ISSN 0021-8944, Journal of Applied Mechanics and Technical Physics, 2015, Vol. 56, No. 4, pp. 715–725. © Pleiades Publishing, Ltd., 2015.
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BOUNDARY ELEMENT SOLUTION OF THE PLANE ELASTICITY PROBLEM FOR AN ANISOTROPIC BODY WITH FREE SMOOTH BOUNDARIES

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UDC 539.3: 519.642.7

Abstract: A boundary singular integral equation of the plane problem was constructed using an approach based on the representation of the unknown Lekhnitskii complex potentials in the form of Cauchy type integrals with unknown densities on the boundary of the region occupied by the body. The contours of the holes and cuts and the shape of the outer boundary are exactly or approximately represented in the form of a sequence of straight and curved (in the form of elliptical arcs) boundary elements. The unknown densities on the boundary elements are approximated by a linear combination of some regular functions or complex functions that have a known singularity. In the numerical solution of the integral equation by the collocation method or by the least-squares method and in the subsequent calculations of the stress–strain state, the integrals of all types along the boundary elements are calculated analytically, which significantly increases the accuracy of the results.

Keywords: elasticity, anisotropy, plane problem, complex singular integral equation, boundary element, analytical integration.

DOI: 10.1134/S0021894415040197

INTRODUCTION

Transition to the complex plane in the solution of two-dimensional problems of elasticity allows the use of effective means of the theory of functions of complex variables, namely, analytic functions and their properties, conformal mappings, and Cauchy and Hadamard type integrals. The methods of complex singular and hypersingular boundary integral equations (BIEs) developed on the basis of these integrals allow solving boundary-value problems for isotropic and anisotropic bodies of finite and infinite dimensions with cuts (cracks) and holes of arbitrary shapes [1–6]. However, when numerical solutions of singular BIEs are obtained using the method of mechanical quadrature [1–4] or the complex boundary-element method CBEM) with numerical integration [5], the error in calculating the Cauchy type integrals at the points near the contour (boundary) of integration is significant. This leads to a decrease in the accuracy of solution of the BIEs at small distances between holes and cracks and the external boundary and a decrease in the accuracy of calculation of stresses near concentrators. To increase the calculation accuracy, it is necessary to use special schemes of numerical integration and modify them by changing the position of each computational point near the boundary. In other versions of the CBEM developed for isotropic materials [5, 7], the choice of standard boundary elements (BEs) and approximating functions makes possible elementwise integration in closed form, and thus provides maximum accuracy and easy computational procedures for any position of computational points with respect to the boundary.

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In this paper, we propose a CBEM based on analytical integration for solving complex and singular BIEs and determining the stress–strain states in the plane elastic problem of an anisotropic body. The method was tested for the case of infinite or finite plates weakened by smooth unloaded holes and cuts. This approach can also be used to solve equations of this type in problems of the bending of anisotropic plates [6].

1. FORMULATION OF THE PROBLEM AND THE SYSTEM OF EQUATIONS

Suppose that a finite or an infinite plate made of a rectilinear anisotropic material occupies an open domain D_0 in the plane $z_0 = x_0 + iy_0$. The line L_0 bounding the domain D_0 consists of M individual contours $L_0^{(m)}$ $(m = 1, \dots, M)$, among which the first $N \geqslant 0$ contours are closed (boundaries of holes), and the remaining $M - N \geqslant 0$ contours are open curvilinear (crack cuts). In the case of a finite plate, the line L_0 also contains the closed outer boundary $L_0^{(0)}$. All individual contours are smooth and have no points of self-intersection. The open domains internal to $L_0^{(m)}$ $(m = 1, \dots, N)$ and the open domain external to $L_0^{(0)}$ will be denoted by $D_0^{(m)}$ $(m = m_0, \dots, N$; for an infinite plate, $m_0 = 0$, and for an infinite plate, $m_0 = 1)$. The plate is in a generalized plane stress state [8]. The crack faces, the edges of the holes, and the outer boundary of the finite plate are free of load; the interior points and parts of the domain D_0 are subjected to forces homogeneously distributed across the thickness which are balanced in the case of a finite plate). An infinite plate is additionally subjected to distributed forces at infinity. We consider the case where the crack faces do not interact with each other and their free overlapping (interpenetration) is formally possible. It is required to determine the stress–strain state of the plate.

The displacements and stresses at the point with the coordinate z_0 are expressed in terms of the analytic functions (complex Lekhnitskii potentials) $\varphi_\nu(z_\nu)$ [8, 9]:

$$(u, v) = 2\,\mathrm{Re}\Big(\sum_{\nu=1}^{2}(p_\nu, q_\nu)\varphi_\nu(z_\nu)\Big) + (u_0 - \omega y, v_0 + \omega x); \tag{1}$$

$$(\sigma_x, \tau_{xy}, \sigma_y) = 2\,\mathrm{Re}\Big(\sum_{\nu=1}^{2}(\mu_\nu^2, -\mu_\nu, 1)\Phi_\nu(z_\nu)\Big), \qquad \Phi_\nu(z_\nu) = \varphi_\nu'(z_\nu); \tag{2}$$

$$z_\nu = x_0 + \mu_\nu y_0, \qquad \nu = 1, 2. \tag{3}$$

Here p_ν and q_ν are constants of the plate material; u_0, v_0, and ω are arbitrary constants, μ_1 and μ_2 are complex or imaginary roots of the characteristic equation with positive imaginary parts (the case $\mu_1 \neq \mu_2$ is considered). In transformation (3), the lines L_0 and contours $L_0^{(m)}$ in the plane z_0 correspond to the line L_ν and contours $L_\nu^{(m)}$ in the plane z_ν $(\nu = 1, 2$ and $m = m_0, \dots, M)$.

For $N + 1 - m_0 > 0$, we assume that in the domains $D_0^{(m)}$ $(m = m_0, \dots, N)$, there are plates (dummy) with the same properties as in the domain D_0. Let the potentials $\varphi_\nu(z_\nu)$ describing the stress–strain state in these plates correspond to the homogeneous load conditions on their boundaries and zero stresses at their interior points. For the piecewise analytic function $\varphi_\nu(z_\nu)$ constructed in this way, the boundary conditions at the points $t_0 \in L_0^{(m)}$ $(m = m_0, \dots, M)$ have the form [8–10]

$$\sum_{\nu=1}^{2}\Big[(1 + i\mu_\nu)\varphi_\nu^{\pm}(t_\nu) + (1 + i\bar{\mu}_\nu)\,\overline{\varphi_\nu^{\pm}(t_\nu)}\,\Big] = C^{(m)\pm}, \qquad t_\nu = \mathrm{Re}\, t_0 + \mu_\nu\, \mathrm{Im}\, t_0, \tag{4}$$

where $\varphi_\nu^{\pm}(t_\nu)$ are the limiting values of $\varphi_\nu(z_\nu)$ when the point t_ν is approached from the left (the plus sign) and from the right (the minus sign) for positive direction of traversing $L_\nu^{(m)}$ (in tracing closed contours, the domain D_0 should remain on the left); $C^{(m)\pm}$ are some complex constants [8–10].

