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Higher-Order Contributions to QCD Amplitudes in Regge Kinematics (Scientific Summary)

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The famous Balitsky–Fadin–Kuraev–Lipatov (BFKL) equation was derived using the hypothesis that amplitudes of non-abelian gauge theories with the adjoint representation of the gauge group in cross-channels are given by the Reggeized gauge boson contributions. The hypothesis is true in the leading logarithmic approximation, wherein the equation was originally derived, and in the next-to-leading one. However, in the next-to-next-to-leading logarithmic approximation this is not so, since in this approximation the Regge cuts begin to contribute. Calculations of their contributions to elastic scattering amplitudes in quantum chromodynamics and their role in derivation of the BFKL equation are discussed.

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1. INTRODUCTION

The Balitsky–Fadin–Kuraev–Lipatov (BFKL) equation [1–4], which is one of the fundamental equations in quantum chromodynamics (QCD), is based on gluon Reggeization, a remarkable property of QCD. Quantum chromodynamics turned out to be a unique field theory in which all elementary particles, both quark and gluon [5–9] are Reggeized.

Reggeization of elementary particles is very important for the theoretical description of high-energy processes. The gluon Reggeization is especially important because it determines the high-energy behavior of non-decreasing with energy cross sections in perturbative QCD.

In the leading logarithmic approximation (LLA), when in each order of perturbation theory only terms with the highest powers of the logarithm of the center-of-mass (c.m.) energy $\sqrt{s}$ are retained, and in the next-to-leading one (NLLA), where terms with less by one than the leading degrees of $\ln s$ are also kept, the gluon Reggeization means that amplitudes with adjoint representation of the color group in cross-channels and negative signature (symmetry with respect to the replacement $s \leftrightarrow u \simeq -s$) are determined by contributions of the gluon pole Regge and have a simple factorized form (pole Regge form). This applies not only to elastic amplitudes, but also to amplitudes in the multi-Regge kinematics (MRK), in which all particles have fixed (not growing with s) transverse momenta and are combined into jets with limited invariant mass of each jet and large (growing with s) invariant masses of any pair of the jets. The gluon Reggeization allows to express an infinite number of such amplitudes through several Reggeon vertices and the Reggeized gluon trajectory. Since they give dominant contributions to discontinuities of amplitudes with fixed momentum transfer in the unitarity relations, it provides a simple derivation of the BFKL equation not only in the LLA, but also in the NLLA.

Now the pole Regge form is proved in all orders of perturbation theory both in the LLA [10], and in the NLLA (see [11, 12] and references therein).

However, it is violated in the NNLLA. First, the violation was observed [13] in the high-energy limit of the two-loop amplitudes for gg, gq, and qq scattering. Later infrared singular terms breaking the pole Regge form were obtained in three loops using the infrared factorization techniques [14–16].

The violation of the pole Regge form should have been expected, because it is well known that Regge poles in the complex angular momenta plane generate Regge cuts. Moreover, in amplitudes with positive signature the Regge cuts appear already in the LLA. In particular, the BFKL Pomeron is the two-Reggeon cut. However, in amplitudes with negative signature Regge cuts must be at least three-Reggeon ones and can appear only in the NNLLA. Therefore, it was natural to expect that the observed violation be due to the cut contributions.

The first explanation of the observed violation was done in [17]. It was shown that the terms violating the pole Regge form can be given by the three-Reggeon cut contributions. However, almost at the same time another explanation was done [18]. The cut contribution there is different from [17] (see also [19, 20]), and besides the cut, the pole-cut mixing is used.

Here we present the results of the calculation of the terms that violate the pole form and their explanation in both approaches.

2. POLE REGGE FORM OF QCD AMPLITUDES AND ITS VIOLATION

For elastic scattering processes $A + B \to A' + B'$ in the Regge kinematic region: $s \simeq -u \to \infty$, t is fixed (i.e., not growing with s) the Reggeization means that scattering amplitudes with the gluon quantum numbers in the t-channel and negative signature are written as

$$\mathcal{A}_{AB}^{A'B'} = \mathcal{A}_{AB}^{R}(s,t) = \Gamma_{A'A}^{c}\left[\left(\frac{-s}{-t}\right)^{j(t)} - \left(\frac{s}{-t}\right)^{j(t)}\right]\Gamma_{B'B}^{c}, \quad (1)$$

where $\Gamma_{P'P}^{c}$ are particle–particle–Reggeon (PPR) vertices or scattering vertices, the superscript c specifies the Reggeon color, and $j(t) = 1 + \omega(t)$ is the Reggeon trajectory.

