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Shape of the 11-Year Cycle of Solar Activity and the Evolution of Latitude Characteristics of the Sunspot Distribution

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Abstract—The solar activity index and parameters of the spatial distribution of sunspots are known to be related. Using these relationships, we propose interrelated approximations for the sunspot number (SN) and the two key latitude characteristics of their distribution in the cycle: the mean latitude of sunspots and the standard deviation of their latitudes. The two parameters of these approximations are the cycle amplitude SN_{max} and the drift of its downward branch relative to the cycle beginning t_0. These approximations specifically take into account the relationship between amplitudinal and spatial properties of the 11-year solar cycle, as well as the universality of the behavior of the activity and mean latitude of sunspots in the declining phase of the cycle. We demonstrate that the pair of parameters SN_{max} and t_0 allows approximation of both the shape of the cyclic curve and the latitude–time diagram for sunspots of this cycle ("Maunder's butterfly").

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1. INTRODUCTION

The search for dependences describing the behavior of the solar activity index in the 11-year solar cycle dates back to the 19th century. Rudolf Wolff tried to describe a dependence by superimposing four sinusoids corresponding to the phases of the motion of the solar system planets (see Waldmeier, 1935); over the next 150 years, there had been multiple attempts to find an appropriate dependence (see, e.g., Vitinskii et al. (1986), Hathaway et al. (1994), and references therein).

These approximations often have a formal mathematical character: finding a function with a small number of parameters that is most consistent with the behavior of the given index. However, in describing the approximating curve, it is useful to take into account the relationships between sunspot distribution features.

It is known that the evolution of solar activity during the 11-year cycle can be divided into two qualitatively different phases. The inclining and declining phases are characterized by a comparatively rapid increase and a slower decrease in activity, respectively. The activity in the inclining phase increases more quickly for higher cycles ("the Waldmeier rule"); on the contrary, in the declining phase its behavior depends weakly on the prehistory of this cycle, but it is well correlated with the current mean latitude of sunspots (Eigenson et al., 1948; Gnevyshev and Gnevysheva, 1949; Hathaway, 2011; Ivanov and Miletsky, 2014). The latter is useful because it allows one to construct two interrelated approximating curves: the first describes the evolution of the activity index in the solar cycle, and the second describes the behavior of the average latitude of sunspots.

Cameron and Schüssler (2016) showed that the specific behavior of solar activity in the declining phase of the cycle may be related to the fact that the decrease of the magnetic field strength at this time happens predominantly due to magnetic diffusion, and they proposed a quantitative model of this mode of activity.

In this study, we propose two consistent approximating functions that describe both the behavior of the activity index in the 11-year cycle and the latitudinal distribution of sunspots in this cycle; the form of these sunspots takes into account the aforementioned relationships and regularities.

2. DATA

The data used by us are the annual-mean values of the recalibrated sunspot number SN for 1700–2016 (WDC-SILSO, Royal Observatory of Belgium, Brussels, http://www.sidc.be/silso/DATA/SN_y_tot_V2.0.txt). Our calculations of the spatial characteristics of sunspots for 1874–2016 are based on the Greenwich sunspot catalog (Greenwich – USAF/NOAA Sunspot Data, http://solarscience.msfc.nasa.gov/greenwch.shtml); the average latitude of sunspot groups φ and the standard deviations σ_φ of these latitudes (regardless of their sign) for the given year are calculated for these data. The time intervals used below correspond to the

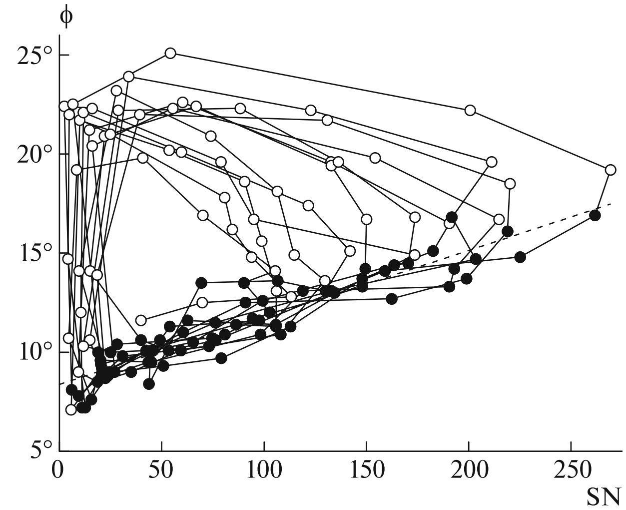

Fig. 1. Relationship between the annual-mean sunspot latitude φ and the SN index for 1874–2016. The black circles indicate the years of the declining phase, and the dotted line stands for regression (1).

