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UNSTEADY SELF-SIMILAR VISCOUS FLOW NEAR A STAGNATION POINT

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Abstract: The problem of unsteady viscous incompressible fluid flow near a stagnation point is considered. Self-similar solutions describing plane and axisymmetric flows are constructed.

Keywords: unsteadiness, self-similarity, stagnation point.

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It is known that on a surface in flow, there is always at least one point at which the velocity components vanish. Such points are called stagnation points. The problem of incompressible viscous fluid near a stagnation point, i.e., the problem of flow near a wall perpendicular to the flow direction has been the subject of many papers. Classical solutions of this problem are given by Schlichting [1]. Steady-state self-similar solutions of the Navier–Stokes equations describing viscous fluid flow in a layer between a solid rotating plane and a free surface parallel to it were derived by Lavrent'eva [2]. An analytical solution of unsteady axisymmetric flow near a stagnation point in flow to a porous wall in the cases of injection and suction were constructed by Shapiro [3]. Sin and Chio [4] studied unsteady self-similar solution describing plane flow toward a stagnation point. Proudman and Johnson [5] constructed an asymptotic solution for large times assuming that the viscous terms can be neglected. However, it has been shown [6] that viscous forces cannot be neglected, so that higher-order terms were added to the asymptotic solution. Kuznetsov and Pukhnachov [7] considered examples of non-self-similar solutions describing unsteady viscous fluid flows in the vicinity of a stagnation point on a solid rectilinear boundary and proposed scenarios of breakdown of the solution of the initial-boundary-value problem in finite time. Examples of self-similar unsteady viscous fluid flows near a moving permeable flat plate and near a rotating disk are given by Gaifullin [8].

This paper is devoted to the construction of self-similar solutions of unsteady Navier–Stokes equations describing viscous fluid flow near a stagnation point. Fluid flow from infinity to the stagnation point and fluid flow from the stagnation point in the case of a flat boundary and in the axisymmetric case are considered.

PLANE FLOW

In the case of plane flow, a fluid flowing from infinity impinges on a wall placed across the flow and then flows along it in the opposite direction, away from the stagnation point. We denote by u and v the velocity projection onto the x and y Cartesian axes. The x axis is aligned with the wall, the y axis is perpendicular to the wall, and the origin is located at the stagnation point. In this case, the Navier–Stokes equations admit an exact solution of the form

$$u = xq(y,t), \qquad v = -\int_0^y q(s,t)\,ds, \qquad p = -\frac{1}{2}\,\rho x^2 a(t) + Q(y,t),$$

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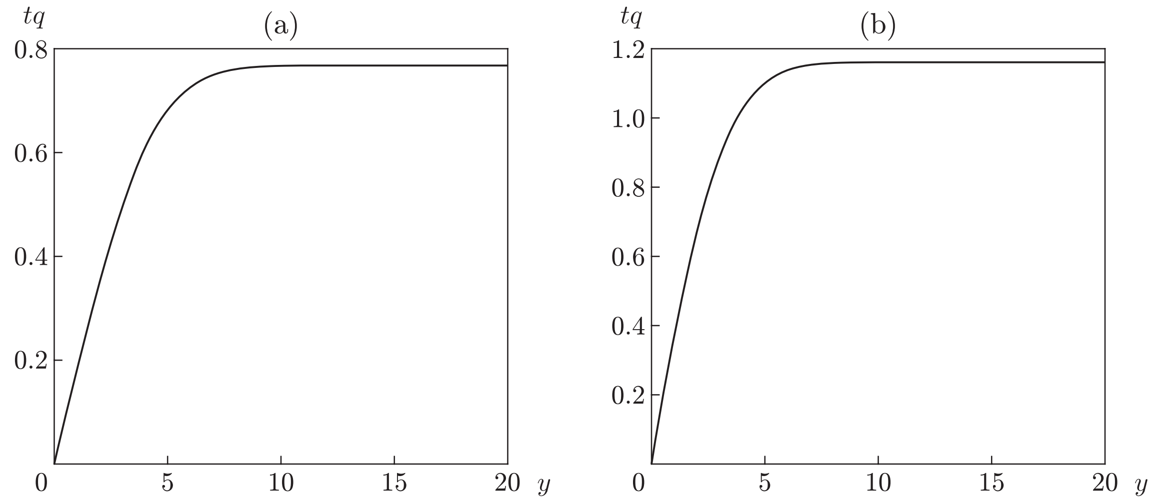