The important property of Regge poles is the factorization of their contributions

$$\mathcal{A}_{gg}^{g'g'}\mathcal{A}_{qq}^{q'q'} = \left(\mathcal{A}_{gq}^{g'q'}\right)^2, \quad (2)$$

which is fulfilled because three amplitudes are expressed in terms of two Reggeon–particle–particle vertices.

The first observation of the violation of this form in the NNLLA was done [13] when comparing the two-loop gg, gq, and qq scattering amplitudes in the high-energy limit. The first terms of the NNLLA are non-logarithmic two-loop terms. It was found in [13] that the restrictions imposed on them by the factorization condition (2) are not fulfilled.

Consideration of the violation in three loops was performed only for infrared singular terms using the infrared factorization techniques [14–16]. The violation by two-loop non-logarithmic terms was confirmed and infrared singular single-logarithmic terms at three loops violating the pole Regge form were found.

For comparison of Regge and infrared factorizations, a non-factorizing remainder function was introduced and the scattering amplitudes with adjoint representation of the color group in the t-channel and negative signature were written as

$$\mathcal{A}_{AB}^{A'B'} = \mathcal{A}_{AB}^{R}(s,t) + \Gamma_{A'A}^{(0)c}\frac{s}{t}\Gamma_{B'B}^{(0)c}\mathcal{R}_{AB}, \quad (3)$$

where $\mathcal{A}_{AB}^{R}(s,t)$ is defined in (1), the superscript (0) denotes the lowest order and $\mathcal{R}_{AB}$ is the remainder function. With the three-loop NNLLA accuracy it can be presented as

$$\mathcal{R}_{AB} = \left(\frac{\alpha_s}{\pi}\right)^2\left[R_{AB}^{(0)} + \left(\frac{\alpha_s}{\pi}\right)R_{AB}^{(1)}\ln s\right]. \quad (4)$$

For the two-loop contribution, using [13], one has:

$$R_{qq}^{(0)} = \frac{\pi^2}{4\epsilon^2}\left(1 - \frac{3}{N_c^2}\right)(1 - \epsilon^2\zeta(2)), \quad (5)$$

$$R_{gg}^{(0)} = -\frac{3\pi^2}{2\epsilon^2}(1 - \epsilon^2\zeta(2)), \quad (6)$$

$$R_{qg}^{(0)} = -\frac{\pi^2}{4\epsilon^2}(1 - \epsilon^2\zeta(2)), \quad (7)$$

where $\epsilon = (D-4)/2$, D is the space-time dimension. In $R_{AB}^{(0)}$, only terms vanishing at $\epsilon \to 0$ are omitted. The values $R_{AB}^{(1)}$ were obtained using the infrared factorization, so that the terms of zero order in ϵ are also omitted:

$$R_{qq}^{(1)} = -\left(\frac{\alpha_S}{\pi}\right)^3\frac{\pi^2}{\epsilon^3}\frac{2N_c^2 - 5}{12N_c}\left(1 - \frac{3}{2}\epsilon^2\zeta(2)\right) + \mathcal{O}(\epsilon^0), \quad (8)$$

$$R_{gg}^{(1)} = \left(\frac{\alpha_S}{\pi}\right)^3\frac{\pi^2}{\epsilon^3}\frac{2}{3}N_c\left(1 - \frac{3}{2}\epsilon^2\zeta(2)\right) + \mathcal{O}(\epsilon^0), \quad (9)$$

$$R_{qg}^{(1)} = \left(\frac{\alpha_S}{\pi}\right)^3\frac{\pi^2}{\epsilon^3}\frac{N_c}{24}\left(1 - \frac{3}{2}\epsilon^2\zeta(2)\right) + \mathcal{O}(\epsilon^0). \quad (10)$$

It is necessary to say that the three-loop results (8)–(10) were obtained using so called dipole form [21–23] of the soft anomalous dimension matrix. It turns out that this form is correct only up to two loops. Recently the "quadrupole correction" appearing in three loops was calculated [24]. However, in the NNLLA this correction turns out to contribute only to amplitudes with positive signature [18], so that it does not change the results (8)–(10).

3. REGGE CUT CONTRIBUTIONS

The nonzero remainder function $\mathcal{R}_{AB}$ was explained using the three-Reggeon cuts [17, 18]. It could be considered as a good new if the explanations were the same or at least similar. Unfortunately, it is not so. The differences in the explanations start from the approaches used. The approach used in [17] (see also [19, 20]) can be called diagrammatic, since it starts from Feynman diagrams. Contrary, the approach used in [18] is not related to Feynman diagrams and is based on Wilson line representation of high-energy scattering amplitudes. Both approaches explain the violation of the pole form in three loops but in different ways.

