

$$L_f[f_m(\eta) - \chi_m f_{m-1}(\eta)] = h_f R_f^m(\eta), \quad L_g[g_m(\eta) - \chi_m g_{m-1}(\eta)] = h_g R_g^m(\eta),$$

$$L_\theta[\theta_m(\eta) - \chi_m \theta_{m-1}(\eta)] = h_\theta R_\theta^m(\eta),$$

$$f_m(0) = f_m'(0) = f_m'(\infty) = 0, \quad g_m(0) = g_m'(0) = g_m'(\infty) = 0, \quad \theta_m(0) = \theta_m(\infty) = 0,$$

$$R_f^m(\eta) = f'''_{m-1}(\eta) - 2\sum_{k=0}^{m-1} f'_{m-1-k} f'_k - 2\sum_{k=0}^{m-1} g'_{m-1-k} f'_k + \sum_{k=0}^{m-1} (f_{m-1-k} f''_k + g_{m-1-k} f''_k)$$

$$+ K_1\Big(6\sum_{k=0}^{m-1} f'_{m-1-k} f'''_k + 3\sum_{k=0}^{m-1} g''_{m-1-k} f''_k - 3\sum_{k=0}^{m-1} f''_{m-1-k} f''_k + \sum_{k=0}^{m-1} \eta g'''_{m-1-k} f''_k$$

$$+ 4\sum_{k=0}^{m-1} g'_{m-1-k} f'''_k + 2\sum_{k=0}^{m-1} \eta g''_{m-1-k} f'''_k - \sum_{k=0}^{m-1} f_{m-1-k} f''''_k - \sum_{k=0}^{m-1} g_{m-1-k} f''''_k - \sum_{k=0}^{m-1} \eta g'_{m-1-k} f''''_k\Big),$$

$$R_g^m(\eta) = g'''_{m-1}(\eta) - 2\sum_{k=0}^{m-1} g'_{m-1-k} g'_k - 2\sum_{k=0}^{m-1} g'_{m-1-k} f'_k + \sum_{k=0}^{m-1} (f_{m-1-k} g''_k + g_{m-1-k} g''_k)$$

$$+ K_1\Big(6\sum_{k=0}^{m-1} g'_{m-1-k} g'''_k + 3\sum_{k=0}^{m-1} f''_{m-1-k} g''_k - 3\sum_{k=0}^{m-1} g''_{m-1-k} g''_k + \sum_{k=0}^{m-1} \eta f'''_{m-1-k} g''_k +$$

$$+ 4\sum_{k=0}^{m-1} f'_{m-1-k} g'''_k + 2\sum_{k=0}^{m-1} \eta f''_{m-1-k} g'''_k - \sum_{k=0}^{m-1} g_{m-1-k} g''''_k - \sum_{k=0}^{m-1} f_{m-1-k} g''''_k - \sum_{k=0}^{m-1} \eta f'_{m-1-k} g''''_k\Big),$$

$$R_\theta^m(\eta) = \theta''_{m-1} + \Pr \sum_{k=0}^{m-1} (\theta'_{m-1-k} f_k + \theta'_{m-1-k} g_k) - \Pr A \sum_{k=0}^{m-1} (f'_{m-1-k}\theta_k + g'_{m-1-k}\theta_k).$$

Solving these problems, we obtain

$$f_m(\eta) = f_m^*(\eta) + C_1 + C_2\,\mathrm{e}^{\eta} + C_3\,\mathrm{e}^{-\eta}, \quad g_m(\eta) = g_m^*(\eta) + C_4 + C_5\,\mathrm{e}^{\eta} + C_6\,\mathrm{e}^{-\eta},$$

$$\theta_m(\eta) = \theta_m^*(\eta) + C_7\,\mathrm{e}^{\eta} + C_8\,\mathrm{e}^{-\eta},$$

where f_m^*, g_m^*, and θ_m^* are special solutions.

CONVERGENCE ANALYSIS

Obviously, series (7) contain the auxiliary parameters h_f, h_g, and h_θ, which control the convergence of HAM solutions. To determine suitable ranges for h_f, h_g, and h_θ, we constructed h-curves of the 10th order of approximation (Fig. 1). It is seen that admissible values of h_f, h_g, and h_θ are in the intervals $-0.75 \leqslant h_f \leqslant -0.30$, $-0.75 \leqslant h_g \leqslant -0.30$, and $-0.8 \leqslant h_\theta \leqslant -0.4$. It should be noted that our series solutions converge in the entire range of η at $h_f = h_g = h_\theta = -0.5$ (Table 1).

