ISSN 0021-3640, JETP Letters, 2020, Vol. 112, No. 6, pp. 370–375. © Pleiades Publishing, Inc., 2020.
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METHODS
OF THEORETICAL PHYSICS

Mirror Pairs of Quintic Orbifolds

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Received September 3, 2020; revised September 3, 2020; accepted September 3, 2020

Two constructions of mirror pairs of Calabi–Yau manifolds are compared by example of quintic orbifolds $\mathfrak{Q}$. The first, Berglund–Hubsch–Krawitz, construction is as follows. If X is the factor of the hypersurface $\mathfrak{Q}$ by a certain subgroup H' of the maximum allowed group SL, the mirror manifold Y is defined as the factor by the dual subgroup H'^T. In the second, Batyrev, construction, the toric manifold T containing the mirror Y as a hypersurface specified by zeros of the polynomial W_Y is determined from the properties of the polynomial W_X specifying the Calabi–Yau manifold X. The polynomial W_Y is determined in an explicit form. The group of symmetry of the polynomial W_Y is found from its form and it is tested whether it coincides with that predicted by the Berglund–Hubsch–Krawitz construction.

DOI: 10.1134/S002136402018006X

Calabi–Yau manifolds appear when six of ten dimensions are compactified in the superstring theory [1], which is a way to unify the Standard Model and Quantum Gravity. The effective theory is specified in terms of the so-called special geometry arising in the moduli space [2] of Calabi–Yau manifolds and their mirrors. We study mirror pairs by example of three cases of quintic orbifolds. We consider Calabi–Yau manifolds defined as hypersurfaces in the weighted projective space

$$\mathbb{P}^4_{(k_1,k_2,k_3,k_4,k_5)} = \{(x_1,\dots,x_5)\in\mathbb{C}^5\backslash\{0\} | x_i \sim \lambda^{k_i}x_i\}. \quad (1)$$

Let M be the matrix of degrees of the quasihomogeneous polynomial $W_M(x) := \sum_{i=1}^5\prod_{j=1}^5 x_j^{M_{ij}}$, i.e.,

$$W_M(\lambda^{k_i}x) = \lambda^d W_M(x), \quad (2)$$

which specifies a hypersurface by the equation $W_M(x) = 0$. The matrix M is invertible and has a chain or loop Fermat type [3], which, together with the equality

$$d = \sum_i^5 k_i, \quad (3)$$

is the condition that the hypersurface is a Calabi–Yau manifold.

The whole family of Calabi–Yau manifolds X_M is obtained by the deformation of the initial polynomial W_M by allowed monomials:

$$W(x,\varphi) = W_M(x) + \sum_{l=1}^h \varphi_l \prod_{i=1}^5 x_i^{S_{li}}. \quad (4)$$

Here, φ_l are moduli of the complex structure of the manifold X_M.

We note that it follows from Eqs. (1) and (2) that, if

$$\lambda = \omega, \quad \omega^d = 1, \quad (5)$$

the polynomial $W_M(x)$ is invariant under the action of the group of transformations Q_M, which acts as

$$x_i \mapsto \omega^{k_i}x_i. \quad (6)$$

The group Q_M is called the quantum symmetry group. The polynomial W_M can have the maximal symmetry group $\mathrm{Aut}(M)$, $Q_M \subset \mathrm{Aut}(M)$, and its order is [3]

$$|\mathrm{Aut}(M)| = \det M. \quad (7)$$

The group $\mathrm{Aut}(M)$ is generated by the generators $q(M)_i, i = 1,\ldots,5$, which act at each coordinate x_j as

$$q_i(M) : x_j \mapsto e^{2\pi i B_{ji}} x_j, \tag{8}$$

where the matrix B is inverse to the matrix M.

A generator of the quantum group Q_M is the product $\prod_{i=1}^{5} q_i(M)$, which acts on x_i as in Eq. (6). The group $\mathrm{Aut}(M)$ has subgroups that conserve the holomorphic 3-form Ω nonvanishing on X_M or, equivalently, conserve $\prod_{i=1}^{5} x_i$. Such subgroups are called allowed. Let $SL(M)$ be the maximal allowed subgroup.