3.1. Diagrammatic Approach

3.1.1. Appearance of the cut. Due to the signature conservation, the cut with negative signature must be the three-Reggeon one. Since QCD Reggeon is the Reggeized gluon, the three-Reggeon cut first contributes to amplitudes corresponding to the Feynman diagrams with three gluons in the t-channels, which differ in permutations σ (taking values a, b, c, d, e, f) of the gluon vertices. The amplitudes $\mathcal{A}_{AB}^{A'B'}$ can be written as the sum over the permutations of products of color factors and color-independent matrix elements:

$$\mathcal{A}_{AB}^{A'B'} = (\chi_{A'}^{*})_{\alpha'}(\chi_{B'}^{*})_{\beta'} \times \sum_{\sigma}\left(C_{AB}^{(0)\sigma}\right)_{\alpha\beta}^{\alpha'\beta'}(\chi_A)^{\alpha}(\chi_B)^{\beta}M_{AB}^{(0)\sigma}(s,t), \tag{11}$$

where χ denote the color parts of the wavefunctions, α and β (α' and β') are color indices of incoming (outgoing) projectile A and target B, respectively.

Here the same letters are used for quark and gluon color indices; it should be remembered, however, that there is no difference between upper and lower indices (running from 1 to $N_c^2 - 1$) for gluons, whereas for quarks lower and upper indices (running from 1 to N_c) refer to mutually related representations.

The color factors are

$$\left(C_{AB}^{(0)\sigma}\right)_{\alpha\beta}^{\alpha'\beta'} = \left(\mathcal{T}_A^{c_1}\mathcal{T}_A^{c_2}\mathcal{T}_A^{c_3}\right)_{\alpha}^{\alpha'}\left(\mathcal{T}_B^{c_1^\sigma}\mathcal{T}_B^{c_2^\sigma}\mathcal{T}_B^{c_3^\sigma}\right)_{\beta}^{\beta'}, \tag{12}$$

where $\mathcal{T}^a$ are the color group generators in corresponding representations, $[\mathcal{T}^a, \mathcal{T}^b] = if_{abc}\mathcal{T}^c$; for all representations, $(\mathcal{T}^a)_c^b = -if_{abc}$ for gluons, $(\mathcal{T}^a)_{\alpha}^{\alpha'} = (t^a)_{\alpha}^{\alpha'}$ for quarks; $\mathrm{Tr}(\mathcal{T}^a_P\mathcal{T}^b_P) = \mathcal{T}_P\delta_{ab}$, $T_q = 1/2$, $T_g = N_c$.

The color factors can be decomposed into irreducible representations $\mathbf{R}$ of the color group in the t-channel:

$$\left(C_{AB}^{(0)\sigma}\right)_{\alpha\beta}^{\alpha'\beta'} = \sum_{\mathbf{R}}\left[\mathcal{P}_{AB}^{\mathbf{R}}\right]_{\alpha\beta}^{\alpha'\beta'}\mathcal{G}(\mathbf{R})_{AB}^{(0)\sigma}, \tag{13}$$

where

$$\left[\mathcal{P}_{AB}^{\mathbf{R}}\right]_{\alpha\beta}^{\alpha'\beta'} = \sum_{\mathbf{R}}\left[\mathcal{P}_{AB}^{\mathbf{R},n}\right]_{\alpha}^{\alpha'}\left[\mathcal{P}_{B}^{\mathbf{R},n*}\right]_{\beta'}^{\beta}, \tag{14}$$

$\hat{P}^{\mathbf{R},n}$ is the wavefunction on the state n in the representation $\mathbf{R}$ in the color index space with the normalization

$$[\mathcal{P}_P^{\mathbf{R},n}]_{\beta}^{\alpha}[\mathcal{P}_P^{\mathbf{R}',n'*}]_{\beta}^{\alpha} = T_P\delta_{\mathbf{R},\mathbf{R}'}\delta_{n,n'}, \tag{15}$$

so that

$$\mathcal{G}(\mathbf{R})_{AB}^{(0)\sigma} = \frac{1}{N_{\mathbf{R}}T_AT_B}\left(\mathcal{T}_A^{c_1}\mathcal{T}_A^{c_2}\mathcal{T}_A^{c_3}\right)_{\alpha}^{\alpha'} \times\left(\mathcal{T}_B^{c_1^\sigma}\mathcal{T}_B^{c_2^\sigma}\mathcal{T}_B^{c_3^\sigma}\right)_{\beta}^{\beta'}\left[\mathcal{P}_{AB}^{\mathbf{R}*}\right]_{\alpha\beta}^{\alpha'\beta'}, \tag{16}$$

$N_{\mathbf{R}}$ is the dimension of the representation $\mathbf{R}$.

In contrast to the Reggeon, which contributes only to amplitudes with the adjoint representation of the color group (color octet in QCD) in the t-channel, the cut can contribute to various representations.