From Eq. (4) and its conjugate equality, we exclude $\overline{\varphi_2(t_2)}$ [10]. Then, differentiating the equation with respect to the variable t_2, by analogy with [1], we write the boundary conditions on the edges of the boundary contours in the form

$$\begin{gathered} a_0\,\frac{dt_1}{dt_2}\,\Phi_1^{\pm}(t_1) + b_0\,\frac{d\bar{t}_1}{dt_2}\,\overline{\Phi_1^{\pm}(t_1)} + \Phi_2^{\pm}(t_2) = 0, \\ a_0 = (\mu_1 - \bar{\mu}_2)/(\mu_2 - \bar{\mu}_2), \qquad b_0 = (\bar{\mu}_1 - \bar{\mu}_2)/(\mu_2 - \bar{\mu}_2). \end{gathered} \tag{5}$$

The unknown functions $\Phi_\nu(z_\nu)$ in all domains are represented in the following form [1, 2, 4]:

$$\Phi_\nu(z_\nu) = \Phi_\nu^0(z_\nu) + \Phi_\nu^1(z_\nu); \tag{6}$$

$$\Phi_\nu^1(z_\nu) = \frac{1}{2\pi i}\int\limits_{L_\nu} \frac{\omega_\nu(s_\nu)\,ds_\nu}{s_\nu - z_\nu}, \qquad s_\nu = \operatorname{Re} s_0 + \mu_\nu \operatorname{Im} s_0, \qquad s_0 \in L_0, \quad \nu = 1, 2. \tag{7}$$

Here the functions $\Phi_\nu^0(z_\nu)$ describing the stress state of an infinite solid plate (with no holes and cuts) caused by exposure to a given load are assumed to be known [8, 9] or can be obtained by integrating the solution of the problem of action of concentrated forces; the functions $\Phi_\nu^1(z_\nu)$ define the perturbation of the stress–strain state due to the presence of the boundary L_ν; $\omega_\nu(t_\nu)$ are unknown complex density functions.

We substitute (6) and (7) into the boundary conditions (5). Summing and subtracting the two limiting equalities and using the properties of the limiting values of the Cauchy type integrals, we obtain the singular integral equation for the densities $\omega_1(t_1)$ and $\omega_2(t_2)$ and the equation describing their relationship:

$$a_0 \frac{dt_1}{dt_2}\int\limits_{L_1} \frac{\omega_1(s_1)\,ds_1}{s_1 - t_1} - b_0 \frac{d\bar{t}_1}{dt_2}\overline{\int\limits_{L_1} \frac{\omega_1(s_1)\,ds_1}{s_1 - t_1}} + \int\limits_{L_2} \frac{\omega_2(s_2)\,ds_2}{s_2 - t_2} = -2\pi i\Big(a_0 \frac{dt_1}{dt_2}\,\Phi_1^0(t_1) + b_0 \frac{d\bar{t}_1}{dt_2}\,\overline{\Phi_1^0(t_1)} + \Phi_2^0(t_2)\Big); \tag{8}$$

$$\omega_2(t_2) = -a_0 \frac{dt_1}{dt_2}\,\omega_1(t_1) - b_0 \frac{d\bar{t}_1}{dt_2}\,\overline{\omega_1(t_1)}, \qquad t_\nu \in L_\nu, \quad \nu = 1, 2. \tag{9}$$

Using Eq. (9), Eq. (8) can be transformed to an equation with one unknown function $\omega_1(t_1)$ [2] and an explicit Cauchy kernel [1, 4], but for numerical implementation, it is advisable to use (8) and (9). The condition of uniqueness of displacements in tracing each contour $L_0^{(m)}$ leads to the following additional relations [1, 2, 4]:

$$\int\limits_{L_1^{(m)}} \omega_1(s_1)\,ds_1 = 0, \qquad m = m_0, \ldots, M. \tag{10}$$

In general, in each of the domains D_0 or $D_0^{(m)}$ $(m = m_0, \ldots, N)$, the functions $\Phi_1(z_1)$ and $\Phi_2(z_2)$ satisfying the conditions of the first primary problems are determined up to arbitrary terms that can be fixed by specifying one (for each domain) real parameter [9]. Representation of the functions $\Phi_\nu(z_\nu)$ in the form (6), (7) provides their definiteness in the domain $D_0^{(0)}$ (finite plate) or in the domain D_0 (infinite plate) since $\Phi_\nu^0(z_\nu)$ are assumed to be known definite functions, and $\Phi_\nu^1(\infty) = 0$ is independent of the type of the desired density function. To uniquely determine $\Phi_\nu(z_\nu)$ in the other domains and, therefore, to uniquely determine the unknown functions $\omega_\nu(t_\nu) = \Phi_\nu^+(t_\nu) - \Phi_\nu^-(t_\nu)$, it is necessary to impose some fixing conditions such as those proposed in [9]: $\operatorname{Im}\Phi_1(z_1^{(m)}) = C^{(m)}$ ($C^{(m)}$ are arbitrary real constants) which, in this case, can be reduced to

$$\operatorname{Re}\int\limits_{L_1} \frac{\omega_1(s_1)\,ds_1}{s_1 - z_1^{(m)}} = 0, \qquad z_1^{(m)} = \operatorname{Re} z_0^{(m)} + \mu_1 \operatorname{Im} z_0^{(m)}, \qquad m = m_0, \ldots, N, \tag{11}$$

where $z_0^{(m)} \in D_0^{(m)}$ $(m = 1, \ldots, N)$; for a finite plate, $z_0^{(0)} \in D_0$.