Fig. 1. Solution of the unsteady problem by the relaxation method $a(t) = -3/(16t^2)$ (a) and $a(t) = 3/(16t^2)$ (b).

where $a(t)$ is a given function of time. To determine $q(y,t)$, we pose the boundary-value problem

$$q_t + q^2 - q_y \int_0^y q(s,t)\,ds - a(t) = \nu q_{yy}, \qquad y > 0, \quad t > 0; \tag{1}$$

$$q(0,t) = 0, \quad t > 0, \qquad q \to f(t), \quad y \to \infty, \quad t > 0; \tag{2}$$

$$q(y,0) = q_0(y), \qquad y > 0. \tag{3}$$

Here $f(t)$ is the solution of the Riccati equation

$$\frac{df}{dt} + f^2 - a(t) = 0, \qquad f(0) = f_0 = \lim_{y\to\infty} q_0(y) \tag{4}$$

describing ideal fluid flow. In the flow regime from the stagnation points along the solid wall, $(y = 0)$ $f_0 > 0$. In this case, if in the expression for the pressure gradient, $p_x = -\rho x a(t)$ and $a > 0$ at $t > 0$, the solution of the Cauchy problem for Eq. (4) exists for all $t > 0$. If $a < 0$, then the solution of the Cauchy problem can break down in finite time, even for $f_0 < 0$, which corresponds to fluid flow to the stagnation point along the wall at the time $t = 0$ [7].

If the function q is independent of t, we have the classical steady-state problem. It is known that a steady-state solution is possible only in the fluid flow regime from a stagnation point. Otherwise, the pressure gradient on the plane is larger than zero, which is responsible for the absence of the analog of the Hiemenz solution. However, in this case, it is possible to construct a solution of the problem by considering the unsteady problem but assuming that the pressure gradient decreases with time. The unsteady problem (1)–(4) is considered in [9], where it is shown that if the pressure gradient is a periodic function of time, there may be both periodic motion and breakdown of the solution in finite time. If the initial data of the unsteady problem contain a counterflow zone and if the pressure gradient on the solid plane is negative, such zones disappear in finite time.

The constant factor ν can be eliminated from Eq. (1) using a suitable affine transformation. Therefore, without loss of generality, we assume that $\nu = 1$.

Analysis of the dimension of the quantities in Eqs. (1)–(4) shows that there is a similar variable $\xi = y/\sqrt{t}$ and the system of equations of the problems admits a self-similar representation of the solution.

Problem (1)–(4) is solved numerically by the relaxation method. The results of the calculation for the function $a(t) = -3/(16t^2)$ are presented in Fig. 1a and for the function $a(t) = 3/(16t^2)$ in Fig. 1b. As $t \to \infty$, the solution of the unsteady problem converges to a self-similar solution obtained below.