Possible representations for quark–quark and quark–gluon scattering are only singlet (**1**) and octet (**8**), whereas for the gluon–gluon scattering there are singlet (**1**), symmetric $\mathbf{8_s}$ and antisymmetric $\mathbf{8_a}$ octet, **10**, **10***, and **27**. Accounting of Bose statistic of gluons and symmetry of the representations **1**, $\mathbf{8_s}$, and **27** gives that possible representations in amplitudes with negative signature are $\mathbf{8_a}$, **10**, and **10*** for gluon–gluon scattering, **1** and **8** for quark–quark scattering and only **8** for the quark–gluon scattering. It makes sense to say that for $N_c > 3$ the situation essentially is not changed because only extra symmetric representation appears for two-gluon system.

The important thing is the existence itself of the representations of the color group other than the adjoint one in amplitudes with negative signature, which means existence of singularities different from the Regge pole in the complex angular momentum plane.

The projection operators for the octet channel are (below, we omit the subscript a in $\mathbf{8_a}$):

$$\left[\mathcal{P}_{gg}^{\mathbf{8}}\right]_{a'b'}^{ab} = -f_{aa'c}f_{bb'c} \tag{17}$$

for gluon–gluon scattering,

$$\left[\mathcal{P}_{gq}^{\mathbf{8}}\right]_{ab}^{a'b'} = -if_{aa'}^{c}(t^c)_{\beta}^{\beta'} \tag{18}$$

for gluon–quark scattering, and

$$\left[\mathcal{P}_{qq}^{\mathbf{8}}\right]_{ab}^{a'b'} = (t^c)_{\alpha}^{\alpha'}(t^c)_{\beta}^{\beta'} \tag{19}$$

for quark–quark scattering.

The channels **10** and **10*** exist only for gluon–gluon scattering. The projection operators are

$$\left[\mathcal{P}_{gg}^{\mathbf{10}}\right]_{a'b''}^{ab} = \frac{N_c}{4}\Bigg(\delta_{ab}\delta_{a'b'} - \delta_{ab'}\delta_{a'b} - 2\frac{f_{aa'c}f_{bb'c}}{N_C} + if_{ba'c}d_{b'ac} + id_{ba'c}f_{b'ac}\Bigg), \tag{20}$$

$$\left[\mathcal{P}_{gg}^{\mathbf{10}}\right]_{a'b''}^{ab} = \left(\left[\mathcal{P}_{gg}^{\mathbf{10}}\right]_{a'b''}^{ab}\right)^{*}. \tag{21}$$

Finally, the channel **1** exists in the negative signature only for quark–quark scattering; the projection operator is

$$\left[\mathcal{P}^{\mathbf{1}}_{qq}\right]^{\alpha'\beta'}_{\alpha\beta} = \frac{1}{2N_c}\delta^{\alpha'}_{\alpha}\delta^{\beta'}_{\beta}. \quad (22)$$

For the representations $\mathbf{R}$ different from adjoint the color coefficients $\mathcal{G}(\mathbf{R})^{(0)\sigma}_{AB}$ do not depend on σ; they are equal

$$\mathcal{G}(\mathbf{10}+\mathbf{10}^*)^{(0)\sigma}_{gg} = -\frac{3}{4}N_c,$$
$$\mathcal{G}(\mathbf{1})^{(0)\sigma}_{qq} = \frac{(N_c^2-4)(N_c^2-1)}{8N_c^2}, \quad (23)$$

irrespective of σ.

Therefore, the momentum dependent factors for them summed up to the eikonal amplitude

$$\sum_{\sigma} M^{(0)\sigma}_{AB}(s,t) = A^{eik} = g^6\frac{s}{t}\left(\frac{-4\pi^2}{3}\right)\mathbf{q}^2 A_2(q_\perp), \quad (24)$$

where

$$A_2(q_\perp) = \int\frac{d^{2+2\epsilon}l_1 d^{2+2\epsilon}l_2}{(2\pi)^{2(3+2\epsilon)}\mathbf{l}_1^2\mathbf{l}_2^2(\mathbf{q}-\mathbf{l}_1-\mathbf{l}_2)^2}. \quad (25)$$

This result is very important, because contribution of the cut must be gauge invariant, whereas $M^{(0)\sigma}_{AB}$ taken separately are gauge dependent.