There is the chain of embeddings $Q_M \subseteq SL(M) \subseteq \mathrm{Aut}(M)$.

Now, we can define an orbifold in X_M as follows. We choose in $SL(M)$ a certain subgroup $H(M)$ that includes (or equals) Q_M. Further, the factor-group by Q_M is obtained from it as

$$H'(M) := H(M)/Q_M. \tag{9}$$

Then, the orbifold is defined as

$$Z(M, H) := X_M / H'(M). \tag{10}$$

A similar construction can be made for the matrix M^T and four groups are obtained: $Q_{M^T} \subseteq H(M^T) \subseteq SL(M^T) \subseteq \mathrm{Aut}(M^T)$, where $H(M^T)$ is a certain subgroup $SL(M^T)$, and the orbifold $Z(M^T, H^T)$.

Now, there is the question: Can these groups be chosen, and, if so, can they be chosen such that $Z(M, H)$ and $Z(M^T, H^T)$ are mirrors of each other?

The Berglund–Hubsch–Krawitz construction [4, 5] is as follows. Let the group $H(M)$ leave invariant the monomials

$$e_l(x) = \prod_{i=1}^{5} x_i^{S_{li}} \tag{11}$$

for a certain choice of S_{li}. These monomials, together with $W_M(x)$, specify the orbifold $Z(M, H)$. Then, by definition, the generators of the group $H(M^T)$ are

$$q(l) := \prod_{i=1}^{5} q_i(M^T)^{S_{li}}, \tag{12}$$

where S_{li} are the same as in Eq. (11) and the generators of the group $SL(M^T)$ are specified by their action on x_j

$$q_i(H^T) : x_j \mapsto e^{2\pi i B_{ij}} x_j. \tag{13}$$

According to Eq. (13), the generators of the group $H(M^T)$ act on the coordinates as

$$q(l) : x_j \mapsto e^{2\pi i B_{ij} S_{li}} x_j. \tag{14}$$

We now define monomials as $e_m^T := \prod_{i=1}^{5} z_i^{R_{mi}}$ from the requirement of their invariance under the action of the group $H(M^T)$.

It is convenient to reproduce this requirement in the form

$$\langle l, m\rangle := B_{ij} S_{li} R_{mj} \in \mathbb{Z}. \tag{15}$$

Below, we consider quintic orbifolds for which $B_{ij} = \delta_{ij}/5$. In this case, (15) is transformed to the condition

$$\sum_i S_{li} R_{mi} = 0 (\mathrm{mod}\ 5). \tag{16}$$

The Berglund–Hubsch–Krawitz hypothesis implies that the polynomial W_{M^T}, together with the superposition of e_m^T, specifies the mirror orbifold $Z(M^T, H^T)$.

The Chiodo–Ruan theorem [6] states the mirror symmetry of the orbifolds $Z(M, H)$ and $Z(M^T, H^T)$ in the cohomology level.

In the simplest case, we can set $H(M) = Q_M$. Then, the orbifold is the initial Calabi–Yau manifold X_M, i.e., $Z(M, Q_M) = X_M$. Then, the mirror Calabi–Yau manifold denoted as Y_M is the orbifold by the group $SL'(M^T) := SL(M^T)/Q_{M^T}$:

$$Y_M \equiv Z(M^T, SL^T) = X_{M^T}/SL'(M^T). \tag{17}$$

According to the second, Batyrev, construction of mirror pairs of Calabi–Yau manifolds [7, 8], the mirror Y of the initial orbifold X is specified as a hypersurface in the toric manifold. First, we describe the construction of the toric manifold in X. The polynomial specifying the family X_M can be represented in the form

$$\begin{aligned} W_X(x, \varphi) &= \sum_{i=1}^{5}\prod_{j=1}^{5} x_j^{M_{ij}} + \sum_{l=1}^{h_X}\varphi_l \prod_{j=1}^{5} x_j^{S_{jl}} \\ &= \sum_{a=1}^{5+h_X} C_a(\varphi)\prod_{j=1}^{5} x_j^{V_{aj}}. \end{aligned} \tag{18}$$