DISCUSSION

Figures 2–5 depict the effects of the velocity ratio parameter α and the viscoelastic parameter K_1 on the velocity components $f'(\eta)$ and $g'(\eta)$. It is observed from Figs. 2 and 3 that $f'(\eta)$ decreases with increasing α, whereas $g'(\eta)$ increases. As α increases, the side surface starts to move in the y direction; thus, the velocity component $g'(\eta)$ increases, and the velocity component in the original stretching direction $f'(\eta)$ decreases. Figures 4 and 5 are plotted to analyze the effect of the viscoelastic parameter K_1 on the velocities $f'(\eta)$ and $g'(\eta)$, respectively. It is seen that the fluid velocity is an increasing function of K_1.

Figures 6–9 are sketched to illustrate the effects of the velocity ratio parameter α, viscoelastic parameter K_1, temperature exponent A, and Prandtl number on the temperature $\theta(\eta)$. Figure 6 shows the influence of the velocity ratio parameter α on the temperature $\theta(\eta)$. It is found that the temperature $\theta(\eta)$ decreases from unity to zero as the dimensionless distance η increases from zero to infinity. Figure 7 depicts the effect of K_1 on the temperature

Table 1. Convergence of series solutions for different orders of approximation for $K_1 = 0.1$, $A = 0.2$, $\Pr = 1.2$, $\alpha = 0.2$, $h_f = -0.5$, $h_g = -0.6$, and $h_\theta = -0.7$

Order of approximation	$-f''(0)$	$-g''(0)$	$-\theta'(0)$
1	1.16889	0.229867	0.885200
5	1.21057	0.228658	0.820787
10	1.21049	0.228600	0.822115
15	1.21045	0.228600	0.822280
20	1.21045	0.228601	0.822282
25	1.21045	0.228601	0.822282
30	1.21045	0.228601	0.822282

Table 2. Values of $-f''(0)$, $-g''(0)$, and $f(\infty) + g(\infty)$ for $K_1 = 0$ and different values of α

α	Data of Liu et al. [32]			Present results		
	$-f''(0)$	$-g''(0)$	$f(\infty)+g(\infty)$	$-f''(0)$	$-g''(0)$	$f(\infty)+g(\infty)$
0	1.28180856	0	0.90564383	1.28180	0	0.90564
0.5	1.56988846	0.78494423	1.10918263	1.56988	0.78494	1.10918
1.0	1.81275105	1.81275105	1.28077378	1.81275	1.81275	1.28077

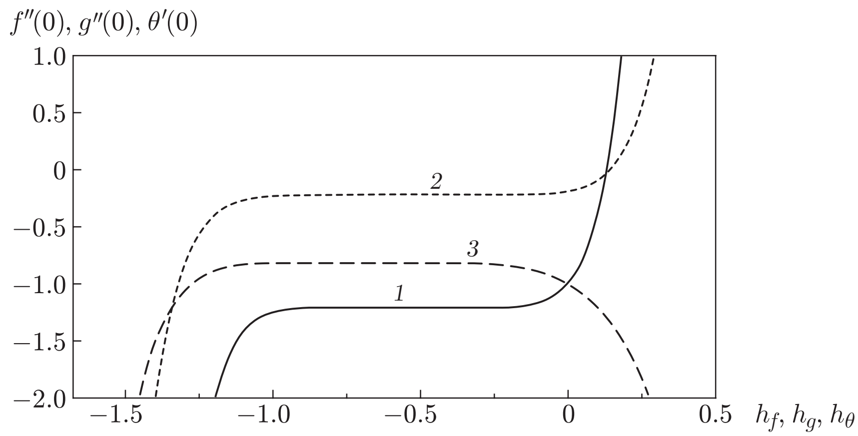

Fig. 1. h-Curves for the functions $f''(0)$ (*1*), $g''(0)$ (*2*), and $\theta'(0)$ (*3*) for $K_1 = 0.3$, $\beta_1 = 0.2$, $A = 0.2$, $\Pr = 1.0$, and $\alpha = 0.3$.

$\theta(\eta)$. It is seen that the fluid temperature is a decreasing function of K_1. The effect of the temperature exponent A on the temperature $\theta(\eta)$ is illustrated in Fig. 8. It is seen that the value of $\theta(\eta)$ decreases from unity to zero as A increases. Figure 9 is plotted to illustrate the effect of the Prandtl number on the temperature $\theta(\eta)$. It is noticed that the fluid temperature decreases with increasing $\Pr$. Thus, if $\Pr$ increases, then the thermal diffusivity decreases. This leads to a decrease in the energy transfer ability and in the thermal boundary layer thickness.