2. BOUNDARY-ELEMENT APPROXIMATION. STANDARD ELEMENTS

In the numerical solution of system (8)–(11), the boundary line L_0 is accurately or approximately represented in the form of a sequence $J = J_M$ of boundary elements L_{0j}, while maintaining the smoothness of the contours of $L_0^{(m)}$:

$$L_0 = \bigcup_{m=m_0}^{M} L_0^{(m)} \approx \bigcup_{m=m_0}^{M}\Big(\bigcup_{j=J_{m-1}+1}^{J_m} L_{0j}\Big) = \bigcup_{j=1}^{J} L_{0j},$$

$$L_{0j} \cap L_{0n} = \varnothing, \qquad j, n = 1, \ldots, J, \quad j \neq n.$$

Here J_m is the maximum element number on the contour $L_0^{(m)}$ ($J_{-1} = 0$ for $m_0 = 0$ and $J_0 = 0$ for $m_0 = 1$). The boundary element (BE) L_{0j} and its start and end points (A_{0j} and B_{0j}) in the plane z_0 are transformed to the BE $L_{\nu j}$ and the points $A_{\nu j}$ and $B_{\nu j}$ in the plane z_ν ($\nu = 1, 2$).

The standard BE L_{0j} is an element of simple form for which, under some specially selected conformal mappings of the plane z_1 and z_2, the images of the elements L_{1j} and L_{2j} in the corresponding transformed planes coincide. Application of these mappings to relations (7)–(11) written for the discretized boundary provides transition from two variables (t_1, t_2 or s_1, s_2) to one variable in each BE. In each standard BE, the densities are approximated by functions of a new variable of such form that at all points of the element, conditions (9) are satisfied and in the numerical solution of system (8), (10) (11) and in the determination of stresses (2) using formula (7), all integrals are calculated element-wise in closed form.

Let $\zeta_{\nu j} = \zeta_{\nu j}(z_\nu)$ and $z_\nu = z_{\nu j}(\zeta_{\nu j}) = Q_{\nu j} Z_{\nu j}(\zeta_{\nu j})$ be mutually reciprocal one-valued conformal mappings of the entire plane z_ν to the entire auxiliary plane $\zeta_{\nu j}$, or its part, and vice versa ($Q_{\nu j}$ is a dimensional coefficient, where $\nu = 1, 2$, and j is the BE number); in general, $\zeta_{1j}(z_1) \neq \zeta_{2j}(z_2)$. Then, the point of an arbitrary nth BE $t_{\nu n} \in L_{\nu n}$ is mapped to the point $\tau_{\nu jn}$ of the corresponding contour $\Lambda_{\nu jn}$ in the plane $\zeta_{\nu j}$:

$$\zeta_{\nu j}(t_{\nu n}) = \tau_{\nu jn} \in \Lambda_{\nu jn}, \qquad z_{\nu j}(\tau_{\nu jn}) = t_{\nu n} \in L_{\nu n}, \qquad j, n = 1, \ldots, J,$$

and, in general, $\tau_{1jn} \neq \tau_{2jn}$. We require that at $n = j$,

$$\begin{gathered} \tau_{1jj} = \zeta_{1j}(t_{1j}) = \tau_{2jj} = \zeta_{2j}(t_{2j}) = \tau_j \in \Lambda_j = \Lambda_{1jj} = \Lambda_{2jj}, \\ z_{1j}(\tau_j) = t_{1j}, \qquad z_{2j}(\tau_j) = t_{2j}. \end{gathered} \tag{12}$$

The points $s_{\nu j}$, $A_{\nu j}$, and $B_{\nu j}$ belonging to the line $L_{\nu j}$ are mapped to the points σ_j, α_j, and β_j on the contour Λ_j:

$$\zeta_{\nu j}(s_{\nu j}) = \sigma_j, \quad \zeta_{\nu j}(A_{\nu j}) = \alpha_j, \quad \zeta_{\nu j}(B_{\nu j}) = \beta_j, \qquad \nu = 1, 2. \tag{13}$$

The desired function $\omega_\nu(t_\nu)$is approximated at the jth BE by the function $\omega_{\nu j}(t_{\nu j})$, which is represented in the form of a linear combination of some independent functions $u_{jp}(\tau_j)/Z'_{\nu j}(\tau_j)$ with unknown coefficients $c_{\nu jp}$:

$$\omega_{\nu j}(t_{\nu j}) = \omega_{\nu j}[z_{\nu j}(\tau_j)] = \frac{1}{Z'_{\nu j}(\tau_j)} \sum_{p=1}^{q_j} c_{\nu jp} u_{jp}(\tau_j), \qquad j = 1, \ldots, J, \quad \nu = 1, 2. \tag{14}$$

2.1. Straight BE

For the straight element with origin at the point A_{0j} and end at the point B_{0j}, as the mappings $z_{\nu j}(\zeta_{\nu j})$ ($\nu = 1, 2$) we use the linear functions

$$\begin{gathered} z_{\nu j}(\zeta_{\nu j}) = Q_{\nu j} Z_{\nu j}(\zeta_{\nu j}), \qquad Z_{\nu j}(\zeta_{\nu j}) = \zeta_{\nu j} + (B_{\nu j} + A_{\nu j})/(B_{\nu j} - A_{\nu j}), \\ Q_{\nu j} = (B_{\nu j} - A_{\nu j})/2, \\ \alpha_j = -1 \leqslant \operatorname{Re} \tau_j \leqslant 1 = \beta_j, \qquad \operatorname{Im} \tau_j = 0, \qquad Z'_{\nu j}(\zeta_{\nu j}) = Z'_{\nu j}(\tau_j) = 1. \end{gathered} \tag{15}$$

The transformations $\zeta_{\nu j}(z_\nu)$ reverse to (15) satisfy conditions (12) and (13).