The degrees of the monomials of the initial polynomial (18) specifying the orbifold X are vectors in the lattice $\mathbb{Z}^5$, namely, $(\mathbf{V}_a)_j = V_{aj}$. They can also be represented in the matrix form $V^T = (M^T|S^T)$. In view of quasihomogeneity, $\sum_{j=1}^{5} V_{aj} k_j = d$. Consequently, the vectors $(\mathbf{V}_a)_j$ are located in four-dimensional lattice $\mathbb{Z}^4$

and constitute a Batyrev polygon [8]. At the same time, the same vectors $\mathbf{V}_a$ are the edges of the fan [7], which determines the toric manifold T. Since $h_X + 5$ vectors $\mathbf{V}_a$ lie in the four-dimensional space, they should satisfy h_X linear relations

$$\sum_{a=1}^{h_X+5} Q_{la}\mathbf{V}_a = 0, \quad l = 1,\dots,h_X. \tag{19}$$

These relations determine the weights of the action of the Abelian group $(\mathbb{C}^*)^{h_X}$ in $\mathbb{C}^{h_X+4}$

$$y_a \to \lambda^{Q_{la}} y_a, \quad a = 1,\dots,h_X+4, \quad l = 1,\dots,h_X, \tag{20}$$

which specifies the toric manifold

$$T := \frac{\left(\mathbb{C}^{h_X+4} - Z\right)}{(\mathbb{C}^*)^{h_X}}, \tag{21}$$

where the manifold Z is invariant under this action [7].

The next step is the construction of the hypersurface in T.

The mirror Calabi–Yau manifold Y is specified by the zeros of a certain quasihomogeneous polynomial [9–11] $W_Y(y_1,\dots,y_{h_X+4})$, i.e.,

$$Y := \{(y_1,\dots,y_{h_X+4}) \in T | W_Y(y) = 0\}. \tag{22}$$

We find the explicit form of the polynomial W_Y from the homogeneity conditions:

$$\begin{aligned} &W_Y(\lambda^{Q_{l1}}y_1,\dots,\lambda^{Q_{l,h_X+4}}y_{h_X+4}) \\ &= \lambda^d W_Y(y_1,\dots,y_{h_X+4}), \quad i = 1,\dots,h_X. \end{aligned} \tag{23}$$

We represent the polynomial W_Y in the form of some monomials $L_i(y)$, which will be found below:

$$W_Y(y) = \sum_{i=1}^{h_Y+5} \tilde{C}_i(\psi)L_i(y), \quad L_i(y) = \prod_{a=1}^{h_X+4} y_a^{n_a^i}. \tag{24}$$

From Eqs. (23) and (24), we obtain the systems of equations for the exponents n_a (for brevity, we omit the index i):

$$\sum_{a=1}^{h_X+4} Q_{la}n_a = \delta_{l,h_X}d, \quad l = 1,\dots,h_X. \tag{25}$$

We note that six solutions of the system (25) are always known. Since

$$\sum_{i=1}^{5} V_{ai}k_i = d, \quad \sum_{a=1}^{h_X+5} Q_{la}V_{ai} = 0, \quad Q_{h_X,h_X+5} = -d, \tag{26}$$

these 5 + 1 solutions of the system (25) have the form

$$n_a^i = V_{ai}, \quad i = 1,\dots,5, \tag{27}$$

$$n_a^6 = 1, \tag{28}$$

where $a = 1,\dots,h_X+4$.