In the Reggeized gluon channel, the color coefficients $\mathcal{G}(R)^{(0)\sigma}_{AB}$ depend on σ. However, this dependence has a specific form. Let $\sigma = a$ ($\sigma = f$) relates to the diagram without u (s) channel cuts. Note that since $M^{(0)a}_{AB}$ and $M^{(0)f}_{AB}$ are connected by the $s \leftrightarrow u$ replacement, only the sum $\mathcal{G}(\mathbf{8})^{(0)a}_{AB} + \mathcal{G}(\mathbf{8})^{(0)f}_{AB}$ enters in the amplitudes with negative signature. It turns out that

$$\frac{1}{2}[\mathcal{G}(\mathbf{8})^{(0)a}_{AB} + \mathcal{G}(\mathbf{8})^{(0)f}_{AB}] = \mathcal{G}(\mathbf{8})^{(0)}_{AB} + \frac{N_c^2}{8}, \quad (26)$$

whereas for all other diagrams ($\sigma = b, c, d, e$) $\mathcal{G}(\mathbf{8})^{(0)\sigma}_{AB} = \mathcal{G}(\mathbf{8})^{(0)}_{AB}$. The Regge pole factorization requires the equality

$$\mathcal{G}(\mathbf{8})^{(0)}_{gg} + \mathcal{G}(\mathbf{8})^{(0)}_{qq} = 2\mathcal{G}(\mathbf{8})^{(0)}_{gq}. \quad (27)$$

Since

$$\mathcal{G}(\mathbf{8})^{(0)}_{gg} = \frac{3}{2}, \quad \mathcal{G}(\mathbf{8})^{(0)}_{gq} = \frac{1}{4},$$
$$\mathcal{G}(\mathbf{8})^{(0)}_{qq} = \frac{1}{4}\left(-1+\frac{3}{N_c^2}\right), \quad (28)$$

it is evident that the pole factorization is violated.

However, it is seen also that the terms violating the pole factorization have σ-independent color coefficients, so that momentum factors for them summed up to the eikonal amplitude.

However, the color factors (28) cannot be fully attributed to the cut contributions. Indeed, separation of the pole and cut contributions is impossible in the two-loop approximation because of the ambiguity of the allocation of the part of the amplitudes violating the factorization: it is always possible to write

$$\mathcal{G}(\mathbf{8})^{(0)}_{AB} = \mathcal{G}(\mathbf{8})^{(0)\mathrm{cut}}_{AB} + \mathcal{G}(\mathbf{8})^{(0)\mathrm{pole}}_{AB},$$

with $\mathcal{G}(\mathbf{8})^{(0)\mathrm{pole}}_{AB}$ satisfying the factorization condition but otherwise arbitrary.

3.1.2. Three loops. The separation becomes possible in higher loops, due to different energy dependence of the pole and cut contributions. The energy dependence of the pole contribution is determined (besides the factor s) by the Regge factor $\exp(\omega(t)\ln s)$, where $1+\omega(t)$ is the gluon trajectory,

$$\omega(t) = -g^2N_c\mathbf{q}^2\int\frac{d^{2+2\epsilon}l}{2(2\pi)^{(3+2\epsilon)}\mathbf{l}^2(\mathbf{q}-\mathbf{l})^2}, \quad (29)$$

whereas for the three-Reggeon cut, it is $\exp(\hat{\mathcal{H}}\ln s)$, where

$$\hat{\mathcal{H}} = \hat{\omega}_1 + \hat{\omega}_2 + \hat{\omega}_3 + \hat{\mathcal{H}}_r(1,2) + \hat{\mathcal{H}}_r(1,3) + \hat{\mathcal{H}}_r(2,3), \quad (30)$$

$\hat{\omega}_i$ stands for the ith Reggeon trajectory and $\hat{\mathcal{H}}_r(m,n)$ is the real part of the BFKL kernel describing interaction between Reggeons m and n. The explicit form of the real part of the kernel describing interaction between two Reggeons with transverse momenta $\mathbf{q}_1$ and $\mathbf{q}_2$ and color indices c_1 and c_2 is

$$[\mathcal{H}_r(\mathbf{q}_1,\mathbf{q}_2;\mathbf{k})]^{c_1'c_2'}_{c_1c_2} = -T^a_{c_1c_1'}T^a_{c_2c_2'}\frac{g^2}{(2\pi)^{D-1}}\left[\frac{\mathbf{q}_1^2\mathbf{q}_2'^2 + \mathbf{q}_2^2\mathbf{q}_1'^2}{\mathbf{k}^2} - \mathbf{q}^2\right], \quad (31)$$

where $\mathbf{q}_1+\mathbf{q}_2 = \mathbf{q}_1'+\mathbf{q}_2' = \mathbf{q}$, $\mathbf{q}_1-\mathbf{q}_1' = \mathbf{q}_2'-\mathbf{q}_2 = \mathbf{k}$.

