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ADVECTIVE FLOW IN A ROTATING LIQUID FILM

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Abstract: This paper presents a new exact solution of the Navier–Stokes equations in the Boussinesq approximation that describes thermocapillary advective flow in a slowly rotating horizontal layer of incompressible fluid with free boundaries. Such flow occurs in the case of linear temperature distribution over horizontal coordinates or in the case of heat flux distribution at the layer boundaries. The influence of the Taylor, Marangoni, Grashof, and Biot numbers on the flow and temperature velocity profiles is studied.

Keywords: thermocapillary advection, rotation, exact solution.

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INTRODUCTION

Advective flows occur in a flat horizontal layer of fluid under the effect of a longitudinal temperature gradient [1]. A feature of these flows is the absence of the vertical velocity component, with the flow velocity vector being directed perpendicularly to the buoyancy force. This property remains unchanged under different boundary conditions for the velocity, as evidenced, for example, by analytical and numerical studies of thermocapillary advective flow that occurs in horizontal layers of fluid and gas with a free boundary between them in microgravity [2].

In a case where the temperature at the boundary layer is a linear function ($T_1 = Ax$, where x is the longitudinal coordinate and A is the permanent horizontal temperature gradient at the layer boundaries), the flow is described analytically, being an exact solution of the Navier–Stokes equations in the Boussinesq approximation. Advective flow in the presence of gravity was first described analytically in [3]. In [4], advective flows occurring in a flat horizontal layer with rigid boundaries or with a free upper boundary are presented as an exact solution of the Navier–Stokes equations. A review of such plane-parallel advective flows under various boundary conditions is given in [5, 6]. Stationary and usually closed flows with zero consumption are considered therein. Exact solutions of the Oberbeck–Boussinesq equations that describe motion in a horizontal band and a circular rotating pipe when the longitudinal temperature gradient is time-dependent are presented in [7].

Advective flows in a rotating layer of fluid directed perpendicular to its axis of rotation were first described analytically in [8]. The object of that study was a thin flat layer of rotating fluid with a solid lower boundary and a free nondeformable upper boundary. The exact solution was used to derive quasi-two-dimensional equations for the velocity and temperature averaged across the layer, and advective waves and solitons were investigated, with only the case of fast rotation being considered. Generalization to the case of conducting fluid was made in [9].

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A wide class of new advective flows in a rotating flat layer of incompressible fluid is presented in [10], where a procedure is also proposed for obtaining exact solutions using complex functions both in the case of slow and fast rotation, i.e., for any value of the Taylor number. Particularly, advective flow in a rotating horizontal layer of fluid with rigid boundaries whose stability was studied in [11–13] and flow with a free upper boundary whose stability was touched upon in [14] are described in [10]. A review of the research on the influence of rotation on the stability of advective flows is given in [15]. As in the case of no rotation, advective flows have no vertical velocity component and the velocity vector directed along the normal to the buoyancy force has two horizontal components.

Advective flow in a flat layer with a longitudinal temperature gradient emerging in zero-gravity under the influence of linear high-frequency vibrations is described analytically in [16]. It is shown in [10] that advective flows formed under the influence of longitudinal vibrations are similar to flows occurring under the effect of rotation. Plane-parallel thermocapillary flow in a rotating layer of fluid with a free upper boundary, which is a special case of the solution described in [18, 19], and the stability of this flow are studied in [17]. A theoretical analysis of two-dimensional stationary axially symmetric thermocapillary convection occurring in a slowly rotating circular cylindrical container with a free upper boundary with heating of the outer walls under the force of gravity is presented in [20]. Thermocapillary flow of silicone oil (Prandtl number Pr $= 0.011$) in a slowly rotating small circular cell in which with the outer surface of its side wall is heated and the inner surface of its side wall is cooled has been studied by numerical simulation [21]. The numerical calculation results enable a more complete description of the formation of wave structures, the vibrational behavior of hydrothermal waves, and the conditions of their origin.

Advective thermocapillary flow in zero-gravity in a slowly rotating thin layer of an incompressible fluid with two flat free boundaries whose heat transfer satisfies the Newton law is analytically described in [22]. The object of the study was the flow stability at Pr $= 6.7$. The behavior of finite amplitude perturbations outside the stability region was investigated numerically.

This work is a continuation of [22] and devoted to thermocapillary flow in a slowly rotating layer with free boundaries under gravity.