The polynomial W_Y consists of only $h_Y + 5$ invariant monomials whose degrees are found from the solutions (25) with the imposition of additional conditions of nonnegativity of degrees $n_a^i \geq 0$. Here, $h_Y := h_{21}^Y$ is the Hodge number of the Calabi–Yau manifold Y. Thus, it is necessary to solve the system

$$\begin{aligned} &\sum_{a=1}^{h_X+4} Q_{la}n_a = \delta_{l,h_X}d, \quad n_a \geq 0, \\ &l = 1,\dots,h_X, \quad a = 1,\dots,h_X+4. \end{aligned} \tag{29}$$

Using the symmetry of the action of the torus, one can make a change to the following (projective) coordinates invariant under the action of $(\mathbb{C}^*)^{h_X}$:

$$z_j = \prod_{a=1}^{h_X+4} y_a^{V_{aj}/5}. \tag{30}$$

Thus, we reduce the toric manifold to the projective space $\mathbb{P}^4$. In coordinates z_j, the polynomial W_Y has the form

$$W_Y(z) = \sum_{i=1}^{5}\prod_{j=1}^{5} x_j^{M_{ji}} + \sum_{m=1}^{h_Y}\psi_m e_m^T(z). \tag{31}$$

It is seen that the mirror Y is determined by the transposed matrix M^T.

From the form of deformations $e_m^T(z)$ of the complex moduli space of the manifold Y, we find the group of symmetry of the polynomial W_Y and can verify whether it coincides with the group H^T obtained in the Krawitz procedure.

Below, we apply these two constructions to quintic orbifolds. The whole family in this case is specified by the following equation in the projective space:

$$\begin{aligned} \mathfrak{Q} = \Bigg\{&(x_1,\dots,x_5) \\ &\in \mathbb{P}^4 \Bigg| W_{\mathfrak{Q}}(x,\varphi) := W_0(x) + \sum_{s=1}^{101}\varphi_l e_l(x) = 0\Bigg\}, \end{aligned} \tag{32}$$

where

$$\begin{aligned} W_0(x) &:= x_1^5 + x_2^5 + x_3^5 + x_4^5 + x_5^5, \\ e_l &= \prod_{i=1}^{5} x_i^{S_{li}}, \quad 0 \leq S_{li} \leq 3, \end{aligned} \tag{33}$$

and φ_l are the moduli of the complex structure of the manifold $\mathfrak{Q}$.

In the case of quintic, the matrix of degrees is diagonal and is $M = 5I$, where I is the identity matrix, and $W_M = W_{M^T} \equiv W_0$. The initial polynomial $W_0(x)$ has the quantum symmetry group $Q_{5I} = \mathbb{Z}_5[1,1,1,1,1]$. The full symmetry group of the monomial is $\text{Aut}(5I) = \mathbb{Z}_5^5$

and its generators act as $q_i(5I): x_i \mapsto e^{2\pi i/d} x_i$. The group conserving the product $\prod_{i=1}^{5} x_i$ has the form

$$SL(5I) = \mathbb{Z}_5^4 = \mathbb{Z}_5[1,0,0,0,4] \oplus \mathbb{Z}_5[0,1,0,0,4] \oplus \mathbb{Z}_5[0,0,1,0,4] \oplus \mathbb{Z}_5[0,0,0,1,4]. \quad (34)$$

The group $SL(5I)$ is sometimes called the maximal allowed group of the polynomial W_0. It contains the diagonal $Q_{5I} = \mathbb{Z}_5[1,1,1,1,1]$, and the factor group is

$$SL'(5I) := SL(5I)/Q_{5I} = \mathbb{Z}_5[0,1,0,0,4] \oplus \mathbb{Z}_5[0,0,1,0,4] \oplus \mathbb{Z}_5[0,0,0,1,4]. \quad (35)$$

We deal with quintic orbifolds $X = \mathcal{Q}/H'$, which are specified by the following equation in $\mathbb{P}^4$:

$$W_X(x,\varphi) := x_1^5 + x_2^5 + x_3^5 + x_4^5 + x_5^5 + \sum_{l=1}^{h_X} \varphi_l e_l(x) = 0. \quad (36)$$

Here, $h_X := h_{21}^X$ is the Hodge number. The monomials e_l are invariant under the action of the group H. According to the procedure described above, we construct the mirror Calabi–Yau orbifold

$$Y := \mathcal{Q}/H'^T, \quad (37)$$

or, in terms of the initial polynomial,

$$W_Y(z,\varphi) := z_1^5 + z_2^5 + z_3^5 + z_4^5 + z_5^5 + \sum_{m=1}^{h_Y} \psi_m e_m^T(z) = 0. \quad (38)$$

The monomials $e_m^T(z)$ are in turn invariant under the action of the group H^T.