Greenwich catalog (1874–2016; i.e., cycles 11(12)–23) unless otherwise noted.

3. RELATIONSHIP BETWEEN SN, φ, AND σ_φ

Figure 1 shows the correlation between the annual-mean SN values and the mean latitude of sunspots φ. In the declining cycle phase (black circles), there is a dependence between these indices (Ivanov and Miletsky, 2014) with a correlation coefficient of R = 0.93, which can be described by the linear regression

$$\varphi(\mathrm{SN}) = a_\varphi + b_\varphi \mathrm{SN} = 8.37° + 0.0338°\mathrm{SN}. \quad (1)$$

The standard deviations of latitudes σ_φ are also related to the SN (Ivanov et al., 2011) (Fig. 2a). If the years of activity minima and their adjacent years (indicated by empty circles in the figure), when the old and new 11-year cycles often overlap and it is difficult to determine their parameters unambiguously, are disregarded, this dependence can be described by the regression

$$\sigma_\varphi(\mathrm{SN}) = a_\sigma + b_\sigma \mathrm{SN} = 4.01° + 0.0150°\mathrm{SN} \quad (R = 0.87). \quad (2)$$

Obviously, one can also construct a third linear regression describing the correlation between σ_φ and φ in the declining cycle phase (Fig. 2b):

$$\sigma_\varphi(\varphi) = 0.459\varphi - 0.0401° \quad (R = 0.91). \quad (3a)$$

Since the free term of this linear regression is small, it can be dropped without loss of accuracy:

$$\sigma_\varphi(\varphi) = p\varphi = 0.456\varphi. \quad (3b)$$

4. SN APPROXIMATION

Cameron and Schüssler (2016) proposed a mechanism describing the behavior of activity in the declining phase. Assuming that the intensity of the toroidal magnetic field in this phase decreases mainly due to diffusion through the equator and that the latitudinal field profiles can be approximately described by normal distributions, these authors obtained for description of the SN a function F that satisfies the equation

$$\frac{dF}{dt} = -C\frac{\varphi}{\sigma_\varphi^3}F. \quad (4)$$

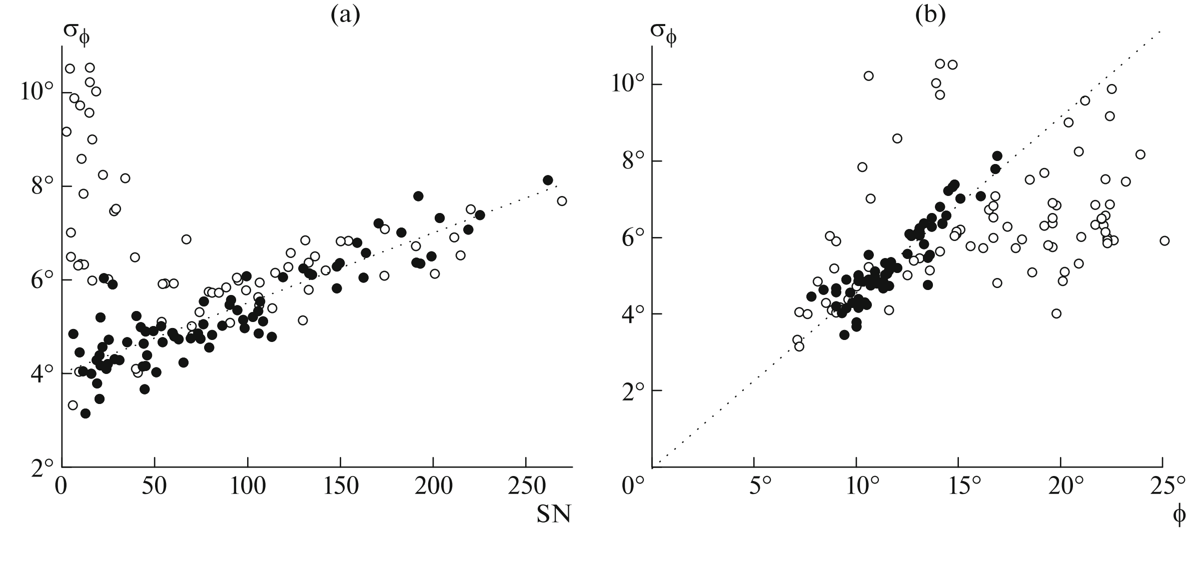

Fig. 2. Relationship between the annual-mean activity parameters for 1874–2016: (a) (SN, σ_φ), the dotted line by regression (2) and (b) (φ, σ_φ), the dotted line by regression (3). The regressions in both diagrams were constructed for years that are more than one year away from cycle minima (black circles).