1. FORMULATION OF THE PROBLEM

We consider a flat layer of incompressible fluid of thickness $2h$ that slowly rotates with constant angular velocity Ω_0 in the rotating Cartesian coordinate system $Oxyz$. The axis of rotation coincides with the vertical coordinate axis directed upward. The Froude number $\Omega_0^2 l/g \ll 1$ (l is the horizontal scale of the fluid motion and the g is the acceleration of gravity) [23] is assumed to be small so that the centrifugal force is negligible at a sufficiently large distance from the vertical axis. Both layer boundaries are free, assumed to be flat during slow rotation, and under the effect of the Marangoni tangential thermocapillary force.

The surface tension coefficient depends linearly on the fluid temperature T:

$$\sigma = \sigma_0 - \gamma(T - T_1)$$

(γ is the temperature coefficient of surface tension).

Choosing h, h^2/ν, $\gamma A h/(\rho_0 \nu)$, Ah, and γA (ν is the kinematic viscosity and ρ_0 is the average density) as the size scales of the size, time, velocity, temperature, and pressure, respectively, we obtain equations for the thermocapillary flows in dimensionless form

$$\frac{\partial u}{\partial t} + \mathrm{Mn}\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right) - \sqrt{\mathrm{Ta}}\, v = -\frac{\partial p}{\partial x} + \Delta u; \quad (1)$$

$$\frac{\partial v}{\partial t} + \mathrm{Mn}\left(u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right) + \sqrt{\mathrm{Ta}}\, u = -\frac{\partial p}{\partial y} + \Delta v; \quad (2)$$

$$\frac{\partial w}{\partial t} + \mathrm{Mn}\left(u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}\right) = -\frac{\partial p}{\partial z} + \Delta w + WT; \quad (3)$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0; \quad (4)$$

$$\frac{\partial T}{\partial t} + \mathrm{Mn}\left(u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} + w\frac{\partial T}{\partial z}\right) = \frac{1}{\mathrm{Pr}}\,\Delta T \tag{5}$$

with the boundary conditions

$$z = \pm 1\colon \qquad \frac{\partial u}{\partial z} = -\frac{\partial T}{\partial x}, \quad \frac{\partial v}{\partial z} = -\frac{\partial T}{\partial y}, \quad w = 0, \quad p = 0, \tag{6}$$

where p is the deviation of the external pressure from the hydrostatic pressure, u, v, and w are the velocity vector components, t is time, $\mathrm{Mn} = \gamma A h^2/(\rho_0 \nu^2)$ is the Marangoni number, $\mathrm{Ta} = (2\Omega_0 h^2/\nu)^2$ is the Taylor number, $W = \mathrm{Gr}/\mathrm{Mn}$, $\mathrm{Pr} = \nu/\chi$, $\mathrm{Gr} = g\beta A h^4/\nu^2$ is the Grashof number, β is the thermal expansion coefficient, and χ is the thermal diffusivity. The boundary conditions for the temperature are determined below.

2. ADVECTIVE FLOW

Accounting for boundary conditions (6), the exact solution of system (1)–(5) is sought in the form

$$u = u_0(z), \quad v = v_0(z), \quad w \equiv 0, \quad T = T_0 \equiv x\tau_0(z) + \tau_1(z), \quad p = p_0(x, z). \tag{7}$$

Substituting expressions (7) into system(1)–(5), we obtain the following system of partial differential equations for the pressure and the horizontal velocity and temperature components:

$$\frac{\partial p_0}{\partial z} = W T_0; \tag{8}$$

$$-\sqrt{\mathrm{Ta}}\, v_0(z) = -\frac{\partial p_0}{\partial x} + u_0''(z); \tag{9}$$

$$\sqrt{\mathrm{Ta}}\, u_0(z) = v_0''(z); \tag{10}$$

$$\mathrm{Mn}\,\mathrm{Pr}\, u_0(z)\tau_0(z) = x\tau_0''(z) + \tau_1''(z). \tag{11}$$