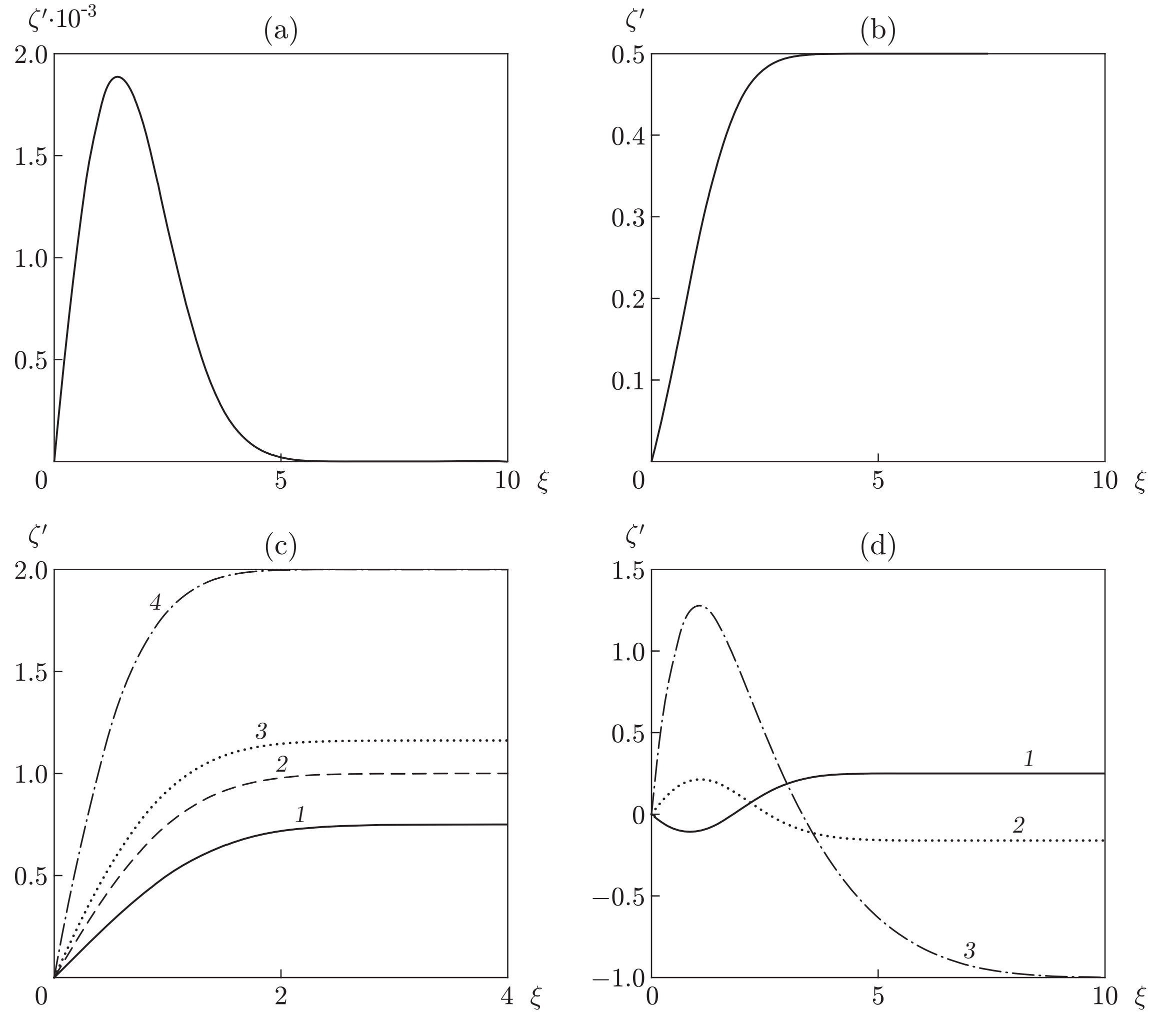

Fig. 2. Distribution of the longitudinal velocity in the plane problem for different values of the parameters k and m: (a) $k = 0$ and $m = 0$; (b) $k = -1/4$ and $m = 1/2$; (c) $k = -3/16$ and $m = 3/4$ (*1*), $k = 0$ and $m = 1$ (*2*), $k = 3/16$ and $m = (2 + \sqrt{7})/4$ (*3*), and $k = 2$ and $m = 2$ (*4*); (d) $k = -3/16$ and $m = 1/4$ (*1*), $k = 3/16$ and $m = (2 - \sqrt{7})/4$ (*2*), and $k = 2$ and $m = -1$ (*3*).

The self-similar solution of problem (1)–(3) will be sought in the form

$$u = \frac{x}{t}\,\zeta'(\xi), \qquad v = -\frac{1}{\sqrt{t}}\,\zeta(\xi), \qquad \xi = \frac{y}{\sqrt{t}}\,.$$

This solution corresponds to the function $a(t) = k/t^2$. The function ζ is the solution of the boundary-value problem

$$\zeta''' = (\zeta' - 1)\zeta' - (\zeta + \xi/2)\zeta'' - k, \qquad \xi > 0,$$

$$\zeta(0) = \zeta'(0) = 0, \qquad \zeta' \to m, \quad \xi \to \infty,$$

$$m^2 - m - k = 0.$$

The quadratic equation for m has two different real roots for $k > -1/4$ and one root for $m = 1/2$ with $k = -1/4$. Figure 2 shows graphs of $\zeta'(\xi)$ for different values of the parameters k and m. The case shown in Fig. 2a is of greatest interest. The vanishing of the parameters k and m implies that the pressure gradient $p_x = 0$ and the longitudinal velocity at infinity also vanishes. It should be noted that as $\xi \to \infty$, the pressure increases linearly, and the transverse component of the velocity tends to a constant. The results of calculation for $k = -1/4$ and $m = 1/2$ are presented in Fig. 2b, the distributions of the longitudinal velocity for various values of the parameter k and a positive root m in Fig. 2c, and for a negative root m in Fig. 2d.