Here,

$$C = \left(\frac{180^\circ}{\pi}\right)^2 \sqrt{\frac{2}{\pi}} \left(\frac{\eta}{R^2}\right)_{\text{eff}} \exp\left(-\frac{\varphi^2}{2\sigma_\varphi^2}\right),$$

where η is the coefficient of turbulent magnetic diffusion, R^2 is the characteristic square of the radius of the magnetic field generation region, and $\left(\frac{\eta}{R^2}\right)_{\text{eff}}$ is the effective ratio of the coefficients in this region. With relation (3b), the exponent and, hence, C are constants. Then, Eq. (4) is a differential equation of the first order, which, in view of (2) and (3), can be written as

$$\frac{dF}{dt} = -\frac{A}{(F + \alpha)^2} F, \tag{5}$$

where $A = \frac{C}{pb_\sigma^2}$, $\alpha = \frac{a_\sigma}{b_\sigma}$, and its solution is

$$F_2(t; t_0) = f(A(t_0 - t)), \tag{6}$$

where $f = g^{-1}$ is the inverse to the function

$$g(y) = \alpha^2 \log y + 2\alpha y + y^2/2.$$

Thus, the activity in the declining phase can be described by curve (6), which has the same form for all cycles for fixed A and α, and the parameter t_0 determines the shift of the declining branch of this cycle along the time axis (hereafter, time t is assumed to be counted from the cycle minimum). It was noted earlier that it is such a universal behavior of the activity index (unrelated to the cycle intensity) that is typical for the declining cycle phase. In this case, the parameter $\alpha = 248$ is equal to the ratio of the coefficients of linear regression (2), and the second empirical parameter $A = 4.17 \times 10^4$ year^{-1}, which determines the extension of the standard curve $f(x)$ along the ordinate axis, was found by minimizing the sum of standard deviations between the observed index and solution (6) in the cycle declining phases for the entire SN data series (1700–2016).

Unlike the cycle declining phase, the behavior of activity in the inclining phase depends strongly on cycle intensity (according to the Waldmeier rule). The approximating function for this phase must have a minimum at $t = 0$, sufficiently rapid growth, and a maximum ($\text{SN}_{\max}$) at some moment $t = T_{\max}$. With no aim of describing in detail the behavior of activity in the inclining phase, we choose the simplest function of this form

$$F_1(t; \text{SN}_{\max}, T_{\max}) = \text{SN}_{\max} \sin^2 \frac{\pi t}{2T_{\max}}. \tag{7}$$

The function approximating SN behavior in the full cycle is proposed to be a piecewise-defined function F taking values of F_1 and F_2 at the time intervals $0 \le t \le t^*$ and $t \ge t^*$, respectively. Here, the boundary point t^* is chosen such that F is continuous and differentiable on the domain of definition. This can be achieved by requiring that the corresponding curves be tangent to one another at t^*:

$$F_1(t^*) = F_2(t^*),$$
$$F_2'(t^*) = F_2'(t^*).$$

It is rather clear (Fig. 3a) that, if $\text{SN}_{\max} < F_2(0)$, the point t^* exists and is unique on the interval from $T_{\max}$ to $2T_{\max}$. We do not strictly prove this fact; for our purposes, it is sufficient that this point can be found for all cycles of the full SN data series (see Table 1). The condition imposed on F_1 unambiguously relates the parameters t_0, $\text{SN}_{\max}$, and $T_{\max}$. We have the freedom of choice: we can choose t_0 and $\text{SN}_{\max}$ or t_0 and $T_{\max}$ as a pair of parameters specifying the function F. We choose the first option: $\text{SN}_{\max}$ is assumed to be equal to the observed cycle amplitude and the length $T_{\max}$ is uniquely expressed in terms of t_0 and $\text{SN}_{\max}$ (thus, it does not have to be equal to the observed length of the inclining phase $T_{\max,\text{obs}}$, though it is close to it).