Figures 10–12 are plotted to demonstrate the influence of the velocity ratio parameter α, temperature exponent A, viscoelastic parameter K_1, and Prandtl number on the local Nusselt number $-\theta'(0)$. It is seen from Figs. 10–12 that the local Nusselt number increases with increasing α, $\Pr$, A, and K_1.

The values of $-f''(0)$, $-g''(0)$, and $-\theta'(0)$ for $h_f = h_g = -0.6$, and $h_\theta = -0.7$ are compared in Table 2. It should be noted that the values of $-f''(0)$ and $-g''(0)$ converge in the 10th order of approximation, and the values of $-\theta'(0)$ converge in the 20th order of approximation. The values of $-f''(0)$, $-g''(0)$, and $-\theta'(0)$ obtained in [32] are also listed in Table 2. The solutions of the present work are in good agreement with those in [32].

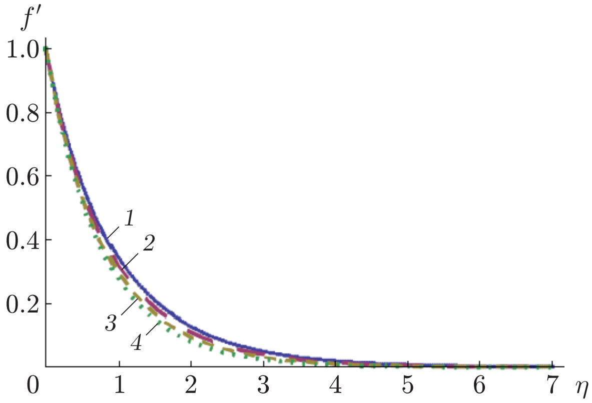

Fig. 2.

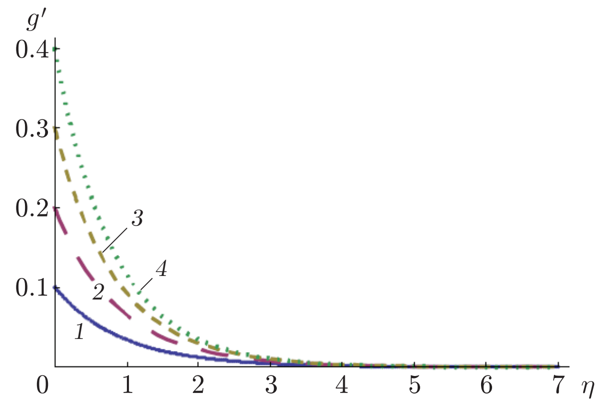

Fig. 3.

Fig. 2. Velocity f' versus the parameter η for $K_1 = 0.1$ and $\alpha = 0$ (*1*), 0.2 (*2*), 0.4 (*3*), and 0.6 (*4*).

Fig. 3. Velocity g' versus the parameter η for $K_1 = 0.1$ and $\alpha = 0.1$ (*1*), 0.2 (*2*), 0.3 (*3*), and 0.4 (*4*).

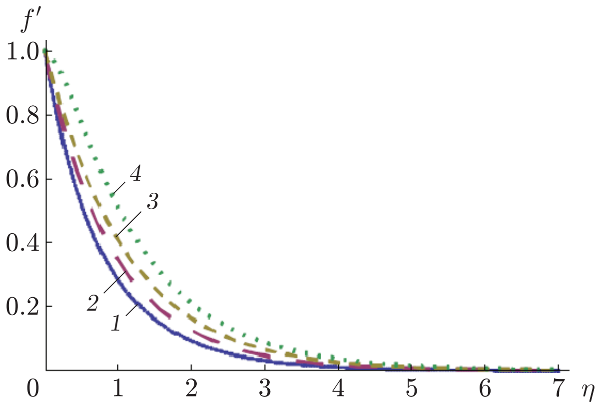

Fig. 4.

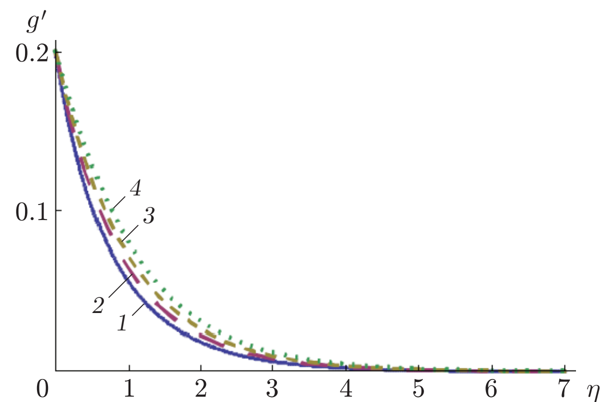

Fig. 5.