Given (15) and the results of [11], which lead to general expressions for the densities $\omega_\nu(t_\nu)$ near the ends of the cuts, we write representation (14) for the straight BE as a function of its position on the contour [middle BE ($\lambda = 0$), initial ($\lambda = 1$), or end ($\lambda = 2$) element of the open contour] is written as

$$\begin{gathered} \omega_{\nu j}(t_{\nu j}) = \sum_{p=1}^{q_j} c_{\nu jp} u_{jp}(\tau_j) = \sum_{k=1}^{q_j^{1\lambda}} c_{\nu jk}^{1\lambda} u_{jk}^{1\lambda}(\tau_j), \qquad q_j = q_j^{1\lambda}, \\ u_{jk}^{10}(\tau_j) = \tau_j^{k-1}, \qquad u_{jk}^{11}(\tau_j) = (1 + \tau_j)^{k-3/2}, \qquad u_{jk}^{12}(\tau_j) = u_{jk}^{11}(-\tau_j) = (1 - \tau_j)^{k-3/2}. \end{gathered} \tag{16}$$

For the indicated functions $u_{jk}^{1\lambda}(\tau_j)$, substituting $\omega_{\nu j}(t_{\nu j})$ from (16) into Eq. (9), we obtain the conditions of its precise satisfaction at all points L_{0j}:

$$c_{2jk}^{1\lambda} = -a_0 (Q_{1j}/Q_{2j}) c_{1jk}^{1\lambda} - b_0 (\bar{Q}_{1j}/Q_{2j}) \bar{c}_{1jk}^{1\lambda}, \qquad k = 1, \ldots, q_j^{1\lambda}, \quad \lambda = 0, 1, 2. \tag{17}$$

2.2. Elliptical BE

Consider the curved BE located on an elliptical arc with center at the point c_{0j} and semiaxes a_j and b_j:

$$t_{0j} = t_{0j}(\theta_j) = (a_j \cos\theta_j + i b_j \sin\theta_j)\exp i\psi_j + c_{0j},$$
$$t_{0j}(\theta_j^A) = A_{0j}, \qquad t_{0j}(\theta_j^A + \Delta\theta_j) = B_{0j}. \tag{18}$$

Here ψ_j is the angle measured counterclockwise from the positive x_0 direction to the semi-axis a_j, θ_j is a variable parameter, and θ_j^A and $\Delta\theta_j \neq 0$ are given constants.

As the mappings $z_{\nu j}(\zeta_{\nu j})$ $(\nu = 1, 2)$ we use the following mappings [9, 12]:

$$z_{\nu j}(\zeta_{\nu j}) = Q_{\nu j} Z_{\nu j}(\zeta_{\nu j}), \qquad Z_{\nu j}(\zeta_{\nu j}) = \zeta_{\nu j} + \varepsilon_{\nu j}/\zeta_{\nu j} + (\operatorname{Re} c_{0j} + \mu_\nu \operatorname{Im} c_{0j})/Q_{\nu j},$$

$$Q_{\nu j} = [(a_j - i\mu_\nu b_j)\cos\psi_j + (\mu_\nu a_j + i b_j)\sin\psi_j]\exp[i(\theta_j^A + \Delta\theta_j/2)]/2, \tag{19}$$

$$\varepsilon_{\nu j} = \frac{(a_j + i\mu_\nu b_j)\cos\psi_j + (\mu_\nu a_j - i b_j)\sin\psi_j}{(a_j - i\mu_\nu b_j)\cos\psi_j + (\mu_\nu a_j + i b_j)\sin\psi_j}\exp[-i\,(2\theta_j^A + \Delta\theta_j)],$$

$$\tau_j = \exp[i(\theta_j - \theta_j^A - \Delta\theta_j/2)], \quad \alpha_j = \exp(-i\Delta\theta_j/2), \quad \beta_j = \exp(i\Delta\theta_j/2) = \bar{\alpha}_j = \alpha_j^{-1}.$$

Transformation (19) conformally maps the unit circle and its exterior $|\zeta_{\nu j}| \geqslant 1$ $(\nu = 1, 2)$ onto the contour and the exterior of the ellipse which, in the plane z_ν, corresponds to the ellipse (18) at $0 \leqslant \theta_j < 2\pi$, and maps the ring $|\sqrt{\varepsilon_{\nu j}}\,| < |\zeta_{\nu j}| < 1$ onto the interior of this ellipse with the straight cut connecting its foci. The transformations $\zeta_{1j}(z_1)$ and $\zeta_{2j}(z_2)$, which are reverse to (19), satisfy conditions (12) and (13).

Using (19), we rewrite expression (14) for the elliptical BE in the form

$$\omega_{\nu j}(t_{\nu j}) = (1 - \varepsilon_{\nu j}\tau_j^{-2})^{-1}\Big\{c_{\nu j0}^{2\lambda} u_{j0}^{2\lambda}(\tau_j) + \sum_{k=1}^{q_j^{2\lambda}} [c_{\nu jk}^{2\lambda} u_{jk}^{2\lambda}(\tau_j) + c_{\nu j,-k}^{2\lambda} u_{j,-k}^{2\lambda}(\tau_j)]\Big\}, \qquad \lambda = 0, 1, 2. \tag{20}$$

For the indicated representation of the function $\omega_{\nu j}(t_{\nu j})$ in the form (20), to exactly satisfy Eq. (9) at all points of the BE L_{0j}, it is only sufficient that at all $k = 0, \ldots, q_j^{2\lambda}$ $(\lambda = 0, 1, 2)$, the following conditions be specified:

$$u_{j,-k}^{2\lambda}(\tau_j) = -(d\bar{\tau}_j/d\tau_j)\overline{u_{jk}^{2\lambda}(\tau_j)} = \tau_j^{-2}\,\overline{u_{jk}^{2\lambda}(\tau_j)} \qquad (\bar{\tau}_j = \tau_j^{-1}); \tag{21}$$

$$c_{2j,\pm k}^{2\lambda} = -a_0(Q_{1j}/Q_{2j})c_{1j,\pm k}^{2\lambda} + b_0(\bar{Q}_{1j}/Q_{2j})\bar{c}_{1j,\mp k}^{2\lambda}. \tag{22}$$

Given (21) and the results of [11], we write the functions $u_{j,\pm k}^{2\lambda}(\tau_j)$ as

$$u_{j,\pm k}^{20}(\tau_j) = \tau_j^{k-1}, \qquad k = 0, \ldots, q_j^{20},$$

$$u_{jk}^{21}(\tau_j) = (\alpha_j - \tau_j)^{k-3/2}, \quad u_{j,-k}^{21}(\tau_j) = \tau_j^{-2}(\alpha_j^{-1} - \tau_j^{-1})^{k-3/2}, \quad k = 1, \ldots, q_j^{21}, \qquad u_{j0}^{21}(\tau_j) = 0,$$

$$u_{j,\pm k}^{22}(\tau_j) = -\overline{u_{j,\pm k}^{21}(\tau_j^{-1})}, \quad k = 1, \ldots, q_j^{22}, \qquad u_{j0}^{22}(\tau_j) = 0.$$

It can be shown that if in an infinite plate with a single free elliptical hole, its contour consists of one or several BEs, solution (20) with the functions $u_{j,\pm 1}^{20}(\tau_j) = \tau_j^{\pm 1 - 1}$ for properly selected coefficients $c_{\nu j,\pm 1}^{20}$ corresponds [in view of (6) and (7)] to the exact solution for the potentials $\Phi_\nu(z_\nu)$ for different types of uniform loading of this plate at infinity; solution (20) with the functions of $u_{j,\pm 2}^{20}(\tau_j) = \tau_j^{\pm 2 - 1}$ for properly chosen coefficients $c_{\nu j,\pm 2}^{20}$ corresponds to the exact solution for $\Phi_\nu(z_\nu)$ for different types of loading of this plate at infinity under a linear law, etc. [8, 9].