Thus, we have described the class of piecewise-defined smooth functions $F(t;\, t_0,\, \text{SN}_{\max})$ with two parameters (see Fig. 3b) describing the behavior of the 11-year cycle in both phases. Table 1 gives the approximation parameters for the cycles of the full SN data series, as well as the rms approximation errors Δ. Figure 4 shows the approximating curves for the Greenwich epoch.

In terms of approximation accuracy, the proposed functions may compete well with other well-known two-parameter formulas. For example, when the SN is approximated by functions of the Stewart–Panofsky type $S_3(t;\, C,\, \beta) = Ct^3\exp(-\beta t)$ (Steward and Panofsky (1938); see also Vitinskii et al. (1986)) the rms error on the full SN data series is 14.6, and for the functions chosen by us it is 17.8. Here, our approximation takes precedence over S_3, because the form of decreasing branches of functions F is consistent with the universal law of activity decrease in the declining phase (Fig. 3b).

Although the functions $F(t; t_0, \text{SN}_{\max})$ depend upon two parameters, values of these parameters corresponding to the full SN data series are not completely independent. In this case, the correlation between $\text{SN}_{\max}$ and t_0 is $R = 0.49$, increasing up to $R = 0.73$ if only the Greenwich cycles starting with the 12th cycle are taken into account (see Fig. 5). The Greenwich epoch is characterized by the following linear relationship between the parameters of the functions:

$$t_0(\text{SN}_{\max}) = 13.2 + 0.00862\ \text{SN}_{\max}\ (R = 0.73). \tag{8}$$

In the pre-Greenwich epoch, the corresponding correlation decreases to $R = 0.40$, which is caused, probably, by increased error in determining of the SN in this epoch.

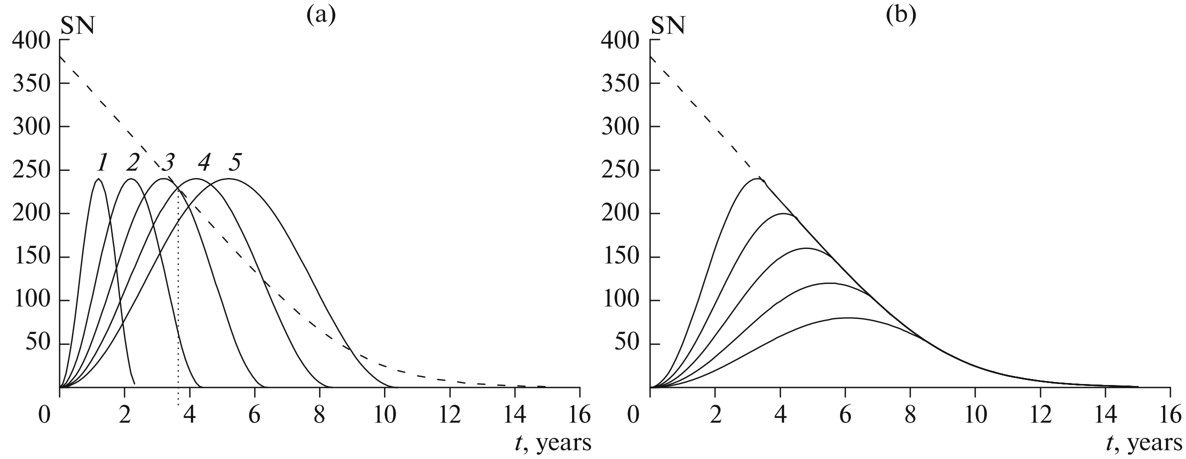

Fig. 3. (a) Search for the conjugation point of the curves of F_1 and F_2. The curves indicated by solid lines and marked with the numbers from *1* to *5* correspond to functions $F_1(t; SN_{max}, T_{max})$ with a fixed $SN_{max} = 240$ and successively increasing T_{max}; curve F_1 number *3* is tangent to the curve $F_2(t; t_0)$ at the point with abscissa t^*. (b) Solid lines are the approximating curves $F(t; t_0, SN_{max})$ corresponding to the cycle amplitudes $SN_{max} = 80, 120, 160, 200, 240$. The dotted line on both graphs denotes the function $F_2(t; t_0)$ for $t_0 = 15$ years. In the graph (b), this function describes the behavior of activity common for all SN_{max} in the cycle declining phase.

