ISSN 0016-7932, Geomagnetism and Aeronomy, 2018, Vol. 58, No. 1, pp. 113–118. © Pleiades Publishing, Ltd., 2018.
Original Russian Text © V.N. Morozov, 2018, published in Geomagnetizm i Aeronomiya, 2018, Vol. 58, No. 1, pp. 119–124.

Penetration of Nonstationary Ionospheric Electric Fields into Lower Atmospheric Layers in the Global Electric Circuit Model

V. N. Morozov

Voeikov Main Geophysical Observatory, St. Petersburg, 194018 Russia

e-mail: vn.morozov@inbox.ru

Received July 28, 2016; in final form, January 16, 2017

Abstract—The problem of the penetration of nonstationary ionospheric electric fields into the lower atmospheric layers is considered based on the model of the global electric circuit in the Earth's atmosphere. For the equation of the electric field potential, a solution that takes into account exponential variation in the electrical conductivity with height has been obtained. Analysis of the solution made it possible to reveal three cases of the dependence of the solution on height. The first case (the case of high frequencies) corresponds to the Coulomb approximation, when the electrical conductivity of the atmosphere can be neglected. In the case of low frequencies (when the frequency of changes in the ionosphere potential is less than the quantity reciprocal to the time of electric relaxation of the atmosphere), a quasi-stationary regime, in which the variation in the electric potential of the atmosphere is determined by the electric conduction currents, occurs. In the third case, due to the increase in the electrical conductivity of the atmosphere, two spherical regions appear: with the Coulomb approximation in the lower region and conduction currents in the upper one. For these three cases, formulas for estimating the electric field strength near the Earth's surface have been obtained.

DOI: 10.1134/S0016793217050140

1. INTRODUCTION

The problem of the penetration of ionospheric electric fields into the lower atmospheric layers was considered in different approximations in a series of works by Russian and foreign investigators. For example, in (Roble and Hays, 1979), this problem was considered in a spherical coordinate system with the origin in the Earth's center in the stationary approximation. In this case, ionospheric fields originated from an ionospheric generator that arose due to solar and lunar tides. In (Morozov and Troshichev, 2008), the transformation of ionospheric electric fields caused by the action of the magnetospheric generator that arose during the interaction between the solar wind and magnetic field of the Earth was considered.

At the same time, it is interesting to consider a nonstationary variant of the problem. For example, in (Morozov, 2012), the nonstationary problem of the penetration of ionospheric electric fields into lower atmospheric layers was considered. The electric potential distribution at the ionosphere level was specified and, in the one-dimensional case, an expression for the electric field strength near the Earth's surface was obtained; from that expression, it followed that the stationary state was established with a time delay with respect to the establishment of this state in the ionosphere.

In this paper, the three-dimensional problem of the distribution of the electric field potential in the atmosphere with exponentially increasing electrical conductivity is considered under the assumption of the harmonic time variation of the potential at the lower boundary of the ionosphere. The obtained analytical solution at different frequency intervals is used to find the vertical component of the electric field strength near the Earth's surface.

2. STATEMENT AND SOLUTION OF THE PROBLEM

To study the problem of the penetration of nonstationary electric fields into the lower atmospheric layers, we proceed from the following equation presented in (Morozov, 2005):

$$\frac{1}{4\pi}\frac{\partial}{\partial t}\Delta\varphi + \nabla(\lambda\nabla\varphi) = 0, \tag{1}$$

where φ is the potential of the electric field of the atmosphere; λ is its electrical conductivity; and t is time.

We assume that the electrical conductivity of the atmosphere increases by the exponential law

$$\lambda = \lambda_0 e^{\alpha(r-R)}, \tag{2}$$

where R is the Earth's radius; λ_0 is the electrical conductivity near the Earth's surface; and $\alpha = (0.2–0.3)\ \text{km}^{-1}$.