The boundary conditions are written in the form

$$z = -1\colon \qquad u_0' = 1, \quad v_0' = 0, \quad p_0 = 0; \tag{12}$$

$$z = 1\colon \qquad u_0' = -1, \quad v_0' = 0, \quad p_0 = 0. \tag{13}$$

Integrating the pressure equation (8) across the layer and taking into account the boundary conditions (12), (13), we obtain

$$p_0 = W \int_{-1}^{z} T_0\, d\zeta.$$

This equation is valid only if the temperature averaged across the layer is zero: $\int_{-1}^{1} T_0\, d\zeta = 0$, i.e., $\int_{-1}^{1} \tau_0(\zeta)\, d\zeta = 0$, and $\int_{-1}^{1} \tau_1(\zeta)\, d\zeta = 0$. It follows from Eq. (11) that

$$\tau_0''(z) = 0, \qquad \mathrm{Mn}\,\mathrm{Pr}\, u_0(z)\tau_0(z) = \tau_1''(z). \tag{14}$$

Here, in view of vanishing of the temperature averaged across the layer, $\tau_0(z) = z$. This means that there are several possible boundary conditions for T_0 and, accordingly, for $\tau_0(z)$ and $\tau_1(z)$.

1. The temperature at the boundaries along the layer can be determined in the form of a linear function in the set coordinate system:

$$z = \mp 1\colon \qquad T_0 = \mp x \tag{15}$$

[in this case $\tau_0(\mp 1) = \mp 1$, $\tau_1(\mp 1) = 0$, and $T_1 = \mp x$].

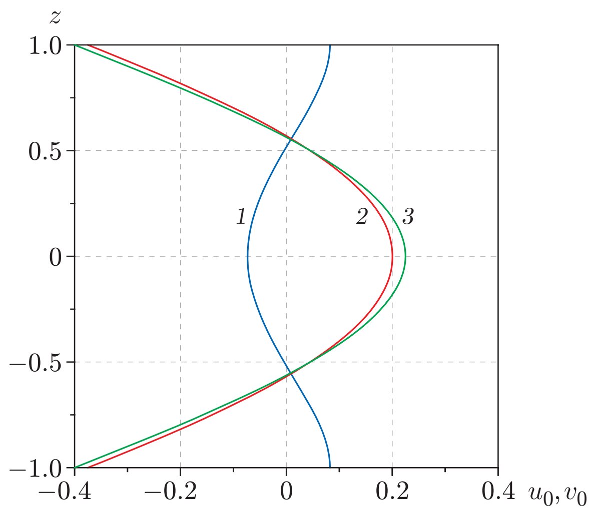

Fig. 1. Profiles of the velocity components $u_0(z)$ (*1* and *3*) and $v_0(z)$ (*2*): (*1, 2*) with rotation ($W = 3$, $\mathrm{Ta} = 10$); (*3*) without rotation.

2. Heat fluxes are set at the boundaries in the form

$$z = \mp 1: \qquad \frac{\partial T_0}{\partial z} = x. \tag{16}$$

Then, $\tau_0'(\mp 1) = 1$, $\tau_1'(\mp 1) = 0$, and $T_1 = \mp x + \tau_1(\mp 1)$.

3. Heat transfer occurs at the boundaries according to Newton's law:

$$z = \mp 1: \qquad \frac{\partial T_0}{\partial z} = \pm \operatorname{Bi}(T_0 \pm A_1 x). \tag{17}$$

Here $\operatorname{Bi} = bh/\varkappa$ is the Biot number, b is the heat transfer coefficient, and $\varkappa$ is the thermal conductivity. Then, $\tau_1'(\mp 1) = \pm \operatorname{Bi} \tau_1(\mp 1)$, if $\operatorname{Bi} \neq 0$, $A_1 = (1 + \operatorname{Bi})/\operatorname{Bi}$, and $T_1 = \pm(1 + \operatorname{Bi})x/\operatorname{Bi} + \tau_1(\mp 1)$.