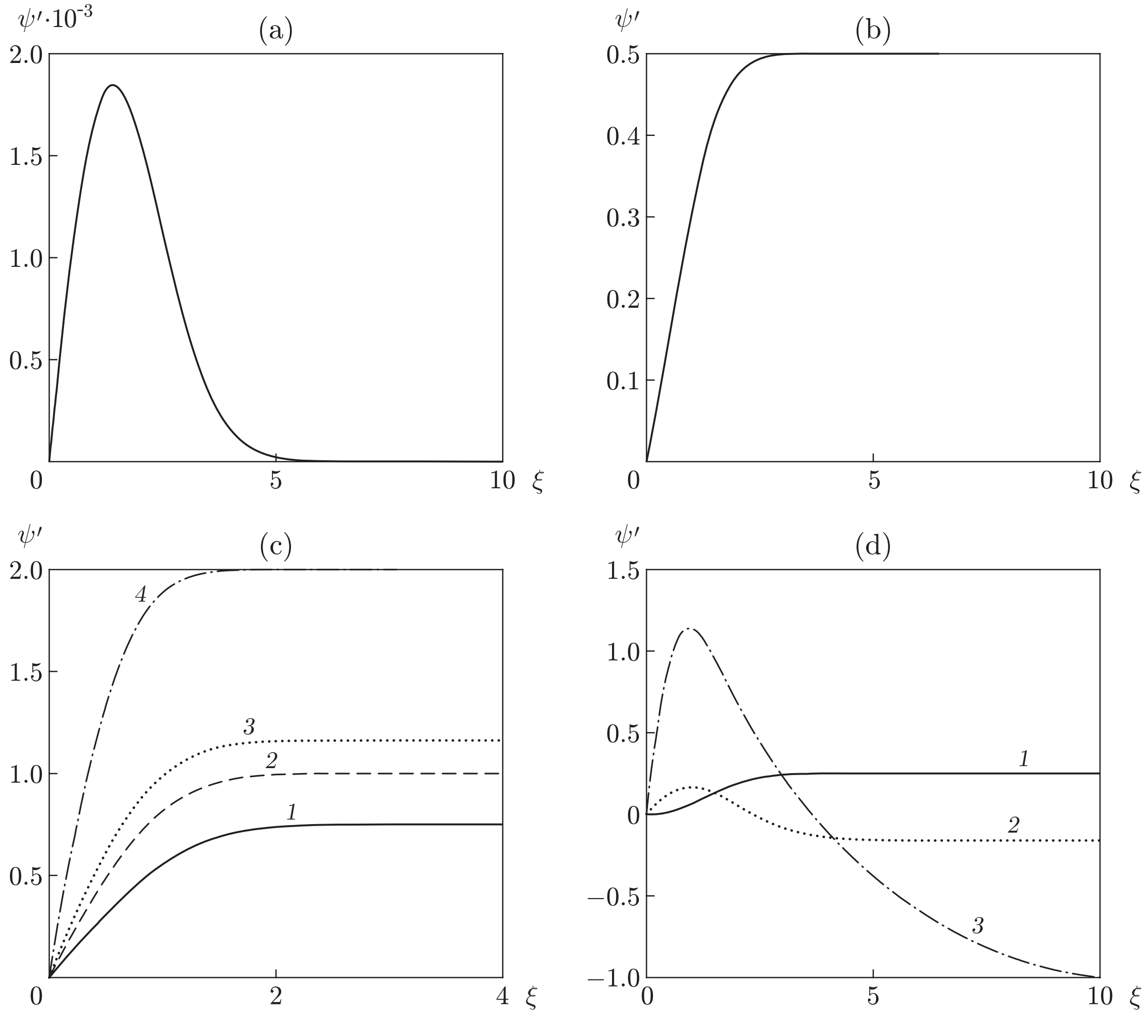

Fig. 3. Distribution of the longitudinal velocity in the axisymmetric problem for different values of the parameters k and m (notation the same as in Fig. 2).