Table 1. Dependence of the transition boundary height h on frequency ω ($\alpha = 0.2$ km, $\tau_0 = (4\pi\lambda_0)^{-1} = 250$ s)

T, s	0.01	0.1	1	10	100	1570
$\omega = 2\pi/T$, s^{-1}	628	62.8	6.28	0.628	0.0628	0.004
$h(T) = h(\omega)$, km	60	50	40	20	10	0

Let us write Eq. (1) in a spherical coordinate system (r, θ, ϕ) with the origin in the center of the Earth:

$$\left[\frac{1}{4\pi}\frac{\partial}{\partial t} + \lambda(r)\right]\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\varphi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\right.$$
$$\left.\times\left(\sin\theta\frac{\partial\varphi}{\partial\theta}\right) + \frac{1}{r^2\sin\theta}\frac{\partial^2\varphi}{\partial\phi^2}\right] + \alpha\lambda(r)\frac{\partial\varphi}{\partial r} = 0. \quad (3)$$

We assume that, at the lower boundary of the ionosphere at $r = R + H$ (H is the lower ionosphere height reckoned from the Earth's surface), the electric field potential varies by the harmonic law

$$\varphi(r = R + H, \theta, \phi, t) = e^{i\omega t}\overline{\varphi}(r = R + H, \theta, \phi),$$
$$\varphi(r = R, \theta, \phi, t) = 0. \quad (4)$$

In Appendix A, the solution of Eq. (3) at boundary conditions (4) is considered. Using the results of Appendix A, let us consider some approximations following from solutions (A.19) and (A.20). In the limiting case $k^2/\alpha^2 \ll 1$, for the solution (A.19), we obtain the representation

$$\overline{\varphi}_{ij}(\omega, z) = \frac{C_{1,ij}}{(-i\omega\tau_0)^{k/\alpha}}e^{kz} + \frac{C_{2,ij}}{(-i\omega\tau_0)^{-k/\alpha}}e^{-kz}. \quad (5)$$

Here, it is assumed that $i(i + 1) \ll \alpha^2R^2 = (1.64 - 3.69) \times 10^6$.

Using boundary conditions (A.17), we obtain the following solution of the problem for frequencies satisfying the condition $4\pi\lambda_0e^{\alpha z} < \omega$

$$\varphi(z, \theta, \phi, t) = \overline{\varphi}(H, \theta, \phi)\frac{z}{H}\exp i\omega t. \quad (6)$$

From (6), we have an expression for the vertical component of the electric field strength

$$E_z = -\frac{\partial\varphi(z, \theta, \phi)}{\partial z} = -\overline{\varphi}(H, \theta, \phi)/H\exp i\omega t. \quad (7)$$

For frequencies satisfying the condition $\omega < 4\pi\lambda_0e^{\alpha z}$, using solution (A.20), we obtain

$$\overline{\varphi}_{ij}(\omega, z) = C_{3,ij}\ln(1 + i\omega\tau_0e^{-\alpha z}) + C_{4,ij}. \quad (8)$$

Using boundary conditions (A.17), we obtain the following solution of Eq. (A.16):

$$\varphi(z, \theta, \phi, t) = \overline{\varphi}(H, \theta, \phi)$$
$$\times\frac{\ln(1 + i\omega\tau_0e^{-\alpha z}) - \ln(1 + i\omega\tau_0)}{\ln(1 + i\omega\tau_0e^{-\alpha H}) - \ln(1 + i\omega\tau_0)}\exp i\omega t. \quad (9)$$

For the vertical component of the electric field strength, we obtain

$$E_z = -\overline{\varphi}(H, \theta, \phi)\alpha\frac{i\omega\tau_0e^{-\alpha z}}{(1 + i\omega\tau_0e^{-\alpha z})[\ln(1 + i\omega\tau_0e^{-\alpha H}) - \ln(1 + i\omega\tau_0)]}\exp i\omega t. \quad (10)$$

Note that, at $|i\omega\tau_0| \ll 1$, from (9) and (10) we obtain formulas for the quasi-stationary case:

$$\varphi(z, \theta, \phi, t) = \overline{\varphi}(H, \theta, \phi)\frac{1 - e^{-\alpha z}}{1 - e^{-\alpha H}}\exp i\omega t,$$
$$E_z = -\overline{\varphi}(H, \theta, \phi)\alpha\frac{e^{-\alpha z}}{1 - e^{-\alpha H}}\exp i\omega t. \quad (11)$$

Comparison of formulas (7) and (11) for the vertical component at $z = 0$ shows that the electric field strength in the quasi-stationary case is larger by a factor of αH (at $H = 80$ km and $\alpha = 0.3$ km^{-1}, we obtain $\alpha H = 24$) than the electric field strength determined by expression (7) and smaller by a factor of 24 than the electric field strength determined by expression (11).