It should be noted that the approximation chosen by us leads naturally to the Waldmeier rule. As mentioned above, the length of the declining phase of the approximating curve T_{max} is functionally associated with t_0 and SN_{max}. If the shift in t_0 were the same for all cycles, T_{max} would decrease with SN_{max} growth (see Fig. 3b) and the correlation between these parameters would be close to –1. However, in real data, t_0 increases with cycle amplitude according to (8); here, the curve of F_2 shifts rightward and T_{max} increases. Thus, there is no full anticorrelation between SN_{max} and T_{max}. Nevertheless, the correlation between these parameters remains sufficiently strong (Fig. 6):

$$T_{max}(SN_{max}) = 6.56 - 0.0129\, SN_{max}\,(R = -0.78). \quad (9)$$

5. APPROXIMATION OF THE CURVE OF MEAN LATITUDES OF SUNSPOTS

It is well known that the mean sunspot latitudes φ decrease monotonically with the evolution of the 11-year

Table 1. Characteristics of functions $F(t; t_0, SN_{max})$ approximating the SN index in individual solar cycles: t_0 and SN_{max} are the function parameters, T_{max} is its time of maximum, t^* is the boundary of intervals, Δ is the rms error of the approximation

Cycle number	t_0, years	SN_{max}	T_{max}, years	t^*, years	Δ	Cycle number	t_0, years	SN_{max}	T_{max}, years	t^*, years	Δ
–4	12.4	96.7	3.9	4.8	21.2	10	14.7	182.2	4.2	4.7	20.1
–3	14.2	105.0	5.2	6.5	13.4	11	14.4	232.0	2.9	3.0	22.5
–2	14.7	203.3	3.7	4.1	27.9	12	14.3	106.1	5.2	6.5	12.2
–1	15.5	185.0	4.7	5.5	18.4	13	14.2	142.0	4.4	5.5	8.6
0	15.0	139.0	5.1	6.4	14.8	14	13.9	105.5	4.9	6.2	13.9
1	15.4	143.2	5.4	6.5	18.0	15	14.1	173.6	3.8	4.2	22.6
2	14.1	176.8	3.7	4.1	18.1	16	14.0	129.7	4.5	5.5	11.9
3	14.4	257.3	2.3	2.4	46.3	17	15.4	190.6	4.5	5.3	8.5
4	15.7	220.0	4.2	4.8	22.2	18	14.8	214.7	3.6	3.8	19.5
5	14.1	79.2	5.5	7.3	6.7	19	15.5	269.3	3.1	3.3	23.1
6	14.4	76.3	5.8	7.7	14.1	20	15.2	150.0	5.1	6.2	19.3
7	15.4	117.4	5.8	7.4	6.2	21	15.2	220.1	3.8	4.3	18.4
8	15.1	227.3	3.6	3.9	21.0	22	14.8	211.1	3.7	4.0	19.1
9	16.5	208.3	5.2	5.9	17.1	23	15.2	173.9	4.7	5.5	12.4