AXISYMMETRIC FLOW

In viscous axisymmetric flow in the presence of a stagnation point, the fluid impinges on a wall placed perpendicular to the flow direction and flows away from the stagnation points along this wall in all radial directions. In cylindrical coordinates (r, φ, z), suppose that the plane $z = 0$ is aligned with the wall, the origin is located at the stagnation point, and the z direction is opposite to the direction of the incoming flow. Let u and w be the velocity components in the radial direction r and in the axial direction z, respectively. Due to the axial symmetry, the circumferential velocity component is zero. The velocity and pressure distributions are given by the formulas

$$u = rq(z,t), \qquad w = -2\int_0^z q(s,t)\,ds, \qquad p = -\frac{1}{2}\rho r^2 a(t) + Q(z,t),$$

where $a(t)$ is a given function of time; the function $q(z,t)$ satisfies the initial-boundary-value problem

$$q_t + q^2 - 2q_z \int_0^z q(s,t)\,ds - a(t) = \nu q_{zz}, \qquad z > 0, \quad t > 0,$$

$$q(0,t) = 0, \quad t > 0, \qquad q \to f(t), \quad z \to \infty, \quad t > 0, \qquad q(z,0) = q_0(z), \quad z > 0,$$

$f(t)$ is the solution of the Riccati equation (4). In this case, we can also assume that $\nu = 1$.

The axisymmetric problem has a self-similar solution

$$u = \frac{r}{t}\,\psi'(\xi), \qquad v = -\frac{2}{\sqrt{t}}\,\psi(\xi), \qquad \xi = \frac{z}{\sqrt{t}},$$

where the function ψ is the solution of the boundary-value problem

$$\psi''' = (\psi' - 1)\psi' - (2\psi + \xi/2)\psi'' - k, \qquad \xi > 0,$$

$$\psi(0) = \psi'(0) = 0, \qquad \psi' \to m, \quad \xi \to \infty,$$

$$m^2 - m - k = 0,$$

and, as in the plane case, $a(t) = k/t^2$ ($k \geqslant -1/4$).

Figure 3 shows graphs of $\psi'(\xi)$ for various values of the parameters k and m. Comparison of Figs. 2 and 3 shows that longitudinal velocity distributions differ only slightly.

CONCLUSIONS

Self-similar unsteady solutions describing plane and axisymmetric viscous incompressible fluid flows in the vicinity of a stagnation point located on the solid boundary are constructed. Fluid flow regimes from infinity to the stagnation point and from it are considered.

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REFERENCES

1. H. Schlichting, *Boundary-Layer Theory* (Braun, New York, 1955).
2. O. M. Lavrent'eva, "Flow of a Viscous Liquid in a Layer on a Rotating Plane," Prikl. Mekh. Tekh. Fiz. **130** (5), 41–48 (1989) [J. Appl. Mech. Tech. Fiz. **30** (5), 706–713 (1989)].
3. A. Shapiro, "An Analytical Solution of the Navier–Stokes Equations for Unsteady Backward Stagnation Point Flow with Injection or Suction," J. Appl. Math. Mech. **86** (4), 281–290 (2006).
4. V. K. Sin and C. K. Chio, "Unsteady Reversed Stagnation-Point Flow over a Flat Plate," ISRN Appl. Math. **2012**, Article ID 430432 (2012).
5. I. Proudman and K. Johnson, "Boundary-Layer Growth near a Rear Stagnation Point," J. Fluid Mech. **12** (2), 161–168 (1962).
6. A. Robins and J. Howarth, "Boundary-Layer Development at a Two Dimensional Rear Stagnation Point," J. Fluid Mech. **56** (1), 161–171 (1972).
7. V. V. Kuznetsov and V. V. Pukhnachov, "New Family of Exact Solutions of the Navier–Stokes Equations," Dokl. Akad. Nauk **425** (1), 40–44 (2009).
8. A. M. Gaifullin, "Self-Similar Unsteady Flow of Viscous Fluid," Izv. Ross. Akad. Nauk, Mekh. Zhidk. Gaza, No. 4, 29–35 (2005).
9. A. G. Petrova, V. V. Pukhnachev, and O. A. Frolovskaya, "Nonstationary Flow near the Critical Point," in *Progress in Continuum Mechanics, Collected Papers of the Int. Conf. Dedicated to the 75th Anniversary of Academician V. A. Levin, Vladivostok, September 28 to October 4, 2014* (Megaprint, Irkutsk, 2014), pp. 385–388.