Finally, let us consider the case where the condition $\left|\frac{4\pi\lambda_0e^{\alpha z}}{\omega}\right| < 1$, being satisfied in a certain variation range of z, goes over into the condition $\left|\frac{4\pi\lambda_0e^{\alpha z}}{\omega}\right| > 1$, at $z_* = h(\omega) = \alpha^{-1}\ln\omega\tau_0$. Let us estimate the quantity z_* for different frequencies by rewriting this expression in the following form: $z_* = h(T) = \alpha^{-1}\ln 2\pi\tau_0/T$, $\omega = 2\pi/T$, T is the period of oscillations; and $\tau_0 = (4\pi\lambda_0)^{-1}$ is the time of the electric relaxation near the Earth's surface, which is taken to be equal to 250 s.

As follows from the consideration of Table 1, solution (A.19) should be used for frequencies exceeding 628 s^{-1}; for frequencies less than 0.004, solution (A.20) should be used. For the intermediate case, it is necessary to consider regions separated by the height $h(T) = h(\omega)$ individually and to unite the obtained solutions.

Using the results of Appendix B, we obtain the following expression for $\varphi_{ij}(\omega, z)$ at $z > h(\omega)$:

Table 2. Dependence of the amplitude of the vertical component of the electric field strength E_z^0 on height $h(\omega)$ near the Earth's surface $E_z^0 = -\overline{\varphi}(R+H,\theta,\phi)\dfrac{1}{h(\omega)}$, $\overline{\varphi}(R+H,\theta,\phi) = 100$ kV

$h(\omega)$, km	60	50	40	30	10
E_z^0, V/m	–1.7	–2	–2.5	–3.3	–10

$$\varphi_{ij}(\omega,z) = [(-1)^{k/\alpha}C_{1,ij} + (-1)^{-k/\alpha}C_{2,ij}] \times [1-(1/2)\ln(1+i\omega\tau_0 e^{-\alpha z})]. \quad (12)$$

Relation (B.4), as well as the second condition (A.17), yield the following expression for the electric field potential at $z > |h(\omega)|$, $h(\omega) = \alpha^{-1}\ln i\omega\tau_0$:

$$\varphi(\omega,z,\theta,\phi) = \varphi(R+H,\theta,\phi)\frac{1-\frac{1}{2}\ln(1+e^{-\alpha(z-h(\omega))})}{1-\frac{1}{2}\ln(1+e^{-\alpha(H-h(\omega))})}\exp i\omega t. \quad (13)$$

In the region $z < h(\omega)$ and $kz \ll 1$, $kh(\omega) \ll 1$ we obtain

$$\varphi(\omega,z,\theta,\phi) = \overline{\varphi}(R+H,\theta,\phi) \times \frac{z}{h(\omega)[1-(1/2)\ln(1+e^{-\alpha(H-h(\omega))})]}\exp i\omega t. \quad (14)$$

Estimates at $H > h(\omega)$ for denominators of (13) and (14) yield the following expressions for the electric field potential in these regions:

$$\varphi(\omega,z,\theta,\phi) = \overline{\varphi}(R+H,\theta,\phi)\frac{z}{h(\omega)}\exp(i\omega t), \quad z < h(\omega),$$
$$\varphi(\omega,z,\theta,\phi) = \overline{\varphi}(R+H,\theta,\phi) \times [1-(1/2)\ln(1+e^{-\alpha(z-h(\omega))})]\exp i\omega t, \quad z > h(\omega). \quad (15)$$

For the vertical component of the electric field strength, using the first expression of (15), we obtain

$$E_z(\omega,z,\theta,\phi) = -\frac{\overline{\varphi}(R+H,\theta,\phi)}{h(\omega)}\exp i\omega t. \quad (16)$$

Thus, the obtained expressions for the electric field potential and electric field strength differ from expressions (6) and (7) in that, for frequencies that are lower than 628 s^{-1} but higher than 0.004 s^{-1}, the ionosphere height H is replaced by the height $h(\omega)$, which is the boundary between the region with small values of electric relaxation times and region where the effect of the electrical conductivity is small. For frequencies that are higher than 628 s^{-1}, the frequency $h(\omega)$ is close to the height H.