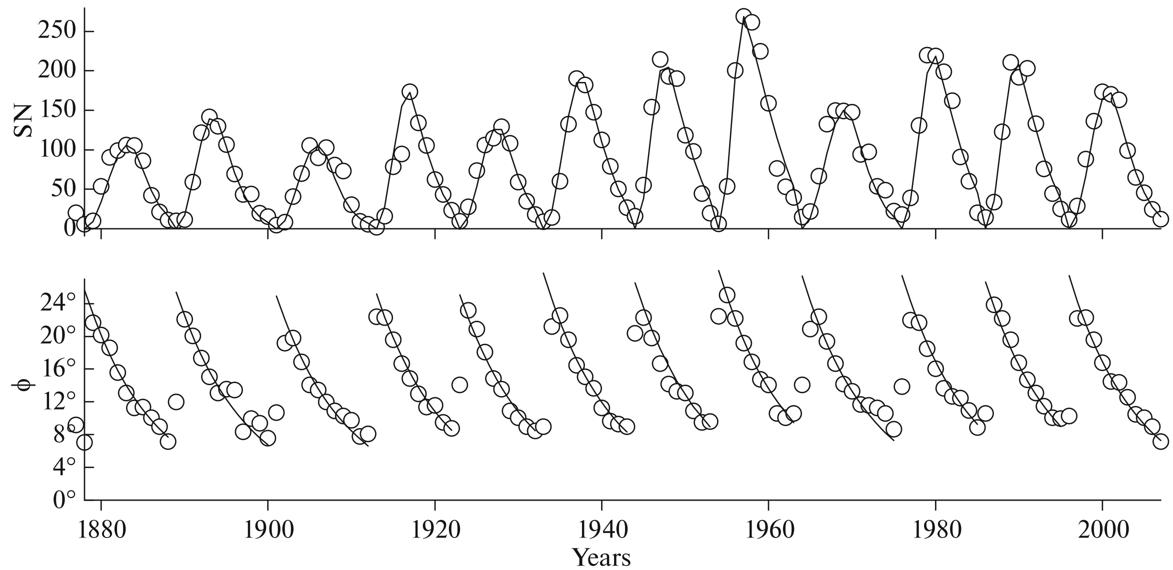

Fig. 4. Annual-mean SN (top) and annual-mean sunspot latitude φ (bottom). The circles correspond to observed values and the lines correspond to the approximations $F(t; t_0, \mathrm{SN}_{max})$ and $F_\varphi(t; \tau(t_0))$, respectively.

cycle ("Spörer's law"), and it has repeatedly been shown that their behavior depends weakly on the cycle amplitude and can be described, for example, by an exponential dependence (Hathaway, 2011; Roshchina and Sarychev, 2011; Ivanov and Miletsky, 2014):

$$F_\varphi(t;\tau) = \varphi_0 \exp\left(\frac{\tau - t}{\beta}\right), \quad (10)$$

where $\varphi_0 = 15°$ and $\beta = 8.3$ years are empirical coefficients common for all cycles and τ is a cycle-dependent parameter that determines the shift of the exponent along the time axis. Table 2 shows the parameters τ and the rms approximation errors $\Delta\varphi$ for the Greenwich epoch cycles.

It follows from relation (1) that τ must be related to the cyclic curve parameters. Indeed, the following regression can be obtained for τ:

$$\tau(t_0) = 0.61t_0 - 4.30 \ (R = 0.80). \quad (11)$$

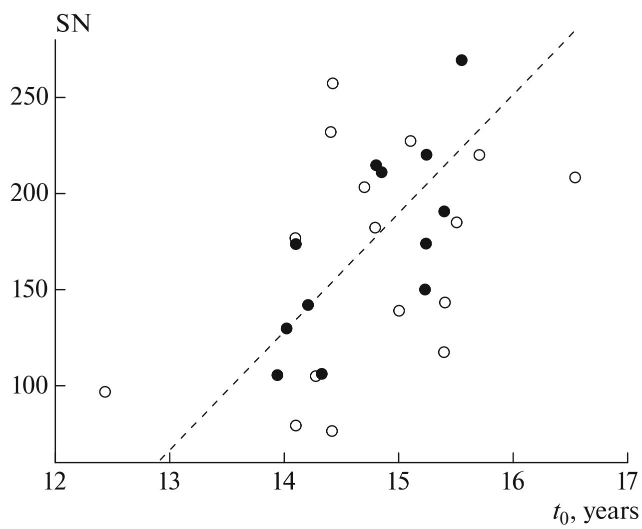

Fig. 5. Relationship between the approximation parameters t_0 and SN_{max}. The black circles indicate the cycles of the Greenwich epoch (starting from the 12th cycle), and the light circles indicate the cycles of the pre-Greenwich epoch. The dotted line is regression (8).

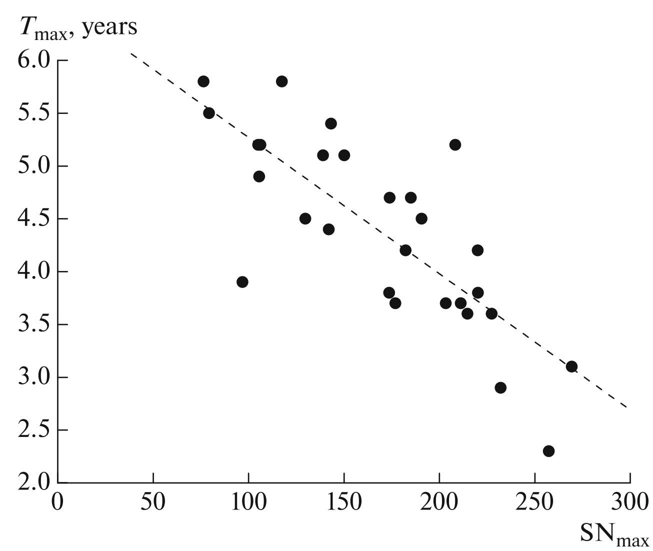

Fig. 6. Relationship between the cycle amplitude SN_{max} and the growth phase length T_{max} for approximating functions to illustrate the Waldmeier rule. The dotted line corresponds to regression (9).