We now perform calculations for three orbifolds.

Example 1. As the first example, we consider the orbifold [12]

$$X = \frac{\mathcal{Q}}{\mathbb{Z}_5[0,1,1,4,4]}, \quad h_X = 17, \quad (39)$$

which is specified by the zeros of the polynomial

$$W_X(x) = x_1^5 + x_2^5 + x_3^5 + x_4^5 + x_5^5 + \sum_{l=1}^{17} \varphi_l e_l(x) = \sum_{a=1}^{22} C_a \prod_{j=1}^{5} x_j^{V_{aj}}. \quad (40)$$

Here, $e_l(x) := \prod_{i=1}^{5} x_i^{S_{li}}$ are invariant under the action of $H = \mathbb{Z}_5[1,1,1,1,1] \oplus \mathbb{Z}_5[0,1,1,4,4]$ and have the form

$$\begin{gathered} e_1 = x_2^3 x_3^2, \quad e_2 = x_2^2 x_3^3, \quad e_3 = x_4^2 x_5^3, \\ e_4 = x_4^3 x_5^2, \quad e_5 = x_1^3 x_2 x_4, \quad e_6 = x_1^3 x_3 x_4, \\ e_7 = x_1^3 x_3 x_5, \quad e_8 = x_1^3 x_2 x_5, \quad e_9 = x_1 x_2^2 x_4^2, \\ e_{10} = x_1 x_3^2 x_5^2, \quad e_{11} = x_1 x_3^2 x_4^2, \quad e_{12} = x_1 x_2^2 x_5^2, \\ e_{13} = x_1 x_2 x_3 x_4^2, \quad e_{14} = x_1 x_2 x_3 x_5^2, \\ e_{15} = x_1 x_2^2 x_4 x_5, \quad e_{16} = x_1 x_3^2 x_4 x_5, \\ e_{17} = x_1 x_2 x_3 x_4 x_5. \end{gathered} \quad (41)$$

According to the Berglund–Hubsch–Krawitz construction, the mirror Calabi–Yau manifold is constructed as the orbifold $Y = \mathcal{Q}/H'^T$. The group H^T is specified by the generators $q(l) = \prod_{i=1}^{5} q_i(5I)^{S_{li}}$, where S_{li} are the degrees of monomials $e_l(x)$ in Eq. (41), and $q_i(5I)$ generate the groups $\mathbb{Z}_5[1,0,0,0,0], \ldots, \mathbb{Z}_5[0,0,0,0,1]$. In other words, the group H'^T is generated by additive groups $\mathbb{Z}_5[S_{l1}, S_{l2}, S_{l3}, S_{l4}, S_{l5}]$ in the modulus of the action $\mathbb{Z}_5[1,1,1,1,1]$. In this case, the group H'^T has two generators and it has the form

$$H'^T = \mathbb{Z}_5[0,1,4,0,0] \oplus \mathbb{Z}_5[0,0,0,1,4]. \quad (42)$$

The explicit form of monomials $e_m(z) = \prod_{i=1}^{5} z_i^{R_{mi}}$ in Eqs. (44) can be alternatively obtained by solving Eq. (16), i.e.,

$$\sum_i S_{li} R_{mi} = 0 (\text{mod } 5). \quad (43)$$

The solutions of Eqs. (43) give the degrees of monomials and have the form

$$\begin{gathered} e_1^T(z) = z_1 z_2^2 z_3^2, \quad e_2^T(z) = z_1 z_4^2 z_5^2, \\ e_3^T(z) = z_1^3 z_4 z_5, \quad e_4^T(z) = z_1^3 z_2 z_3, \\ e_5^T(z) = z_1 z_2 z_3 z_4 z_5. \end{gathered} \quad (44)$$