Table 2 presents the calculation results for the amplitude of the vertical component of the electric field strength near the Earth's surface with respect to the transition boundary height $h(\omega)$ which is a function of the oscillation frequency of the electric field potential at the upper boundary. As seen from Tables 1 and 2, the height $h(\omega)$ decreases with a decrease in the frequency and the electric field strength amplitude increases. In the limiting case of low frequencies with a typical time period on the order of 1 h, estimates by formula (11) yield $E_z^0 = -(20\text{–}30)$ V/m at $\alpha = (0.2\text{–}0.3)$ km^{-1}, i.e., low-frequency oscillations of the ionosphere potential effectively penetrate the ground layer of the atmosphere and, at $\bar{\phi}(R+H,\theta,\varphi) = 100$ kV, the amplitude of the vertical component of the electric field strength near the Earth's surface amounts to 20–30% of the quasi-stationary electric field equal to 100 V/m and determined by the action of thunderstorm generators (Morozov, 2005).

3. DISCUSSION

Thus, the calculations performed above yield expressions for the vertical component of the electric field strength near the Earth's surface and estimates for frequency intervals:

$$E_r = -\frac{\partial\overline{\varphi}}{\partial r} = -\varphi(R+\infty)/R, \quad \omega \gg \frac{R\alpha}{2}\Big/\tau_0, \quad (17)$$

$$E_z = -\frac{\partial\varphi(z,\theta,\phi)}{\partial z} = -\overline{\varphi}(H,\theta,\phi)/H\exp i\omega t, \quad \omega > 1/\tau_0, \quad (18)$$

$$E_z = -\overline{\varphi}(H,\theta,\phi)\alpha\frac{i\omega\tau_0 e^{-\alpha z}}{(1+i\omega\tau_0 e^{-\alpha z})[\ln(1+i\omega\tau_0 e^{-\alpha H}) - \ln(1+i\omega\tau_0)]}\exp i\omega t, \quad \omega < 1/\tau_0, \quad (19)$$

$$E_z = -\overline{\varphi}(H,\theta,\phi)\alpha\frac{e^{-\alpha z}}{1-e^{-\alpha H}}\exp i\omega t, \quad \omega \ll 1/\tau_0. \quad (20)$$

In the case of taking into account the transition from the nonconducting region to the conducting one, we obtain at $z_* = h(\omega)$

$$E_z = -\overline{\varphi}(R+H,\theta,\phi)\frac{\exp i\omega t}{h(\omega)}, \quad \omega \gg 1/\tau_0, \quad z < h(\omega). \quad (21)$$

These expressions can be used to estimate the vertical component of the electric field strength in the above-mentioned frequency intervals.

4. CONCLUSIONS

(1) The solution of the problem about the penetration of nonstationary ionospheric electric fields into lower atmospheric layers has been obtained in an electrostatic approximation with allowance for the exponentially increasing electrical conductivity of the atmosphere. Analysis of the obtained solutions describing the distribution of the electric field potential in the atmosphere with respect to the oscillation frequency of the electric field potential at the ionosphere level revealed three cases for the dependence of the solution on height. The first case (the case of high frequencies) corresponds to the Coulomb approximation, when the electrical conductivity of the atmosphere can be neglected. In the case of low frequencies (when the frequency of changes in the ionosphere potential is less than the quantity reciprocal to the time of electric relaxation of the atmosphere), a quasi-stationary regime, in which the variation in the electric potential of the atmosphere is determined by the electric conduction currents, occurs. In the third case, due to the increase in the electrical conductivity of the atmosphere, two spherical regions appear: with the Coulomb approximation in the lower region and conduction currents in the upper one.

(2) The penetration of the ground layer by ionospheric nonstationary electric fields significantly depends on the oscillation frequency of the electric field potential that arises at the ionosphere level. In the case of high-frequency oscillations, the amplitude of the vertical component of the electric field strength near the Earth's surface is less than 1% of the value corresponding to the quasi-stationary value of the electric field strength (100 V/m). For low-frequency oscillations of the ionosphere potential, this quantity amounts to 20–30% of the quasi-stationary electric field at a given value of the ionosphere potential $\bar{\varphi}(R+H,\theta,\phi) = 100$ kV.

APPENDIX A

The solution of Eq. (3) under boundary conditions (4) is sought in the following form:

$$\varphi(r,\theta,\phi,t) = e^{i\omega t}\bar{\varphi}(r,\theta,\phi). \quad \text{(A.1)}$$

Substituting (A.1) into (3), we obtain the equation for determining the function $\bar{\varphi}(r,\theta,\phi)$:

$$\left[\frac{i\omega}{4\pi} + \lambda(r)\right]\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\bar{\varphi}}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta} \times\left(\sin\theta\frac{\partial\bar{\varphi}}{\partial\theta}\right) + \frac{1}{r^2\sin\theta}\frac{\partial^2\bar{\varphi}}{\partial\phi^2}\right] + \alpha\lambda(r)\frac{\partial\bar{\varphi}}{\partial r} = 0. \quad \text{(A.2)}$$