Table 2. Characteristics of functions $F_\varphi(t; \tau)$ approximating the mean sunspot latitude in individual solar cycles: τ is the function parameter and Δ_φ is the rms error of the approximation

Cycle number	τ, years	Δ_φ
11	4.84	0.89°
12	4.41	0.67°
13	4.54	1.39°
14	4.20	0.76°
15	4.06	0.56°
16	4.24	0.87°
17	5.09	0.58°
18	4.12	0.69°
19	5.03	0.66°
20	5.25	1.17°
21	4.83	0.73°
22	5.04	0.41°
23	5.24	0.58°

Thus, the two parameters t_0 and SN_{max} can be used to describe both the level of cycle activity and one of the characteristics of the spatial distribution of sunspots. Approximations of the mean latitude in cycles corresponding to dependence (10) with the parameter determined from (11) are shown on the bottom panel of Fig. 4.

6. SYNTHESIS OF THE BUTTERFLY DIAGRAM

In a good approximation, the distribution of sunspots by latitude is normal (Ivanov et al., 2011) and the variance of this distribution σ_φ^2 is determined by relation (2); therefore, the approximations obtained above for the SN and φ can also be used to describe the distribution of sunspots by latitude. This means that, knowing the pair of parameters that determine the behavior of the activity level (t_0 and SN_{max}), one can describe in some approximation their spatial distribution as well.

Figure 7 shows two time–latitude diagrams for sunspots ("Maunder's butterflies"). The top diagram was obtained from the observed data, and the bottom diagram was constructed with the use of parameters t_0 and SN_{max} (see Table 1). Here, for each year and hemisphere, we generated a random sequence of sunspot groups, the number of which corresponded to half of the full SN index calculated with the help of the function F and the moments of appearance were distributed uniformly over the year, and the latitudes were distributed according to the normal law with a mean latitude φ and its variance σ_φ obtained from approximation (10) and regressions (11) and (2).

It can be seen that the observed and synthesized "butterfly diagrams" are quite consistent in their form. Some differences between the diagrams are explained by the facts that (a) the actual distribution of sunspots is slightly