Using the general representations (14) for $\omega_{\nu j}(t_{\nu j})$, Eqs. (8), (10), and (11) are transformed into the following system of equations for the unknown coefficients c_{1jp} and c_{2jp}:

$$\sum_{j=1}^{J}\sum_{p=1}^{q_j}\Big\{a_{0n}(\tau_n)I_{1jp}(\tau_{1jn})c_{1jp} - b_{0n}(\tau_n)\,\overline{I_{1jp}(\tau_{1jn})}\,\bar{c}_{1jp} + I_{2jp}(\tau_{2jn})c_{2jp}\Big\}$$

$$\approx -2\pi i\Big\{a_{0n}(\tau_n)\Phi_1^0(t_{1n}) + b_{0n}(\tau_n)\,\overline{\Phi_1^0(t_{1n})} + \Phi_2^0(t_{2n})\Big\}, \tag{23}$$

$$\tau_n = \zeta_{\nu n}(t_{\nu n}), \quad \tau_{\nu jn} = \zeta_{\nu j}(t_{\nu n}), \quad t_{\nu n} = \operatorname{Re} t_{0n} + \mu_\nu \operatorname{Im} t_{0n}, \quad \nu = 1, 2,$$

$$t_{0n} \in L_{0n}, \quad n = 1, \ldots, J;$$

$$\sum_{j=J_{m-1}+1}^{J_m} \sum_{p=1}^{q_j} (Q_{1j}U_{jp})c_{1jp} = 0, \qquad m = m_0, \ldots, M; \tag{24}$$

$$\operatorname{Re}\Big(\sum_{j=1}^{J}\sum_{p=1}^{q_j} I_{1jp}(z_1^{(m)})c_{1jp}\Big) = 0, \qquad m = m_0, \ldots, N. \tag{25}$$

Here

$$a_{0n}(\tau_n) = a_0\,\frac{Q_{1n}}{Q_{2n}}\,\frac{Z'_{1n}(\tau_n)}{Z'_{2n}(\tau_n)}, \qquad b_{0n}(\tau_n) = b_0\,\frac{\bar{Q}_{1n}}{Q_{2n}}\,\frac{\overline{Z'_{1n}(\tau_n)}}{Z'_{2n}(\tau_n)}\,\frac{d\bar{\tau}_n}{d\tau_n},$$
$$I_{\nu jp}(\zeta_{\nu j}) = \int\limits_{\Lambda_j} \frac{u_{jp}(\sigma_j)\,d\sigma_j}{Z_{\nu j}(\sigma_j) - Z_{\nu j}(\zeta_{\nu j})}, \quad \nu = 1, 2, \qquad U_{jp} = \int\limits_{\Lambda_j} u_{jp}(\sigma_j)\,d\sigma_j. \tag{26}$$

For finite q_j, equality (23) is understood as approximate since its exact satisfaction for all $t_{0n} \in L_{0n}$ is generally impossible. Next, the coefficients c_{2jp} in the form of $c_{2jk}^{1\lambda}$ [see (16)] or $c_{2j,\pm k}^{2\lambda}$ (see (20)) can be eliminated, in accordance with (17), (22).

In view of (9), it is sufficient that the continuity conditions for the densities $\omega_{\nu j}(B_{\nu j}) = \omega_{\nu n}(A_{\nu n})$ ($\nu = 1, 2$) at each point of joining of BEs (j and n are the numbers of the previous and next BEs at this point, respectively) be satisfied for $\nu = 1$:

$$\sum_{p=1}^{q_j} \frac{u_{jp}(\beta_j)}{Z'_{1j}(\beta_j)}\,c_{1jp} - \sum_{p=1}^{q_n} \frac{u_{np}(\alpha_n)}{Z'_{1n}(\alpha_n)}\,c_{1np} = 0. \tag{27}$$

If conditions (27) are not satisfied in the approximation of the density functions, the functions $\Phi_\nu(z_\nu)$ and stresses have logarithmic singularities at the points of joining of BEs [13]. To increase the smoothness of the solution at the joining points, it is possible to additionally specify the continuity condition for the derivatives of the densities of the corresponding orders.

3. PARAMETERS OF THE STRESS–STRAIN STATE

The result of the solution of system (23)–(27), using relations (17) and (22) are the coefficients c_{1jp} and c_{2jp}. The stresses are calculated by formulas (2), where, according to (6), (7), (14), and (26) the values of the functions $\Phi_\nu(z_\nu)$ equal

$$\Phi_\nu(z_\nu) = \frac{1}{2\pi i}\sum_{j=1}^{J}\sum_{p=1}^{q_j} c_{\nu jp} I_{\nu jp}(\zeta_{\nu j}) + \Phi_\nu^0(z_\nu), \qquad \nu = 1, 2. \tag{28}$$

At the tips of the cracks, the stress intensity factors (SIF) of the first and second kind K_1 and K_2 are calculated using (28) [3, 14].

The displacements are calculated according to (1) where

$$\varphi_\nu(z_\nu) = \int \Phi_\nu(z_\nu)\,dz_\nu = \int \Big[\frac{1}{2\pi i}\int\limits_{L_\nu} \frac{\omega_\nu(s_\nu)\,ds_\nu}{s_\nu - z_\nu} + \Phi_\nu^0(z_\nu)\Big]\,dz_\nu$$

$$= -\frac{1}{2\pi i}\sum_{m=m_0}^{M}\Big\{\Big[\ln(s_\nu - z_\nu)\int \omega_\nu(s_\nu)\,ds_\nu\Big]_{L_\nu^{(m)}} - \int\limits_{L_\nu^{(m)}}\Big[\int \omega_\nu(s_\nu)\,ds_\nu\Big]\frac{ds_\nu}{s_\nu - z_\nu}\Big\} + \int \Phi_\nu^0(z_\nu)\,dz_\nu, \qquad \nu = 1, 2.$$

Here the insignificant integration constants are omitted; the notation $[\,\cdot\,]_{L_\nu^{(m)}}$ denotes the increment in the expression in parentheses as the point s_ν moves on the contour $L_\nu^{(m)}$ from its initial point to the final one.