The first order color coefficient $\mathcal{G}(\mathbf{R})^{(1)\sigma}_{AB}$ is simply proportional to $\mathcal{G}(\mathbf{R})^{(0)\sigma}_{AB}$. As for the trajectory contributions, it is evident. It is true also for the real part contributions, because the operator $\sum_{i>j=1}^{3}\hat{T}^c(i)\hat{T}^c(j)$ acts on the state $\Psi(\mathbf{R})$, for which, due to color conservation,

$$\left(\sum_{i=1}^{3}\hat{T}^c(i) + \hat{\mathcal{T}}^c(\mathbf{R})\right)\Psi(\mathbf{R}) = 0, \quad (32)$$

where $\hat{\mathcal{T}}^c(\mathbf{R})$ is the color group generator in the representation $\mathbf{R}$. It gives

$$\sum_{i>j=1}^{2}\hat{T}^c(i)\hat{T}^c(j)\Psi(\mathbf{R}) = \frac{1}{2}\left(C_2(\mathbf{R}) - 3C_2(\mathbf{8})\right)\Psi(\mathbf{R}), \quad (33)$$

where $C_2(\mathbf{R})$ is the value of the Casimir operator in the representation $\mathbf{R}$; $C_2(\mathbf{8}) = N_c$. Therefore, in the NNLLA, the three-loop correction is

$$\mathcal{G}(\mathbf{R})^{(\mathrm{cut})}_{AB} g^8 \frac{s}{t}\left(\frac{-4\pi^2}{3}\right)\mathbf{q}^2 \left(\left(\frac{3}{2}N_c - C_2(\mathbf{8})\right)\right. \\ \left.\times A_3^b(q_\perp) - \frac{1}{2}(3N_c - C_2(\mathbf{8}))\, A_3^c(q_\perp)\right)\ln s, \quad (34)$$

where

$$A_3^b(q_\perp) = -\int \frac{d^{2+2\epsilon}l_1 d^{2+2\epsilon}l_2 d^{2+2\epsilon}l_3}{(2\pi)^{3(3+2\epsilon)}\mathbf{l}_1^2\mathbf{l}_2^2\mathbf{l}_3^2(\mathbf{q}-\mathbf{l}_1-\mathbf{l}_2-\mathbf{l}_3)^2}, \quad (35)$$

$$A_3^c(q_\perp) = \int \frac{d^{2+2\epsilon}l_1 d^{2+2\epsilon}l_2 d^{2+2\epsilon}l_3(\mathbf{q}-\mathbf{l}_1)^2}{(2\pi)^{3(3+2\epsilon)}\mathbf{l}_1^2\mathbf{l}_2^2\mathbf{l}_3^2(\mathbf{q}-\mathbf{l}_1-\mathbf{l}_2)^2(\mathbf{q}-\mathbf{l}_1-\mathbf{l}_3)^2}. \quad (36)$$

The calculation of the three-loop corrections [20] shows that the violation of the pole Regge form, analyzed in this approximation with the help of the infrared factorization, can be explained by the pole and cut contributions. In other words, the restrictions imposed by the infrared factorization on the parton scattering amplitudes with the adjoint representation of the color group in the t-channel and negative signature can be fulfilled in the NNLLA at two and three loops if besides the Regge pole contribution there is the Regge cut contribution

$$\mathcal{G}(\mathbf{8})^{(\mathrm{cut})}_{AB} g^6 \frac{s}{t}\left(\frac{-4\pi^2}{3}\right)\mathbf{q}^2 \\ \times\left(A_2(q_\perp) + g^2 N_c \ln s\left(\frac{1}{2}A_3^b(q_\perp) - A_3^c(q_\perp)\right)\right), \quad (37)$$

$$\mathcal{G}(\mathbf{8})^{(\mathrm{cut})}_{gg} = -\frac{3}{2}, \quad \mathcal{G}(\mathbf{8})^{(\mathrm{cut})}_{gq} = -\frac{3}{2}, \\ \mathcal{G}(\mathbf{8})^{(\mathrm{cut})}_{qq} = \frac{3(1-N_c^2)}{4N_c^2}. \quad (38)$$

3.2. Wilson Line Approach

The explanation of the violation of the pole Regge form given in [18] differs from the described above. In this paper, no connection of the three-Reggeon cuts with Feynman diagrams was traced. The color coefficients $\mathcal{G}(\mathbf{R})^{(0)C}_{AB}$ for the cut contributions are taken as

$$\mathcal{G}(\mathbf{R})^{(0)C}_{AB} = \frac{1}{6!N_{\mathbf{R}}T_A T_B}\left(\mathcal{T}^{c_1}_A\mathcal{T}^{c_2}_A\mathcal{T}^{c_3}_A\right)^{\alpha'}_{\alpha} \\ \times\left(\sum_\sigma \mathcal{T}^{c_1^\sigma}_B\mathcal{T}^{c_2^\sigma}_B\mathcal{T}^{c_3^\sigma}_B\right)^{\beta'}_{\beta}\left[\mathcal{P}^{\mathbf{R}*}_{AB}\right]^{\alpha\beta}_{\alpha'\beta'}. \quad (39)$$

As for the momentum dependent part, it is taken equal to A^{eik} (24). For the representations $\mathbf{R}$ different from the Reggeized gluon one it agrees with the diagrammatic approach, since the color coefficients $\mathcal{G}(\mathbf{R})^{(0)\sigma}_{AB}$ do not depend on σ for such representations (see (23)). Therefore, the cut contributions in both approaches are the same for these representations.