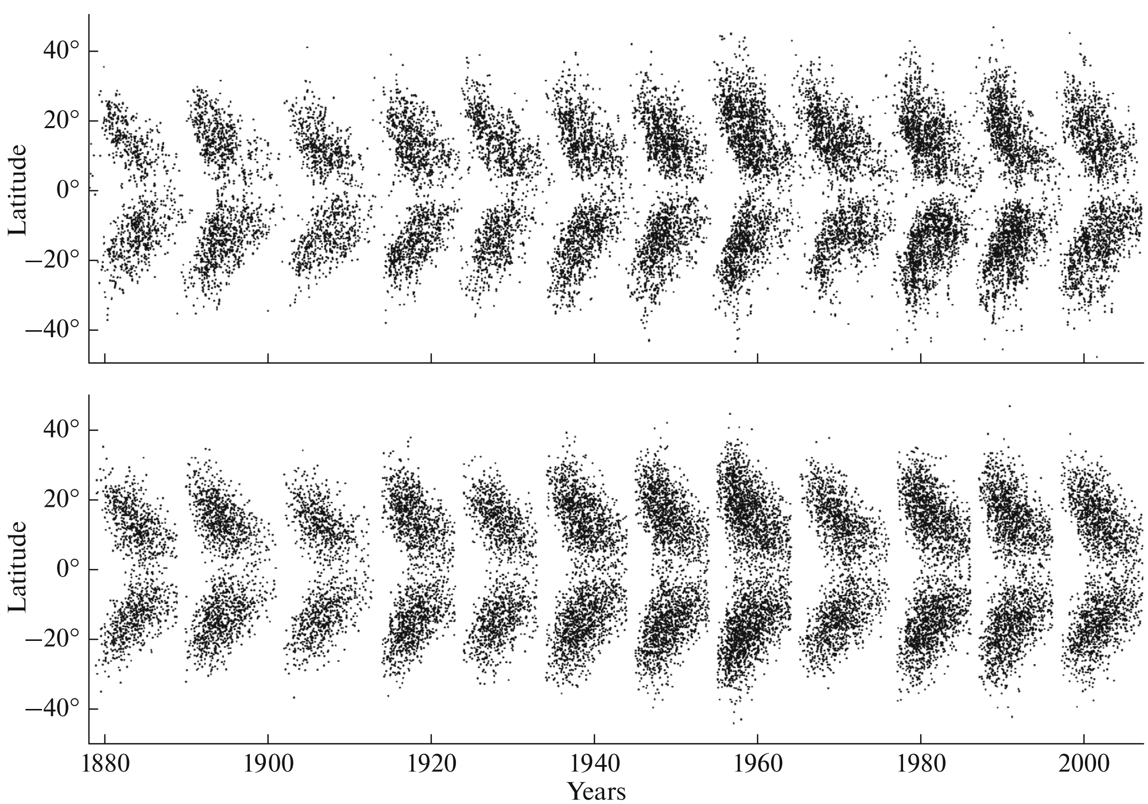

Fig. 7. The observed butterfly diagram (top) and its synthesized version constructed from approximating functions (bottom). For clarity, only one of the ten sunspot groups is shown on each diagram.

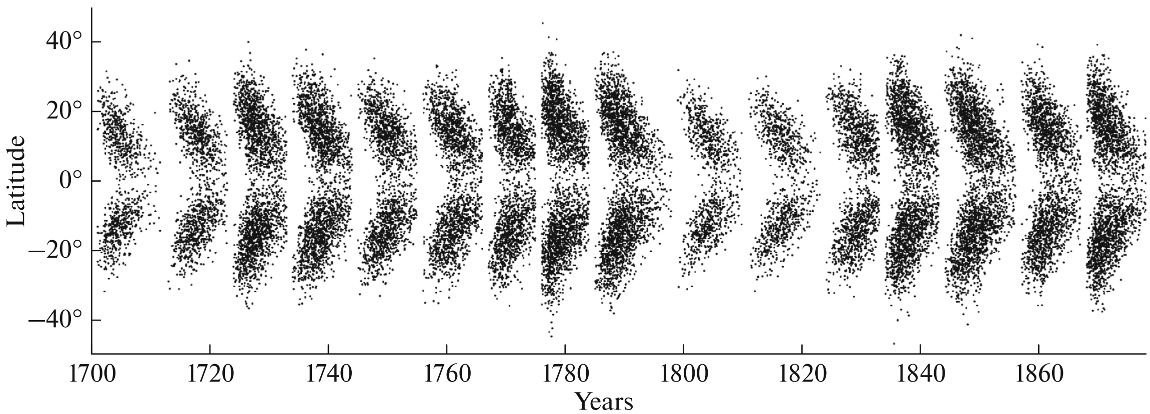

Fig. 8. The synthesized butterfly diagram for the pre-Greenwich epoch. For clarity, only one of the ten sunspot groups is shown on the diagram.

different from the normal distribution (Ivanov et al., 2011) and (b) the mean sunspot latitude at minima is not described by an approximating function (10) (see Fig. 4) because of its strong dependence on the details of the distribution of sunspots in the old and new cycles.

Since the calculation of approximation parameters requires only data for the activity level, we can also construct similar diagrams for the pre-Greenwich epoch, for which there are no systematic data on sunspot latitudes (Fig. 8).

7. CONCLUSIONS

Thus, we have constructed a class of functions $F(t; t_0, SN_{max})$ that approximate the SN behavior and depend on two parameters of the cycle: its height SN_{max} and the shift of the activity curve in the declining phase t_0. The descending branches of these functions have a universal form and differ only by the shift along the time axis. This feature agrees with the observed form of the 11-year cycle of solar activity and can be caused by the special diffusion mode of toroidal magnetic fields in the declining phase of the 11-year solar cycle.

The fact that the activity in the declining phase correlates well with the mean sunspot latitude allowed us to construct another class of approximating functions $F_\varphi(t; \tau)$, describing the behavior of the mean latitudes. These functions depend on a single parameter τ, which can be related to t_0 by relation (11).

Finally, the two parameters of the approximation proposed by us turn out to be sufficient to describe both the form of the cyclic curve (i.e., the activity level behavior) and resulting evolution of characteristics of the latitudinal sunspot distribution (and, as a result, the “butterfly diagram” for the given cycle).

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