The indefinite integral of $\omega_\nu(s_\nu)$ will be considered as definite one with a variable upper limit over each contour $L_\nu^{(m)}$:

$$\int \omega_\nu(s_\nu)\,ds_\nu = \sum_{n=J_{m-1}+1}^{j-1} \int_{A_{\nu n}}^{B_{\nu n}} \omega_{\nu n}(t_{\nu n})\,dt_{\nu n} + \int_{A_{\nu j}}^{s_{\nu j}} \omega_{\nu j}(t_{\nu j})\,dt_{\nu j},$$
$$j = J_{m-1}+1,\ldots,J_m, \qquad m = m_0,\ldots,M. \tag{29}$$

Given (14) and (29), as a result of transformations we obtain

$$\varphi_\nu(z_\nu) = \sum_{j=1}^{J}\Big[\sum_{p=1}^{q_j} C_{\nu jp}W_{\nu jp}(\zeta_{\nu j}) + D_{\nu j}\ln\frac{Z_{\nu j}(\beta_j) - Z_{\nu j}(\zeta_{\nu j})}{Z_{\nu j}(\alpha_j) - Z_{\nu j}(\zeta_{\nu j})}\Big] + \int \Phi_\nu^0(z_\nu)\,dz_\nu; \tag{30}$$

$$C_{\nu jp} = \frac{Q_{\nu j}c_{\nu jp}}{2\pi i}, \qquad W_{\nu jp}(\zeta_{\nu j}) = \int_{\alpha_j}^{\beta_j}\Big[\int_{\alpha_j}^{\sigma_j} u_{jp}(\tau_j)\,d\tau_j\Big]\frac{Z'_{\nu j}(\sigma_j)\,d\sigma_j}{Z_{\nu j}(\sigma_j) - Z_{\nu j}(\zeta_{\nu j})}; \tag{31}$$

$$D_{\nu,J_{m-1}+1} = 0, \qquad D_{\nu j} = D_{\nu,j-1} + \sum_{p=1}^{q_j} C_{\nu,j-1,p}U_{j-1,p},$$

$$\nu = 1,2, \qquad j = J_{m-1}+2,\ldots,J_m, \qquad m = m_0,\ldots,M.$$

4. NUMERICAL METHODS

To solve system (23)–(27) (after exclusion of the factors c_{2jp}), we use the collocation method (CM) or the least-squares method (LSM). The result is a real system of linear algebraic equations for the real and imaginary parts of the coefficients c_{1jp}. The integrals in (23)–(26), (28), (30), and (31) are calculated analytically: Cauchy type integrals and special integrals (in the sense of the Cauchy principal value) are calculated using the formulas proposed in [5] and similar recurrent formulas, and the starting singular integrals are calculated through the limiting values of the corresponding starting Cauchy type integrals using the Sokhotskii–Plemelj formula. The expression $W_{\nu jp}(\zeta_{\nu j})$ in (31) for $u_{jp}(\tau_j) = u_{j0}^{20}(\tau_j)$ is not integrable in closed form; therefore, if it is necessary to calculate the displacements, the functions $u_{j0}^{20}(\tau_j)$ are initially excluded from expansion (20) [the parameter $q_j = 2q_j^{20}$ in (14) becomes even].

5. EXAMPLES OF NUMERICAL CALCULATIONS

Below we consider test problems that show the convergence and accuracy of the proposed method for various types of BEs. In problems 1–3, the plate material is orthotropic with characteristics $E_2/E_1 = 0.7619$, $G_{12}/E_1 = 0.2$, and $\nu_{12} = 0.09$ ($\mu_1 = 0.5383\,i$ and $\mu_2 = 2.1284\,i$) [1], the direction of E_1 coincides with the Ox_0 axis. Splitting into elements was conducted uniformly: in problem 1, along the length of the cut, and in problems 2 and 3, along the parameter in the ordinary parametric equations of the ellipse (elliptical arc). The collocation points in the CM (problems 1 and 3) and LSM (problems 2 and 4) were placed inside each element with some refinement near its ends to reduce the maximum error of the calculations. In the LSM, the number of collocations in the BE was assumed to be equal to twice the number of unknown complex coefficients in expansion (14).

Problems 1 and 2. We consider an infinite orthotropic plate loaded by two concentrated forces $\pm P(1+i)$, applied at the points $\pm l(1+i)$, respectively (P is the force per unit thickness of the plate). The plate is weakened by a straight crack $-l < x_0 < l$, $y_0 = 0$ (problem 1) or an elliptical hole $x_0^2/l^2 + y_0^2/(l/2)^2 < 1$ (problem 2). Tables 1 and 2 list the results of analytical calculations [9] of SIFs, stresses, displacements [in (1), $u_0 = v_0 = \omega = 0$], and

Table 1. Calculated parameters of the stress–strain state in a plate with a straight crack

Parameter	Solution [9]	Δ					
		$J=3$			$J=5$		
		$q_j=2$	$q_j=3$	$q_j=4$	$q_j=2$	$q_j=3$	$q_j=4$
$K_1\sqrt{\pi l}/P$	0.8059	−0.0197	0.0201	0.0037	0.0137	0.0064	0.0004
$K_2\sqrt{\pi l}/P$	0.6487	0.0859	0.0040	−0.0013	0.0362	−0.0001	−0.0004
$\sigma_x(l/2,0)^* \cdot 2l/P$	−1.1564	0.0240	−0.0253	−0.0056	−0.0274	−0.0062	−0.0008
$\sigma_x(0,l/2) \cdot 2l/P$	0.3486	−0.0015	0.0008	0.0001	0.0003	0.0001	$< 5\cdot 10^{-5}$
$\sigma_y(0,l/2) \cdot 2l/P$	−0.0795	−0.0079	−0.0012	−0.0001	−0.0029	−0.0003	$< 5\cdot 10^{-5}$
$\tau_{xy}(0,l/2) \cdot 2l/P$	0.4538	−0.0006	0,0007	$< 5\cdot 10^{-5}$	0,0008	$< 5\cdot 10^{-5}$	$< 5\cdot 10^{-5}$
$\langle p\rangle \cdot 2l/P$	0	0.0673	0.0168	0.0054	0.0327	0.0070	0.0009
$u(l/2,0)^* E_1/P$	0.5169	−0.0117	−0.0031	0.0002	−0.0067	−0.0002	$< 5\cdot 10^{-5}$
$v(l/2,0)^* E_1/P$	1.0132	0.0137	−0.0024	0.0006	−0.0014	$< 5\cdot 10^{-5}$	0.0001
$u(0,l/2)E_1/P$	0.5387	−0.0002	−0.0009	0.0001	−0.0009	$< 5\cdot 10^{-5}$	$< 5\cdot 10^{-5}$
$v(0,l/2)E_1/P$	0.5602	0.0114	−0.0011	−0.0001	0.0020	−0.0001	$< 5\cdot 10^{-5}$

* Values on the upper face of the crack.