Then, the family of the mirror orbifold

$$Y = \frac{\mathcal{Q}}{\mathbb{Z}_5[0,1,4,0,0] \oplus \mathbb{Z}_5[0,0,0,1,4]}, \quad h_Y = 5 \quad (45)$$

is specified as the solutions of the equation

$$W_Y(z) = z_1^5 + z_2^5 + z_3^5 + z_4^5 + z_5^5 + \sum_{m=1}^{5} \psi_m e_m^T(z) = 0. \quad (46)$$

We now apply the Batyrev approach and represent the polynomial W_X in the form

$$W_X(x,\varphi) = \sum_{a=1}^{22} C_a(\varphi) \prod_{j=1}^{5} x_j^{V_{aj}}. \quad (47)$$

The weights of the action of the torus are determined from the vectors $\mathbf{V}_a$. We write them in the matrix form

$$(V)_{aj} := V = \begin{pmatrix} 5I_{5\times5} \\ S \end{pmatrix}, \quad (Q)_{la} := Q = \left(S \mid -5I_{17\times17}\right), \tag{48}$$

where $S \in \mathrm{Mat}(\mathbb{Z})_{17\times5}$ is determined from Eq. (40). Using these weights Q_{la}, we obtain the toric manifold

$$T = \frac{\mathbb{C}^{21} - Z}{\mathbb{C}^{*17}} \tag{49}$$

and determine the polynomial specifying the mirror orbifold Y

$$W_Y(y) = \sum_{i=1}^{h_Y+5} \tilde{C}_i \prod_{a=1}^{21} y_a^{n_a^i}. \tag{50}$$

Imposing the necessary conditions (23), we arrive at the system of equations

$$\sum_{a=1}^{21} Q_{la} n_a = 5, \quad n_b \geq 0, \quad b = 1,\ldots,21. \tag{51}$$

Solving Eqs. (51), we obtain the sets n_a and change the coordinates in T, reducing it to $\mathbb{P}^4$. The polynomial specifying the orbifold Y has the form

$$W_Y(z) = z_1^5 + z_2^5 + z_3^5 + z_4^5 + z_5^5 + \sum_{m=1}^{5} \psi_m e_m^T(z), \tag{52}$$

where the monomials $e_m^T(z)$ are specified by Eqs. (44).

Thus, the two constructions give the same results.

Example 2. We now consider the orbifold

$$X = \frac{\mathcal{Q}}{\mathbb{Z}_5[0,1,2,3,4]}, \quad h_X = 21. \tag{53}$$

The Calabi–Yau manifolds of X are specified by the zeros of the polynomial

$$W_X(x) = x_1^5 + x_2^5 + x_3^5 + x_4^5 + x_5^5 + \sum_{l=1}^{21} \varphi_l e_l(x) = \sum_{a=1}^{26} C_a \prod_{j=1}^{5} x_j^{V_{aj}}, \tag{54}$$

and the deformations of the polynomial $e_l(x)$ are invariant under the action of the group $H = \mathbb{Z}_5[1,1,1,1,1] \oplus \mathbb{Z}_5[0,1,2,3,4]$:

$$e_1 = x_1^3x_3x_4, \quad e_2 = x_1^3x_2x_5, \quad e_3 = x_1^2x_2x_3^2,$$
$$e_4 = x_1^2x_2^2x_4, \quad e_5 = x_1^2x_3x_5^2, \quad e_6 = x_1^2x_4^2x_5,$$
$$e_7 = x_1x_2x_4^3, \quad e_8 = x_1x_2^2x_5^2, \quad e_9 = x_1x_4x_5^3,$$

$$e_{10} = x_1x_3^3x_5, \quad e_{11} = x_1x_2^3x_3, \quad e_{12} = x_1x_3^2x_4^2, \quad e_{13} = x_3^2x_4x_5^2, \quad e_{14} = x_2x_3^3x_4, \quad e_{15} = x_2x_3x_5^3, \tag{55}$$

$$e_{16} = x_2x_4^2x_5^2, \quad e_{17} = x_2^2x_3x_4^2, \quad e_{18} = x_2^2x_3^2x_5,$$
$$e_{19} = x_2^3x_4x_5, \quad e_{20} = x_3x_4^3x_5,$$
$$e_{21} = x_1x_2x_3x_4x_5.$$