Fig. 4. Velocity f' versus the parameter η for $\alpha = 0.1$ and $K_1 = 0$ (*1*), 0.2 (*2*), 0.4 (*3*), and 0.6 (*4*).

Fig. 5. Velocity g' versus the parameter η for $\alpha = 0.1$ and $K_1 = 0$ (*1*), 0.1 (*2*), 0.2 (*3*), and 0.3 (*4*).

CONCLUSIONS

A three-dimensional flow of a viscoelastic fluid with heat transfer is analyzed in this study. The main observations are as follows. As the viscoelastic parameter K_1 increases, the velocity also increases, whereas the temperature decreases. An increase in the Prandtl number leads to reduction of the temperature and boundary layer thickness. With increasing velocity ratio parameter α, the velocity $f'(\eta)$ decreases and $g'(\eta)$ increases. An increase in the temperature exponent A leads to a decrease in the temperature and thermal boundary layer thickness. The local Nusselt number increases with increasing Prandtl number Pr and velocity ratio parameter α.

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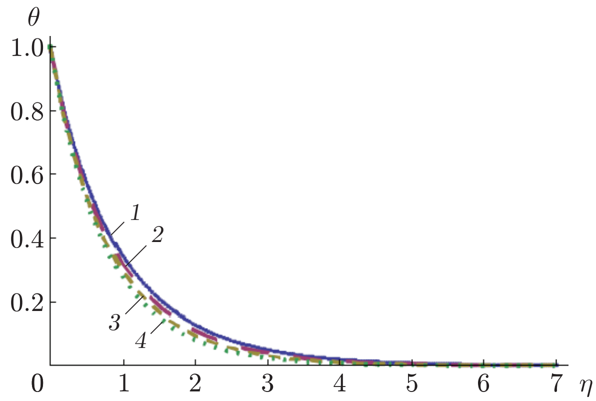

Fig. 6.

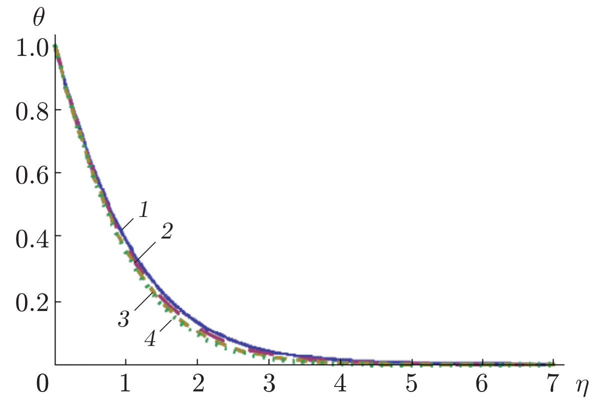

Fig. 7.

Fig. 6. Temperature θ versus the parameter η for $K_1 = 0.1$, $A = 0.2$, $\Pr = 1.2$, and $\alpha = 0$ (*1*), 0.2 (*2*), 0.4 (*3*), and 0.6 (*4*).

Fig. 7. Temperature θ versus the parameter η for $A = 0.2$, $\alpha = 0.2$, $\Pr = 1.2$, and $K_1 = 0$ (*1*), 0.2 (*2*), 0.4 (*3*), and 0.6 (*4*).

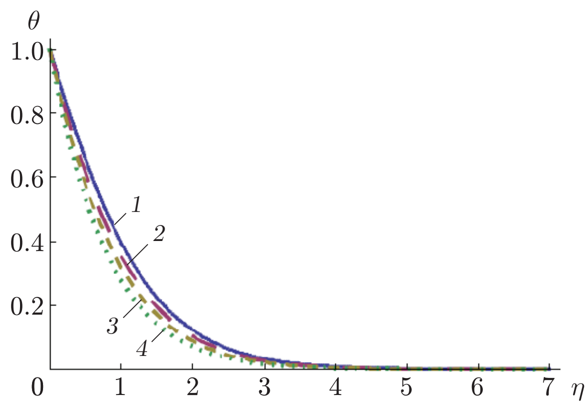

Fig. 8.