Assuming that spatial variations at the lower boundary of the ionosphere are determined by spherical harmonics, we represent the solution of Eq. (6) as an expansion in spherical harmonics $Y_{ij}(\theta,\phi)$ (Jackson, 1965)

$$\bar{\varphi}(r,\theta,\phi) = \sum_{i=0}^{\infty}\sum_{j=-i}^{i}\bar{\varphi}_{ij}(r)Y_{ij}(\theta,\phi), \qquad \bar{\varphi}_{ij}(r) = \int d\Omega\bar{\varphi}(r,\theta,\phi)Y_{ij}^{*}(\theta,\phi) \quad \text{(A.3)}$$

and obtain the following equation for determining components of the function $\bar{\varphi}_{ij}(r)$ entering into expansion (7)

$$\frac{d^2\bar{\varphi}_{ij}}{dr^2} + \left(\frac{2}{r} + \frac{\alpha\lambda_0 e^{\alpha(r-R)}}{\lambda_0 e^{\alpha(r-R)} + \frac{i\omega}{4\pi}}\right)\frac{d\bar{\varphi}_{ij}}{dr^2} - \frac{\mu\bar{\varphi}_{ij}}{r^2} = 0, \quad \text{(A.4)}$$

where $\mu = i(i+1)$ and $i = 0,1,2,\ldots n$.

For the solution of Eq. (A.4), let us first consider the case satisfying the condition $\left|\frac{4\pi\lambda_0 e^{\alpha(r-R)}}{i\omega}\right| \ll \frac{2}{\alpha R} \approx 1.5\times10^{-3}$ at $\alpha = 0.2$ km^{-1} and $R = 6400$ km. If the condition is satisfied, Eq. (A.4) is reduced to the equation

$$\frac{d^2\bar{\varphi}_{ij}}{dr^2} + \frac{2}{r}\frac{d\bar{\varphi}_{ij}}{dr^2} - \frac{\mu\bar{\varphi}_{ij}}{r^2} = 0. \quad \text{(A.5)}$$

The solution of Eq. (A.5) is written in the following form:

$$\bar{\varphi}_{ij}(r) = C_{1,ij}r^{m_1} + C_{2,ij}r^{m_2} = C_{1,ij}r^{\frac{\sqrt{1+4\mu}-1}{2}} + C_{2,ij}r^{-\frac{\sqrt{1+4\mu}+1}{2}}, \quad \text{(A.6)}$$

where $C_{1,\,ij}$ and $C_{2,\,ij}$ are constants.

For the determination of the constants, we use the following boundary conditions:

$$\bar{\varphi}_{ij}(R) = 0, \quad \bar{\varphi}_{ij}(r = R+H) = \bar{\varphi}_{ij}^{0}(R+H). \quad \text{(A.7)}$$

Using the first boundary condition, we obtain for expression (A.6) the representation

$$\bar{\varphi}_{ij}(r) = C_{1,ij}(r^{(\sqrt{1+4\mu}-1)/2} - R^{\sqrt{1+4\mu}}/r^{(\sqrt{1+4\mu}+1)//2}). \quad \text{(A.8)}$$

Let us consider two cases for the solution representation (A.8). Let $r\in[R,\infty)$; it then follows from physical properties of the solution that $C_{1,\,ij} = 0$ for $i \neq 0$, $\mu \neq 0$ and $C_{2,\,ij} = 0$; therefore, $\bar{\varphi}_{ij}(r) = 0$. In a particular case $\mu = 0$ at $i = 0$ we have

$$\bar{\varphi}_{00}(r) = C_{1,00} + C_{2,00}\frac{1}{r}. \quad \text{(A.9)}$$

Using boundary conditions (A.7) at $H = \infty$, we obtain the following expression for $\bar{\varphi}_{00}(r)$:

$$\bar{\varphi}_{00}(r) = \bar{\varphi}_{00}^{\infty}\left(1 - \frac{R}{r}\right). \quad \text{(A.10)}$$

The quantity $\bar{\varphi}_{00}^{\infty}$ is determined by the expression

$$\overline{\varphi}_{00}^{\infty} = \frac{1}{\sqrt{4\pi}}\int_0^{\pi}\sin\theta d\theta\int_0^{2\pi}d\phi\varphi(R+\infty,\theta,\phi). \quad \text{(A.11)}$$