In all cases, the pressure is

$$p_0(x, z) = W\Big(x\,\frac{z^2 - 1}{2} + \int\limits_{-1}^{z} \tau_1(\zeta)\, d\zeta\Big).$$

To determine the horizontal velocity components u and v, we introduce a complex function $M(z) = u_0(z) + iv_0(z)$ (i is the imaginary unit). We multiply Eq. (10) by the imaginary unit and combine the result with Eq. (9). The result is a boundary-value problem for ordinary differential equations with respect to the complex velocity

$$M''(z) - i\sqrt{\mathrm{Ta}}\, M(z) = W\, \frac{z^2 - 1}{2}, \qquad M'(\mp 1) = \pm 1. \tag{18}$$

Solving problem (18), we obtain an analytical expression for the two nonzero velocity components of the thermocapillary flow:

$$M(z) = u_0(z) + iv_0(z) = -\frac{\cosh(\lambda z)}{\lambda \sinh(\lambda)} + \frac{W}{\lambda^2}\Big(\frac{\cosh(\lambda z)}{\lambda \sinh(\lambda)} - \frac{1}{\lambda^2} - \frac{z^2 - 1}{2}\Big), \tag{19}$$

where $\lambda = \sqrt[4]{\mathrm{Ta}/4}\,(1 + i)$. As in [22], the first term in (19) describes the thermocapillary effect and the second term the effect of gravity on the motion of the fluid in the rotating horizontal layer. Analysis of the solution of Eq. (19) showed that the velocity components are parabolic and symmetric with respect to the vertical axis (Fig. 1) and their profile shape barely changes with slow rotation. On the boundaries of the layer, $M(-1) = M(1)$; a boundary layer begins to form with increasing Taylor number near the free boundaries. The velocity reaches a maximum in the middle of the layer where the jet is formed. The maximum velocity is linearly dependent on the parameter W in this range of the Taylor number $0 \leqslant \mathrm{Ta} \leqslant 10$. With increasing Ta, the velocity decreases.

Without rotation (Ta = 0), the horizontal velocity components are equal to

$$u_0(z) = W\,\frac{5z^4 - 30z^2 + 9}{120}, \qquad v_0(z) \equiv 0, \tag{20}$$

and boundary conditions (12) and (13) are satisfied only if $W = 3$. Note that, in expression (20), the velocity function satisfies the closure condition, it is symmetric, $u_0(z) = 0.225$ at the center of the layer, and $u_0(z) = \pm 0.4$ at the boundaries of the layer. If $z = \pm\sqrt{3 - 6\sqrt{5}/5}$, we have $u_0(z) = 0$.

It follows from expression (10) that the velocity component along the x axis equals $u_0(z) = v_0''(z)/\sqrt{\mathrm{Ta}}$. In order to determine the fluid temperature in the layer, we substitute this expression into the second equation of system (14). As a result, we obtain the equation

$$\frac{\mathrm{Mn}\,\mathrm{Pr}}{\sqrt{\mathrm{Ta}}}\, z v_0''(z) = \tau_1''(z).$$

We introduce the notation $f(z) = \int v(z)\,dz$. Given that $M(-1) = M(1)$, $f(-1) = -f(1)$, $\int\limits_{-1}^{1} f(z)\,dz = 0$, and $\int\limits_{-1}^{1} z v(z)\,dz = 0$, we obtain the following temperature distribution for boundary conditions (15):

$$T(z) = xz + \frac{\mathrm{Mn}\,\mathrm{Pr}}{\sqrt{\mathrm{Ta}}}\,[z(v(z) - v(-1)) - 2(f(z) + zf(-1))]. \tag{21}$$

Without rotation (Ta = 0), expression (21) takes the form

$$T(z) = xz + \mathrm{Mn}\,\mathrm{Pr}\,\frac{5z^7 - 63z^5 + 63z^3 - 5z}{1680}. \tag{22}$$

If the heat fluxes at the boundaries of the rotating layer are set in the form (16), the temperature is

$$T(z) = xz + \frac{\mathrm{Mn}\,\mathrm{Pr}}{\sqrt{\mathrm{Ta}}}\,[z(v(z) + v(-1)) - 2f(z)].$$

If Ta = 0,

$$T(z) = xz + \mathrm{Mn}\,\mathrm{Pr}\,\frac{5z^7 - 63z^5 + 63z^3 + 91z}{1680}.$$

If the heat transfer at the boundaries obeys Newton's law in the form (17), we have the formula

$$T(z) = xz + \frac{\mathrm{Mn}\,\mathrm{Pr}}{\sqrt{\mathrm{Ta}}}\left[z\Big(v(z) + \frac{1-\mathrm{Bi}}{1+\mathrm{Bi}}\,v(-1)\Big) - 2\Big(f(z) + z\,\frac{2\,\mathrm{Bi}}{1+\mathrm{Bi}}\,f(-1)\Big)\right],$$

without rotation,

$$T(z) = xz + \mathrm{Mn}\,\mathrm{Pr}\,\frac{5z^7 - 63z^5 + 63z^3 + (91 - 5\,\mathrm{Bi})z/(1+\mathrm{Bi})}{1680}.$$

In all the above-mentioned cases of the boundary conditions, the profile of $\tau_1(z)$ is antisymmetric (Fig. 2).