However, it is not so for the adjoint representation, where the color coefficients $\mathcal{G}(\mathbf{8})^{(0)C}_{AB}$ of [18] turn out to be

$$\mathcal{G}(\mathbf{8})^{(0)C}_{AB} = \mathcal{G}(\mathbf{8})^{(0)}_{AB} + \frac{N_c^2}{24}, \quad (40)$$

where $\mathcal{G}(\mathbf{R})^{(0)}_{AB}$ are given by (28). As for the momentum dependent part, it is also taken equal to A^{eik} (24). It looks strange from the point of view of the diagrammatic approach, because appearance of A^{eik} requires equality of all terms in the sum over σ in (39). In two loops, difference Δ_{AB} between two approaches is such that

$$\Delta_{gg} + \Delta_{qq} = 2\Delta_{gq}. \quad (41)$$

Therefore, it can be attributed to the pole contribution. To do this, it is sufficient to change the two-loop contributions to the gluon–gluon–Reggeon and quark–quark–Reggeon vertices in (1).

However, in three loops the cut contributions turned out to be

$$\mathcal{G}(\mathbf{8}))^{(0)C}_{AB} g^6 \frac{s}{t}\left(\frac{-4\pi^2}{3}\right)\mathbf{q}^2 \\ \times\left(A_2(q_\perp) + g^2 N_c \ln s\left(\frac{1}{2}A_3^b(q_\perp) - A_3^c(q_\perp)\right)\right), \quad (42)$$

and it is impossible to explain the violation of the pole form only by the cut. It becomes possible introducing the Reggeon-cut mixing [18] with the color coefficients

$$\mathcal{G}(\mathbf{8})^{(1)\mathrm{mix}}_{AB} = \frac{1}{6(N_c^2-1)T_A T_B}\sum_\sigma Tr\left(\mathcal{T}^c_g\mathcal{T}^{c_1^\sigma}_g\mathcal{T}^{c_2^\sigma}_g\mathcal{T}^{c_3^\sigma}_g\right) \\ \times\left[\mathcal{T}_A Tr\left(\mathcal{T}^c_B\mathcal{T}^{c_1}_B\mathcal{T}^{c_2}_B\mathcal{T}^{c_3}_B\right) + \mathcal{T}_B Tr\left(\mathcal{T}^c_A\mathcal{T}^{c_1}_A\mathcal{T}^{c_2}_A\mathcal{T}^{c_3}_A\right)\right]. \quad (43)$$

The mixing contributes only in three loops.

It is necessary to note that in the approach used in [18] the cut contribution is not suppressed at large N_c; i.e., it exists in the planar $N = 4$ SYM, in contradiction with the common opinion, that in the high-energy limit the four-point amplitudes in this theory are given by the Reggeized gluon contribution.

3.3. Four Loops in the Diagrammatic Approach

The three-loop results presented above do not allow rejecting any of the approaches. A possible choice between them could be made comparing their results with infrared factorization results. Unfortunately, they are not known yet. Some results are known only in the diagrammatic approach.

In four loops, there are three types of corrections. The first (simplest) one comes from account of the

Regge factors of each of three Reggeons. The second type contains the corrections given by the products of the trajectories and real parts of the BFKL kernel, and the third comes from account of Reggeon–Reggeon interactions. The momentum-dependent part of all these corrections is expressed through the integrals

$$I_i = \int \frac{d^{2+2\epsilon}l_1 d^{2+2\epsilon}l_2 d^{2+2\epsilon}l_3}{(2\pi)^{3(3+2\epsilon)}\mathbf{l}_1^2\mathbf{l}_2^2\mathbf{l}_3^2} F_i \delta^{3+2\epsilon}(\mathbf{q} - \mathbf{l}_1^2 - \mathbf{l}_2^2 - \mathbf{l}_3), \quad (44)$$

where

$$F_a = f_1(\mathbf{l}_1)f_1(\mathbf{l}_2), \quad F_b = f_1(\mathbf{l}_1)f_1(\mathbf{l}_1),$$
$$F_c = f_2(\mathbf{l}_1 + \mathbf{l}_2), \quad F_d = f_1(\mathbf{l}_1 + \mathbf{l}_2)f_1(\mathbf{l}_1 + \mathbf{l}_2), \quad (45)$$
$$F_e = f_1(\mathbf{q} - \mathbf{l}_1)f_1(\mathbf{q} - \mathbf{l}_3),$$