For the spatial component of the electric field potential in the atmosphere, we obtain

$$\overline{\varphi}(r,\theta,\phi) = \frac{1}{4\pi}\left(1-\frac{R}{r}\right)\int_0^{\pi}\sin\theta d\theta\int_0^{2\pi}d\phi\varphi(R+\infty,\theta,\phi). \quad \text{(A.12)}$$

In the case where the expression under the integral sign does not depend on angles, we obtain instead of (A.12) the expression

$$\overline{\varphi}(r,\theta,\phi) = \varphi(R+\infty)\left(1-\frac{R}{r}\right). \quad \text{(A.13)}$$

This expression coincides with that obtained in (Morozov, 2005).

For the electric field strength near the Earth's surface, we obtain the expression

$$E_r = -\frac{\partial\overline{\varphi}}{\partial r} = -\varphi(R+\infty)/R. \quad \text{(A.14)}$$

Let us now turn to the case where the sphericity of the Earth's surface can be neglected. In this case, we have the equation

$$\frac{d^2\overline{\varphi}_{ij}}{dr^2} + \frac{\alpha\lambda_0 e^{\alpha(r-R)}}{\lambda_0 e^{\alpha(r-R)} + \frac{i\omega}{4\pi}}\frac{d\overline{\varphi}_{ij}}{dr} - k^2\overline{\varphi}_{ij} = 0, \quad k^2 = \frac{\mu}{R^2}. \quad \text{(A.15)}$$

Making a change of variables $z = r - R$ in (A.15), we obtain the equation

$$\frac{d^2\overline{\varphi}_{ij}}{dz^2} + \frac{4\pi\alpha\lambda_0 e^{\alpha z}}{4\pi\lambda_0 e^{\alpha z} + i\omega}\frac{d\overline{\varphi}_{ij}}{dz} - k^2\overline{\varphi}_{ij} = 0, \quad k^2 = \frac{\mu}{R^2}. \quad \text{(A.16)}$$

The boundary conditions for the solution of Eq. (A.16) have the form

$$\overline{\varphi}_{ij}(z=0) = 0, \quad \overline{\varphi}_{ij}(z=H) = \overline{\varphi}_{ij}^0. \quad \text{(A.17)}$$

To obtain the solution of Eq. (A.16) under boundary conditions (A.17), we transform this equation by introducing the substitution of variables $u = -\frac{4\pi\lambda_0 e^{\alpha z}}{i\omega}$. Then, Eq. (A.16) is transformed to the following:

$$\frac{d^2\overline{\varphi}_{ij}}{du^2} + \left(\frac{1}{u}+\frac{1}{u-1}\right)\frac{d\overline{\varphi}_{ij}}{du} - \frac{k^2}{\alpha^2 u^2}\overline{\varphi}_{ij} = 0. \quad \text{(A.18)}$$

The solution of Eq. (A.18) is expressed in terms of a function and is represented in the following form when $\left|\frac{4\pi\lambda_0 e^{\alpha z}}{\omega}\right| < 1$ (Kamke, 1971):

$$\begin{aligned}\overline{\varphi}_{ij}(\omega,z) &= \left(-\frac{e^{\alpha z}}{i\omega\tau_0}\right)^{\overline{\alpha}_1}\Bigg[C_{1,ij}F\left(\overline{\alpha}_2,\overline{\beta}_2,\gamma_1,-\frac{e^{\alpha z}}{i\omega\tau_0}\right) \\ &+ C_{2,ij}\left(-\frac{e^{\alpha z}}{i\omega\tau_0}\right)^{1-\gamma_1}F\left(\overline{\alpha}_2-\gamma_1+1,\ \overline{\beta}_2 - \gamma_1+1,\ 2-\gamma_1,-\frac{e^{\alpha z}}{i\omega\tau_0}\right)\Bigg], \quad \overline{\alpha}_1 = \frac{k}{\alpha}, \\ \overline{\alpha}_2 &= \frac{k}{\alpha} + \frac{1+\sqrt{1+4k^2/\alpha^2}}{2}, \\ \overline{\beta}_2 &= \frac{k}{\alpha} + \frac{1-\sqrt{1+4k^2/\alpha^2}}{2}, \quad \gamma_1 = 1+2k/\alpha, \\ \tau_0 &= (4\pi\lambda_0)^{-1}.\end{aligned} \quad \text{(A.19)}$$