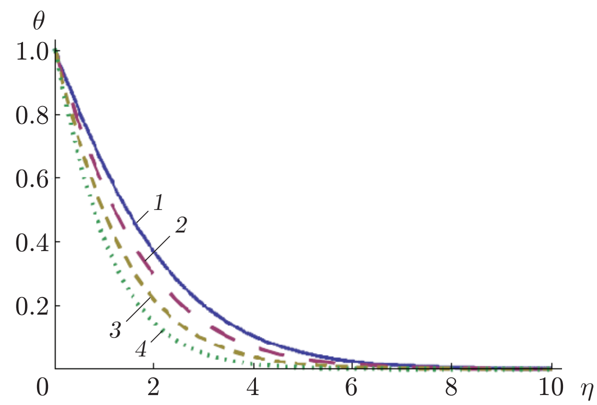

Fig. 9.

Fig. 8. Temperature θ versus the parameter η for $K_1 = 0.2$, $\alpha = 0.2$, $\Pr = 1.2$, and $A = 0$ (*1*), 0.3 (*2*), 0.6 (*3*), and 1.0 (*4*).

Fig. 9. Temperature θ versus the parameter η for $K_1 = 0.2$, $\alpha = 0.2$, $A = 0.2$, and $\Pr = 0.1$ (*1*), 0.3 (*2*), 0.6 (*3*), and 1.0 (*4*).

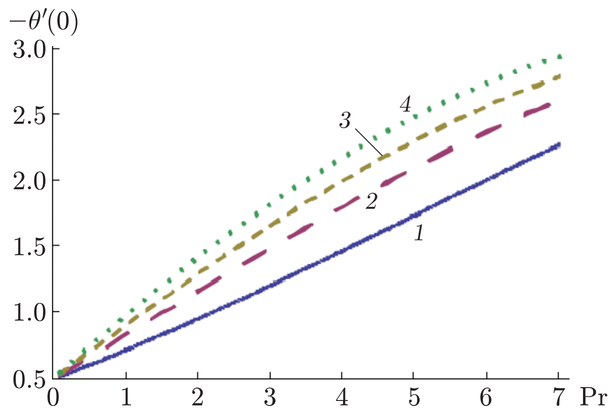

Fig. 10.

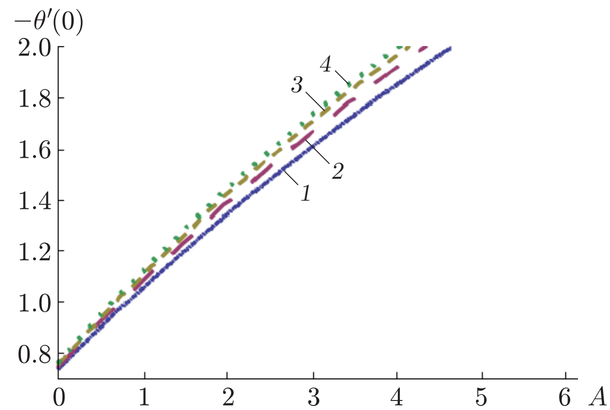

Fig. 11.

Fig. 10. Temperature $-\theta'(0)$ versus the Prandtl number for $K_1 = 0.2$, $\alpha = 0.2$, and $A = 0$ (*1*), 0.3 (*2*), 0.5 (*3*), and 0.7 (*4*).

Fig. 11. Temperature $-\theta'(0)$ versus the temperature exponent A for Pr $= 1.2$, $\alpha = 0.2$, and $K_1 = 0$ (*1*), 0.3 (*2*), 0.6 (*3*), and 1.0 (*4*).

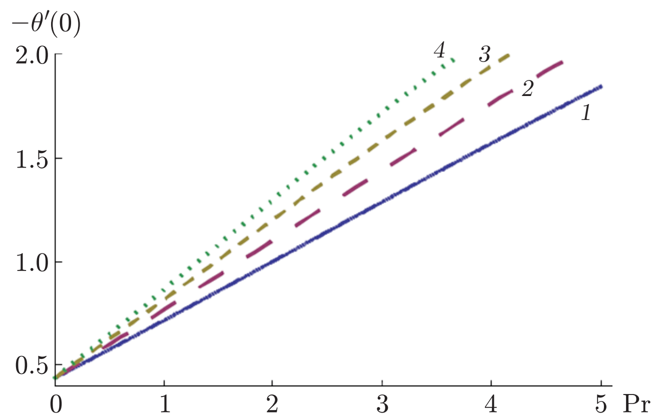

Fig. 12. Temperature $-\theta'(0)$ versus the Prandtl number for $K_1 = 0.2$, $A = 0.2$, and $\alpha = 0$ (*1*), 0.2 (*2*), 0.4 (*3*), and 0.6 (*4*).

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