Similar to the preceding example, the Berglund–Hubsch–Krawitz construction gives the mirror orbifold $Y = \mathcal{Q}/H'^T$, where

$$H'^T = \mathbb{Z}_5[0,1,3,1,0] \oplus \mathbb{Z}_5[0,1,1,0,3]. \tag{56}$$

The only monomial invariant under the action of H^T is $\prod_i z_i$. Then, the mirror Calabi–Yau orbifold

$$Y = \frac{\mathcal{Q}}{\mathbb{Z}_5[0,1,3,1,0] \oplus \mathbb{Z}_5[0,1,1,0,3]}, \quad h_Y = 1 \tag{57}$$

is specified by the zeros of the polynomial

$$W_Y(z) = z_1^5 + z_2^5 + z_3^5 + z_4^5 + z_5^5 + \psi z_1z_2z_3z_4z_5. \tag{58}$$

We now apply the Batyrev construction. The set of vectors $\mathbf{V}_a$, $a = 1,\ldots,26$, and integers Q_{la}, $l = 1,\ldots,21$, are determined similarly. The mirror of Y is specified by the zeros of the polynomial W_Y in the toric manifold

$$T = \frac{\mathbb{C}^{25} - Z}{\mathbb{C}^{*21}}. \tag{59}$$

Imposing similar constraints on the degrees of monomials of the polynomial W_Y and making a change to the projective coordinates, we arrive at its explicit form that coincides with Eq. (58).

Example 3. We consider now the orbifold

$$X = \frac{\mathcal{Q}}{\mathbb{Z}_5[0,0,0,1,4]}, \quad h_X = 25. \tag{60}$$

The whole family of the orbifold of the Calabi–Yau manifold X is specified by the equation

$$W_X = x_1^5 + x_2^5 + x_3^5 + x_4^5 + x_5^5 + \sum_{l=1}^{25} \varphi_l e_l(x) = \sum_{a=1}^{30} C_a \prod_{j=1}^{5} x_j^{V_{aj}}, \tag{61}$$

where the monomials e_l are invariant under the action of $H = \mathbb{Z}_5[1,1,1,1,1] \oplus \mathbb{Z}_5[0,0,0,1,4]$:

$$e_1 = x_1^3x_2^2, \quad e_2 = x_1^3x_3^2, \quad e_3 = x_2^3x_3^2,$$
$$e_4 = x_1^2x_2^3, \quad e_5 = x_1^2x_3^3, \quad e_6 = x_2^2x_3^3,$$
$$e_7 = x_1^3x_2x_3, \quad e_8 = x_1x_2^3x_3, \quad e_9 = x_1x_2x_3^3,$$
$$e_{10} = x_1^2x_2^2x_3, \quad e_{11} = x_1^2x_2x_3^2, \quad e_{12} = x_1x_2^2x_3^2,$$

$$e_{13} = x_1^3x_4x_5, \quad e_{14} = x_2^3x_4x_5, \quad e_{15} = x_3^3x_4x_5, \quad e_{16} = x_1^2x_2x_4x_5, \quad e_{17} = x_1^2x_3x_4x_5, \tag{62}$$

$$e_{18} = x_2^2x_3x_4x_5, \quad e_{19} = x_1x_2^2x_4x_5,$$
$$e_{20} = x_1x_3^2x_4x_5, \quad e_{21} = x_2x_3^2x_4x_5,$$
$$e_{22} = x_1x_4^2x_5^2, \quad e_{23} = x_2x_4^2x_5^2,$$
$$e_{24} = x_3x_4^2x_5^2, \quad e_{25} = x_1x_2x_3x_4x_5.$$