Table 2. Calculated parameters of the stress–strain state in a plate with an elliptical hole

Parameter	Solution [9]	Δ					
		$J=4$			$J=8$		
		$q_j=2$	$q_j=4$	$q_j=6$	$q_j=2$	$q_j=4$	$q_j=6$
$\sigma_y(l,0) \cdot 2l/P$	3.0725	−0.4464	0.1445	0.0243	−0.0308	0.0130	−0.0003
$\sigma_x(0,l/2) \cdot 2l/P$	0.7868	−0.4903	−0.0157	0.0068	−0.1457	−0.0012	0.0001
$\sigma_x(0,l) \cdot 2l/P$	0.7552	0.0121	−0.0025	−0.0001	−0.0035	$< 5\cdot 10^{-5}$	$< 5\cdot 10^{-5}$
$\sigma_y(0,l) \cdot 2l/P$	−0.1158	−0.1547	−0.0003	0.0005	−0.0011	−0.0002	$< 5\cdot 10^{-5}$
$\tau_{xy}(0,l) \cdot 2l/P$	0.3293	−0.0098	0.0035	$< 5\cdot 10^{-5}$	0.0060	0.0004	$< 5\cdot 10^{-5}$
$\langle p\rangle \cdot 2l/P$	0	0.2430	0.0289	0.0062	0.0854	0.0056	0.0005
$u(l,0)E_1/P$	−0.0593	0.0447	−0.0089	−0.0004	−0.0065	−0.0001	$< 5\cdot 10^{-5}$
$v(l,0)E_1/P$	1.1017	0.1227	0.0024	−0.0009	0.0095	0.0001	$< 5\cdot 10^{-5}$
$u(0,l/2)E_1/P$	1.1017	0.0391	0.0033	−0.0002	0.0040	0.0003	$< 5\cdot 10^{-5}$
$v(0,l/2)E_1/P$	0.5251	0.1796	0.0021	−0.0008	−0.0031	0.0002	$< 5\cdot 10^{-5}$
$u(0,l)E_1/P$	0.8620	0.0223	0.0007	−0.0001	−0.0024	$< 5\cdot 10^{-5}$	$< 5\cdot 10^{-5}$
$v(0,l)E_1/P$	0.4942	0.1136	0.0030	−0.0004	−0.0011	0.0003	$< 5\cdot 10^{-5}$

the absolute errors Δ of these quantities in the calculation by the proposed method as functions of the number of elements J and the number of terms of the series q_j in expansion (14). The tables also give the root-mean-square values of the modulus of the total stress $\langle p\rangle$ at the free boundary, which can be treated as integral indicators of the errors of this scheme of the CBEM.

Problem 3. We consider an infinite orthotropic plate with a crack of length $2l$ located along the elliptical arc

$$t_0 = t_0(\eta) = R_1 \cos[(\eta+1)\varphi/2] + iR_2 \sin[(\eta+1)\varphi/2], \qquad -1 \leqslant \eta \leqslant 1,$$

under loading at infinity by the forces $\sigma_x^\infty = 1$ and $\sigma_y^\infty = \tau_{xy}^\infty = 0$. In Fig. 1, solid curves show the values calculated in [1] for the SIF analogs $N_1 = K_1/\sqrt{2\pi l}$ and $N_2 = K_2/\sqrt{2\pi l}$ for this problem. It is seen that the calculation results in [1] and the results of calculations made by the proposed method agree well over the entire range of the parameter φ.

Problem 4. A plate of horseshoe shape with a crack or without cracks is loaded by two pairs of forces. Figure 2 shows a plate with a crack of length $2l = 1.2L$. In the calculations, the plate material is defined as orthotropic with parameters $E_1 = E_2 = E_x = E_y$, $\nu_{12} = 0.3$, $G_{12} = 0.9999E_1/[2(1+\nu_{12})]$, $\mu_1 = 0.9920i$, and $\mu_2 = 1.0081i$ (nearly isotropic material). The calculations were carried out for two types of discretization of the boundaries: $J = 52$, $J_1 = 49$ (J_1 is the number of BEs on the external contour; in Fig. 2, the boundaries of the elements are shown by long dashes) and $J = 100$ and $J_1 = 94$ (additional boundaries of the elements are shown by short dashes). Table 3 lists the values of $\langle p\rangle$ on the free boundary and the results of calculations in three

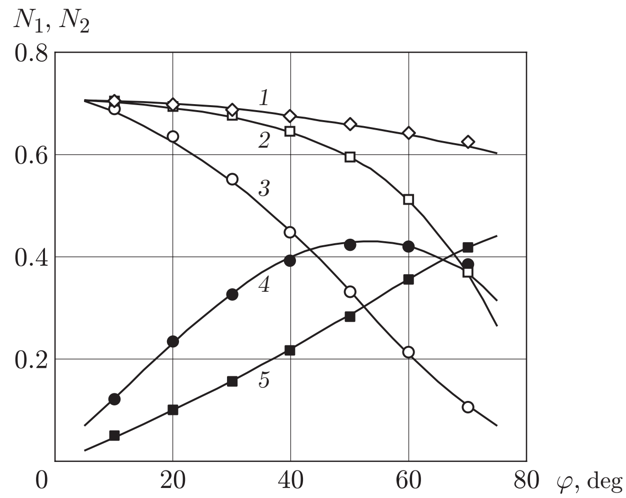

Fig. 1. SIF analogs N_1 and N_2 versus the parameter φ: the curves are the results of calculations [1], and the points are the results of calculations by the proposed method for $q_j = 4$ and $J = 6$; (*1*) values of N_1 at the starting point of the crack ($\eta = -1$) for $R_1/R_2 = 1$; (*2, 3*) the values of N_1 at the end point of the crack ($\eta = 1$) for $R_1/R_2 = 0.4$ (*2*) and 1 (*3*); (*4, 5*) the values of N_2 at the end point of the crack ($\eta = 1$) for $R_1/R_2 = 1$ (*4*) and 0.4 (*5*).