In the presence of rotation with increasing parameter W, the temperature monotonically increases in cases (15) and (21); it monotonically decreases in the considered range of the Taylor number in cases (16) and (17). The maximum temperature decreases monotonically with increasing Ta. As the Biot number becomes greater, the temperature peak at the lower boundary of the layer increases and the local maximum in the middle of the layer reduces and moves toward the middle. This relationship is maintained in the entire considered range of W.

Without rotation of the horizontal layer for the case of boundary conditions for temperature (15) and (22), the temperature in the upper half of the layer is higher than in the bottom (see Fig. 2b); $\tau_1 = 0$ for $z = 0$, $z = \pm 1$, and $z = \pm 0.294715$. Its maximum value $0.000327232\,\mathrm{Mn}\,\mathrm{Pr}$ is reached for $z = -0.166531$. In the case of boundary conditions (16) and (17), the dependence $\tau_1(z)$ is almost linear.

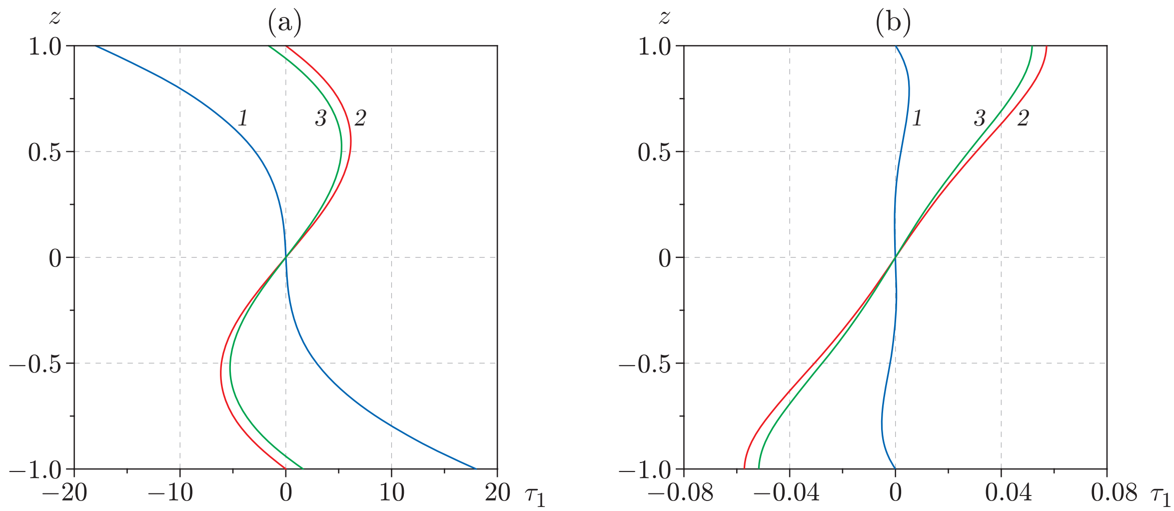

Fig. 2. Profiles of $\tau_1(z)$ with linear temperature distribution and with a heat flux in the form of the linear function (*2*) and with heat exchange (Bi = 0.1) at the layer boundaries (*3*): (a) $W = 3$ and Ta = 10; (b) Ta = 0.

3. CONCLUSIONS

We obtained a new exact solution of the Navier–Stokes equations that analytically describes thermocapillary advective flow in a slowly rotating flat layer of incompressible fluid with free boundaries. It is particularly noteworthy that the solution can also be described without rotation. In the case of slow rotation, it is possible to neglect the centrifugal force and to assume that the boundaries remain flat at a relatively large distance from the axis of rotation. If the outer pressure deviation from the hydrostatic pressure is zero at the free boundaries, the temperature averaged across the layer should also be zero. The profiles of the velocity components are symmetrical relative to the middle of the layer and parabolic at the central part of the layer. The temperature profile is antisymmetric. With an increase in the Taylor number, the maximum velocity and temperature of the thermocapillary flow decrease monotonically.

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