For $|u| > 1$, making a change of variables $u = \frac{1}{u'}$, we obtain the following solution:

$$\begin{aligned}\overline{\varphi}_{ij}(\omega,z) &= \left(-\frac{i\omega\tau_0}{e^{\alpha z}}\right)^{\alpha_1}C_{3,ij}F\left(\overline{\alpha},\overline{\beta},\gamma,-\frac{i\omega\tau_0}{e^{\alpha z}}\right) \\ &+ C_{4,ij}\left(\frac{i\omega\tau_0}{e^{\alpha z}}\right)^{1-\gamma+\alpha_1}F\left(\overline{\alpha}-\gamma+1,\ \overline{\beta}-\gamma+1,\ 2-\gamma,-\frac{i\omega\tau_0}{e^{\alpha z}}\right)\Bigg], \quad \alpha_1 = \frac{1+\sqrt{1+4k^2/\alpha^2}}{2}, \\ \overline{\alpha} &= \frac{k}{\alpha} + \frac{1+\sqrt{1+4k^2/\alpha^2}}{2}, \quad \overline{\beta} = -\frac{k}{\alpha} + \frac{1+\sqrt{1+4k^2/\alpha^2}}{2}, \quad \gamma = 1+\sqrt{1+4k^2/\alpha^2}.\end{aligned} \quad \text{(A.20)}$$

The hypergeometric function $F(\alpha',\beta,\gamma,x)$ is defined by the expression

$$F(\alpha',\beta,\gamma,x) = 1 + \sum_{k=1}^{\infty}\frac{\alpha'(\alpha'+1)\ldots(\alpha'+k-1)\beta(\beta+1)\ldots(\beta+k-1)}{k!\gamma(\gamma+1)\ldots(\gamma+k-1)}x^k. \quad \text{(A.21)}$$

APPENDIX B

To consider the transition of solution (A.19) from the region $u < 1$ to the region $u > 1$, we use the formula (Bateman and Erdélyi, 1984)

$$\begin{aligned}F(a,b,c,u) &= \frac{\Gamma(c)\Gamma(b-a)}{\Gamma(b)\Gamma(c-a)}(-u)^{-a} \times F\left(a,1+a-c,1+a-b,\frac{1}{u}\right) \\ &+ \frac{\Gamma(c)\Gamma(a-b)}{\Gamma(a)\Gamma(c-b)}(-u)^{-b} \times F\left(b,1+b-c,1+b-a,\frac{1}{u}\right),\end{aligned} \quad \text{(B.1)}$$

where $\Gamma(x)$ is the gamma function.

Solution (A.19) can be written in the following form:

$$\varphi_{ij}(\omega,z) = \left(-\frac{1}{i\omega\tau_0}\right)^{k/\alpha} e^{kz} C_{1,ij} F(\alpha_1 + k/\alpha, \bar{\alpha}_1 + k/\alpha, 1 + 2k/\alpha, -\frac{1}{i\omega\tau_0}e^{\alpha z}\Big) + \left(-\frac{1}{i\omega\tau_0}\right)^{-(k/\alpha)} e^{-kz} C_{2,ij} F(\alpha_1 - k/\alpha, \tilde{\alpha} - k/\alpha, 1 - 2k/\alpha, -\frac{1}{i\omega\tau_0}e^{\alpha z}\Big), \quad \text{(B.2)}$$

$$\alpha_1 = \frac{1 + \sqrt{1 + 4k^2/\alpha^2}}{2}, \quad \tilde{\alpha}_1 = \frac{1 - \sqrt{1 + 4k^2/\alpha^2}}{2}.$$

Using (B.1) and (B.2), we obtain a solution in the region $z > |h(\omega)|$, $h(\omega) = \alpha^{-1}\ln i\omega\tau_0$:

$$\varphi_{ij}(\omega,z) = F(\alpha_1 + k/\alpha, \alpha_1 - k/\alpha, 1 + \alpha_1 - \tilde{\alpha}_1, -i\omega\tau_0 e^{-\alpha z})\Bigg[(-1)^{k/\alpha}\frac{\Gamma(1+2k/\alpha)\Gamma(\tilde{\alpha}_1 - \alpha_1)}{\Gamma^2(\tilde{\alpha}_1 + k/\alpha)} \times \left(\frac{1}{i\omega\tau_0}e^{\alpha z}\right)^{-\alpha_1} C_{1,ij} + (-1)^{-(k/\alpha)}\frac{\Gamma(1-2k/\alpha)\Gamma(\tilde{\alpha}_1 - \alpha_1)}{\Gamma^2(\tilde{\alpha}_1 - k/\alpha)} \times \left(\frac{1}{i\omega\tau_0}e^{\alpha z}\right)^{\alpha_1} C_{2,ij}\Bigg] + F(\tilde{\alpha}_1 + k/\alpha, \tilde{\alpha}_1 - k/\alpha, 1 + \tilde{\alpha}_1 - \alpha_1, -i\omega\tau_0 e^{-\alpha z}) \times\Bigg[(-1)^{k/\alpha}\frac{\Gamma(1+2k/\alpha)\Gamma(\alpha_1 - \tilde{\alpha}_1)}{\Gamma^2(\alpha_1 + k/\alpha)}\left(\frac{1}{i\omega\tau_0}e^{\alpha z}\right)^{-\tilde{\alpha}_1} C_{1,ij} + (-1)^{-(k/\alpha)}\frac{\Gamma(1-2k/\alpha)\Gamma(\alpha_1 - \tilde{\alpha}_1)}{\Gamma^2(\alpha_1 - k/\alpha)}\left(\frac{1}{i\omega\tau_0}e^{\alpha z}\right)^{-\tilde{\alpha}_1} C_{2,ij}\Bigg]. \quad \text{(B.3)}$$

The relation between the constants $C_{1,\,ij}$ and $C_{2,\,ij}$ is found using the first boundary condition (A.17). It has the following form:

$$C_{2,ij} = -\frac{(-i\omega\tau_0)^{-(k/\alpha)} F\left(\alpha_1 + k/\alpha,\ \tilde{\alpha}_1 + k/\alpha,\ 1 + 2k/\alpha, -\frac{1}{i\omega\tau_0}e^{\alpha z}\right)}{(-i\omega\tau_0)^{k/\alpha} F\left(\alpha_1 - k/\alpha,\ \tilde{\alpha}_1 - k/\alpha,\ 1 - 2k/\alpha, -\frac{1}{i\omega\tau_0}e^{\alpha z}\right)} C_{1,ij}. \quad \text{(B.4)}$$

Hypergeometric functions entering into (B.3) are calculated with approximate expressions for these functions for $k/\alpha \ll 1$ (Kamke, 1971):

$$F(\alpha_1 + k/\alpha, \alpha_1 - k/\alpha, 1 + \alpha_1 - \tilde{\alpha}_1, -i\omega\tau_0 e^{-\alpha z}) \approx \frac{\ln(1 + i\omega\tau_0 e^{-\alpha z})}{i\omega\tau_0 e^{-\alpha z}}, \quad \text{(B.5)}$$

$$F(\tilde{\alpha}_1 + k/\alpha, \tilde{\alpha} - k/\alpha, 1 + \tilde{\alpha}_1 - \alpha_1, -i\omega\tau_0 e^{-\alpha z}) \approx 1 - \ln(1 + i\omega\tau_0 e^{-\alpha z}).$$

REFERENCES

Bateman, G. and Erdélyi, A., *Higher Transcendental Functions, Hypergeometric Function. Legendre Function, New York: McGraw Hill, 1953*; Moscow: Nauka, 1984.

Jackson, J., *Classical Electrodynamics*, New York: Wiley, 1962; Moscow: Mir, 1965.

Kamke, E., *Differentialgleichungen: Lösungsmethoden und Lösungen, Leipzig: Geest & Portig K.-G., 1959*; Moscow: Nauka, 1971.

Morozov, V. N., Model of the nonstationary electric field in the lower atmosphere, *Geomagn. Aeron. (Engl. Transl.)*, 2005, vol. 45, no. 2, pp. 253–262.

Morozov, V.N., Distribution of the electric field created by ionospheric generator in lower atmospheric layers, *Tr. Gl. Geofiz. Obs. im. A.I. Voeikova*, 2012, no. 565, pp. 205–215.

Morozov, V.N. and Troshichev, O.A., Simulation of variations in the polar atmospheric electric field related to the magnetospheric field-aligned currents, *Geomagn. Aeron. (Engl. Transl.)*, 2008, vol. 48, no. 6, pp. 727–736.

Roble, R.G. and Hays, P.B., A quasi-static model of global atmospheric electricity. II. Electrical coupling between the upper and lower atmosphere, *J. Geophys. Res.*, 1979, vol. 84, no. A12, pp. 7247–7256.

Translated by A. Nikol'skii