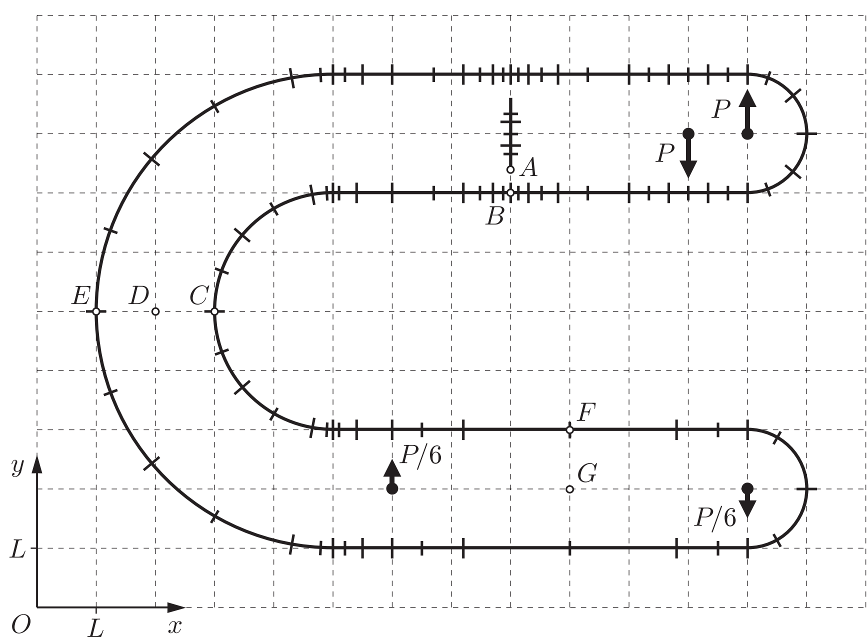

Fig. 2. Diagram of a horseshoe-shaped plate with a crack of length $2l$.

characteristic domains: the SIF at the point A in the presence of a crack [14] or the stress at the point B in the absence of cracks [15], the stresses at the points C, D, and E of a curved rod [15] and at the points F and G in the case of bending of a console loaded at the end [15].

These examples (problems 1, 2, and 4) show that with increasing number of BEs J and (or) the parameter q_j [regardless of the presence or absence of the zero term in expansion (20) for $\lambda = 0$], and also with increasing distance from the calculated point to the boundary, the error decreases.

Analysis of the deviations from zero of the values of the modulus of the total stress vector p on the free boundary of the domain D_0 and all types of stresses inside and on the boundaries of the domains $D_0^{(m)}$ ($m =$

Table 3. Calculated parameters of the stress–strain state in a horseshoe plate

Parameter	Solutions [14, 15]	Δ				
		$J = 52,\ J_1 = 49$			$J = 100,\ J_1 = 94$	
		$q_j = 3$	$q_j = 4$	$q_j = 5$	$q_j = 3$	$q_j = 4$
$K_1(A) \cdot 2L/(3P\sqrt{\pi l}\,)$	0.3196	−0.0046	−0.0005	$< 5 \cdot 10^{-5}$	0.0005	0.0001
$\sigma_x(B)^* \cdot 2L/(3P)$	1.0000	−0.0154	−0.0017	−0.0004	0.0011	−0.0002
$\sigma_y(C) \cdot 2L/(3P)$	1.2925	−0.1095	−0.0088	−0.0014	0.0009	−0.0003
$\sigma_x(D) \cdot 2L/(3P)$	0.1645	−0.0142	−0.0009	−0.0002	$< 5 \cdot 10^{-5}$	$< 5 \cdot 10^{-5}$
$\sigma_y(E) \cdot 2L/(3P)$	−0.8195	0.0587	0.0035	0.0009	−0.0008	0.0001
$\sigma_x(F) \cdot 2L/(3P)$	0.5000	−0.0120	−0.0011	−0.0002	0.0007	$< 5 \cdot 10^{-5}$
$\tau_{xy}(G) \cdot 2L/(3P)$	−0.0833	0.0029	0.0002	$< 5 \cdot 10^{-5}$	−0.0001	$< 5 \cdot 10^{-5}$
$\langle p \rangle \cdot 2L/(3P)$	0	0.0017	0.0003	0.0003	0.0003	0.0001

* Calculation in the absence of cracks.

$m_0, \ldots, N$) allow us to estimate the errors in the calculation of the stresses throughout the domain D_0, and to perform multi-stage adaptive calculations for a smaller (larger) BE size in the regions of the boundary in which the calculations of the previous stage yielded large (small) values of p [5].

Other things being equal, the calculation errors for the BEs adjacent to the point of finite discontinuity of the curvature of the boundary are substantially greater than those for the other elements. Numerical studies for simple problems have shown that as the sizes of the elements decrease in geometrical progression, the density and stress at the point of discontinuity of the curvature tend to finite values and the moduli of their derivatives along the length of the contour increase in arithmetic progression. This is indirect evidence that in the vicinity of the point of discontinuity of the curvature, there are finite asymptotic solutions for the densities and stresses with an infinite discontinuity of their first derivatives at this point. To obtain analytical expressions for this domain, more research is needed. One way to reduce the error when using the functions $u_{jp}(\tau_j)$ of the proposed form is to reduce the sizes of the elements the closest to the point of discontinuity of the curvature.

If the elastic properties of the material, the shape of the boundary, and the position of the boundary elements and the collocations on them are symmetric, there is sometimes a failure of the numerical solution, presumably caused by the occurrence of eigenfunctions due to the symmetric discretization of the continuum equation (8).This negative effect (paradox of symmetry [5]) can be solved by breaking symmetry in arranging collocation points or in dividing the contour into elements.

CONCLUSIONS

Using the ordinary and end boundary elements proposed to solve the BIEs of the plane elastic problem with approximating functions exactly satisfying the relationship of the two required densities allows the required Cauchy type integrals and special and ordinary integrals to be calculated element-wise in closed form. Analytical integration, which is more accurate and simpler than numerical integration, provides more exact solutions of the BIEs and stresses and displacements, in particular, near the boundaries of the body, i.e., near stress concentrators. The proposed method is effective in solving the plane problem for finite or infinite anisotropic elastic bodies ($\mu_1 \neq \mu_2$) and nearly isotropic ($\mu_1 \approx \mu_2 \approx i$) materials with free smooth holes and smooth cuts of arbitrary shape.

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