In [12], the total Hodge number is

$$h_X^{\text{Full}} = h_X + h_X^{\text{Laurent}} = 49. \tag{63}$$

Thus, there are additional 24 Laurent monomials among the deformations of the initial polynomial W_0:

$$\begin{aligned}
&g_1 = x_1^{-1}x_2x_3^3x_4x_5, \quad g_2 = x_1x_2^{-1}x_3^3x_4x_5,\\
&g_3 = x_1^3x_2x_3^{-1}x_4x_5, \quad g_4 = x_1^{-1}x_2^2x_3^2x_4x_5,\\
&g_5 = x_1^2x_2^{-1}x_3^2x_4x_5, \quad g_6 = x_1^2x_2^2x_3^{-1}x_4x_5,\\
&g_7 = x_1^{-1}x_2^3x_3x_4x_5, \quad g_8 = x_1x_2^3x_3^{-1}x_4x_5,\\
&g_9 = x_1^3x_2^{-1}x_3x_4x_5, \quad g_{10} = x_1^{-1}x_3^2x_4^2x_5^2,\\
&g_{11} = x_2^{-1}x_3^2x_4^2x_5^2, \quad g_{12} = x_1^2x_3^{-1}x_4^2x_5^2,\\
&g_{13} = x_1^{-1}x_2x_3x_4^2x_5^2, \quad g_{14} = x_1x_2^{-1}x_3x_4^2x_5^2,\\
&g_{15} = x_1x_2x_3^{-1}x_4^2x_5^2, \quad g_{16} = x_1^{-1}x_2^2x_4^2x_5^2,\\
&g_{17} = x_2^2x_3^{-1}x_4^2x_5^2, \quad g_{18} = x_1^2x_2^{-1}x_4^2x_5^2,\\
&g_{19} = x_1^{-1}x_4^3x_5^3, \quad g_{20} = x_2^{-1}x_4^3x_5^3,\\
&g_{21} = x_3^{-1}x_4^3x_5^3, \quad g_{22} = x_1^{-1}x_2^3x_3^3,\\
&g_{23} = x_1^3x_2^{-1}x_3^3, \quad g_{24} = x_1^3x_2^3x_3^{-1}.
\end{aligned} \tag{64}$$

According to the Berglund–Hubsch–Krawitz construction, the mirror Calabi–Yau manifold is

$$Y = \frac{\mathfrak{Q}}{\mathbb{Z}_5[0,1,4,0,0] \oplus \mathbb{Z}_5[0,3,0,1,1]}, \quad h_Y = 5, \tag{65}$$

and is specified by zeros of the polynomial

$$\begin{aligned} W_Y = \sum_{j=1}^{5} z_j^5 + \psi_1 z_4^2 z_5^3 + \psi_2 z_4^3 z_5^2 + \psi_3 z_1 z_2 z_3 z_5^2 \\ + \psi_4 z_1 z_2 z_3 z_4^2 + \psi_5 z_1 z_2 z_3 z_4 z_5. \end{aligned} \tag{66}$$

In the Batyrev construction, the mirror orbifold is realized as a hypersurface in the manifold

$$T = \frac{\mathbb{C}^{29} - Z}{\mathbb{C}^{*25}}. \tag{67}$$

Calculations similar to the above give the explicit form of the polynomial in the projective coordinates, which coincides with Eq. (66).

Calculations also give two additional monomials $z_4z_5^4$ and $z_4^4z_5$, but they lie in the kernel of the Milnor ring $\mathbb{C}[z_1,\ldots,z_5]\Big/\frac{\partial W_0}{\partial z_i}$. The two constructions give the same results.

To summarize, we have shown that the two constructions of mirror pairs give the same results for the case of quintic orbifolds. It is interesting to understand the relation between these constructions in the general case.

ACKNOWLEDGMENTS

We are grateful to A. Litvinov for stimulating discussions.

FUNDING

This work was performed at the Landau Institute for Theoretical Physics, Russian Academy of Sciences. A. Belavin acknowledges the support of the Ministry of Science and Higher Education of the Russian Federation (state assignment no. 0033-2019-0004). B. Eremin acknowledges the support of the Russian Science Foundation (project no. 18-12-00439).

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Translated by R. Tyapaev