$$f_1(\mathbf{k}) = \mathbf{k}^2 \int \frac{d^{2+2\epsilon}l}{(2\pi)^{(3+2\epsilon)}\mathbf{l}^2(\mathbf{l} - \mathbf{k})^2},$$
$$f_2(\mathbf{k}) = \int \frac{d^{2+2\epsilon}l f_1(\mathbf{l})}{(2\pi)^{(3+2\epsilon)}\mathbf{l}^2(\mathbf{l} - \mathbf{k})^2}. \quad (46)$$

Calculation of the color factors $\mathcal{G}(\mathbf{R})_{AB}^{(2)\sigma}$ is not easy.

Of course, it is trivial for the squared virtual part contributions. The corresponding color factor is

$$\mathcal{G}(\mathbf{R})_{AB,VV}^{(2)\sigma} = \frac{N_c^2}{4}\mathcal{G}(\mathbf{R})_{AB}^{(0)\sigma}. \quad (47)$$

It is not difficult also for the products of the virtual and real parts due to the property (32). It gives

$$\mathcal{G}(\mathbf{R})_{AB,VR}^{(2)\sigma} = \frac{N_c}{2}\left(C_2(\mathbf{R}) - 3N_c\right)\mathcal{G}(\mathbf{R})_{AB}^{(0)\sigma}. \quad (48)$$

However, it is rather complicated for the squared real part. It contains the matrix elements

$$\left\langle \Psi_B^\sigma \left| \sum_{i \neq j = 1}^{3} \hat{T}^c(i)\hat{T}^c(j)\hat{T}^d(i)\hat{T}^d(j) \right| \Psi_A \right\rangle \quad (49)$$

and

$$\left\langle \Psi_B^\sigma \left| \sum_{i \neq j \neq k = 1}^{3} \hat{T}^c(i)\hat{T}^c(j)\hat{T}^d(i)\hat{T}^d(k) \right| \Psi_A \right\rangle. \quad (50)$$

Due to Eq. (32), their difference is equal to

$$\frac{1}{4}\left(C_2(\mathbf{R}) - 3N_c\right)^2\mathcal{G}(\mathbf{R})_{AB}^{(0)\sigma}; \quad (51)$$

therefore, it is sufficient to obtain the first one.

Rather tedious calculations give for its contribution to $\mathcal{G}(\mathbf{8_a})_{AB}^{(s)\sigma}$ to $\mathcal{G}(R)_{AB}^{(2)\sigma}$:

$$\mathcal{G}(\mathbf{8})_{AB}^{(s)b} = \mathcal{G}(\mathbf{8})_{AB}^{(s)c} = \mathcal{G}(\mathbf{8})_{AB}^{(s)d} = \mathcal{G}(\mathbf{8})_{AB}^{(s)e} = \mathcal{G}(\mathbf{8})_{AB}^{(s)}, \quad (52)$$

$$\frac{1}{2}\left(\mathcal{G}(\mathbf{8})_{AB}^{(s)a} + \mathcal{G}(\mathbf{8})_{AB}^{(s)f}\right) = \mathcal{G}(\mathbf{8})_{AB}^{(s)} + \left(\frac{N_c^4}{16} + \frac{3N_c^2}{8}\right). \quad (53)$$

It is important that the terms violating the pole factorization have σ-independent color coefficients, which provides their gauge invariance, as well as in the two and three loops.

4. DISCUSSION

The pole Regge form of QCD amplitudes, which is the basis of the BFKL equation, and which is valid in the LLA and in the NLLA, is violated in the NNLLA. It is natural to think that the cause of the violation are three-Reggeon cuts. This thought is supported by the fact that the observed violation can be explained by these cuts [17, 18]. However, the explanations given in [17] (see also [19, 20]) and [18] are different. In [17], the violation is explained but the cut contribution only, whereas in [18] the Reggeon-cut mixing is also introduced. The approaches [17] and [18] agree in three loops but have to diverge in higher loops. A possible choice between them could be made in four loops.

However, a complete proof that QCD amplitudes with gluon quantum numbers in cross-channels and negative signature are given in the NNLLA by the contributions of the Regge pole and the three-Reggeon cut, with or without mixing, requires or check in each order of perturbation theory (that is evidently impossible) or inventing some method, like bootstrap for the proof of the gluon Reggeization.

The appearance of Regge cuts in amplitudes with negative signature greatly complicates the derivation of the BFKL equation using unitarity relations.

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