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On the Conjugacy of Measurable Partitions with Respect to the Normalizer of a Full Type II_1 Ergodic Group

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Dedicated to the memory of Anatoly Moiseevich Vershik

Abstract. Let G be a countable ergodic group of automorphisms of a measure space (X,μ) and $\mathcal{N}[G]$ be the normalizer of its full group $[G]$. Problem: for a pair of measurable partitions ξ and η of the space X, when does there exist an element $g \in \mathcal{N}[G]$ such that $g\xi = \eta$? For a wide class of measurable partitions, we give a solution to this problem in the case where G is an approximately finite group with finite invariant measure. As a consequence, we obtain results concerning the conjugacy of the commutative subalgebras that correspond to ξ and η in the type II_1 factor constructed via the orbit partition of the group G.

Key words: automorphisms of measurable space, orbit partitions, measurable partition, full group, normalizer, von Neumann factor.

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Introduction

The orbit theory of dynamical systems, which emerged in the late 1960s, had long been the focus of A. M. Vershik's seminar where Anatoly Moiseevich gave a new transparent proof (see [2], [5]) of H. Dye's theorem [9] on the orbit isomorphism of ergodic actions of the group $\mathbb{Z}$ and other groups. At the same seminar, the authors of the present paper were introduced to the connection between orbit theory and the theory of von Neumann algebras.

Let $(X,\mathcal{F},\mu)$ be a Lebesgue space and let $\mathcal{A}(X)$ be the group of all measurable transformations of the measurable space $(X,\mathcal{F})$ that leave the measure μ quasi-invariant. According to the general theory of measurable partitions constructed by V. A. Rokhlin [17], [18], two partitions ξ_1 and ξ_2 are said to be isomorphic if there exists a measure-preserving automorphism $g \in \mathcal{A}(X)$ such that $g\xi_1 = \xi_2$. In particular, according to his Classification Theorem, any two partitions with continuous conditional measures and factor measures are isomorphic.

Let G be a countable ergodic subgroup of $\mathcal{A}(X)$, $[G]$ be the full group of the group G, and $\mathcal{N}[G] = \{g \in \mathcal{A}(X) \mid g[G] = [G]g\}$ be the normalizer of $[G]$ (the properties of normalizers of the full group were studied in [1], [4], [10], [14]). We use the notation $\mathcal{N}[G]$ even though the full group $[G]$ (and hence its normalizer) depend only on the orbit partition of the group. The further elaboration could be reformulated in terms of the corresponding measurable equivalence relation, but this would lengthen the text.

In this paper, we consider the following problem: when are two measurable partitions ξ_1 and ξ_2 of the space X conjugate with respect to the group $\mathcal{N}[G]$ associated to the orbit partition $\theta = \theta(G)$? That is, when does there exist an element $g \in \mathcal{N}[G]$ such that $g\xi_1 = \xi_2$ and $g\theta = \theta$? Thus, we replace the general group $\mathcal{A}(X)$ with a narrower special group $\mathcal{N}[G]$.

We distinguish a broad class of measurable partitions, which we call properly located with respect to $[G]$. For such partitions, we obtain, in a certain sense, a complete solution to this problem in the case where $[G]$ is an approximately finite (a. f.) type II_1 group. Our first main result (Theorems 3.1 and 3.2) shows that, in the above case, the conjugacy problem for properly located partitions is equivalent to the general problem of measure-preserving orbit isomorphisms for groups

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of arbitrary type. In the proof given here, we use the technique of connected partitions developed by A. L. Fedorov [8], [21], [22].

The considered conjugacy problem for countable partitions with respect to the normalizer of an ergodic group is closely related to the conjugacy problem for some commutative subalgebras in a factor. Indeed, to each countable ergodic group, by the well-known construction [1], [13], there corresponds a factor $\mathcal{M}_G$ coinciding with the crossed product $W^*(G, L^\infty(X))$ if G acts freely.

Let j be a canonical embedding $L^\infty(X)$ in $\mathcal{M}_G$ and $\mathcal{M}_G^0 = j(L^\infty(X))$. This subalgebra is a maximal commutative subalgebra in $\mathcal{M}_G$ and is regular, i. e., together with its normalizer, it generates $\mathcal{M}_G$. Such subalgebras are usually called *Cartan* subalgebras.

In the case where the dynamical system (X, μ, G) has a discrete spectrum, Cartan subalgebras have been studied in detail in [15].

To every measurable partition ξ of the space X, there corresponds a commutative subalgebra $\mathcal{M}_G^0(\xi) = j(L^\infty(\xi))$, where $L^\infty(\xi) = L^\infty(X, \xi)$ is the subalgebra of all ξ-measurable functions from $L^\infty(X)$, and j is the canonical embedding of $L^\infty(X)$ in $\mathcal{M}_G$. It is easy to check that if measurable partitions ξ_1 and ξ_2 are conjugate with respect to $\mathcal{N}[G]$, then the subalgebras $\mathcal{M}_G^0(\xi_1)$ and $\mathcal{M}_G^0(\xi_2)$ are conjugate in $\mathcal{M}_G$, i. e., there exists an automorphism $\sigma \in \operatorname{Aut}(\mathcal{M}_G)$ such that $\sigma(\mathcal{M}_G^0(\xi_1)) = \mathcal{M}_G^0(\xi_2)$.

The main question that arises here is whether the conjugacy of $\mathcal{M}_G^0(\xi_1)$ and $\mathcal{M}_G^0(\xi_2)$ in $\mathcal{M}_G$ implies the conjugacy of the partitions ξ_1 and ξ_2 with respect to $\mathcal{N}[G]$. It is clear that the problem is, in fact, whether one can choose an automorphism that conjugates $\mathcal{M}_G^0(\xi_1)$ and $\mathcal{M}_G^0(\xi_2)$ in such a way that it leaves the Cartan subalgebra $\mathcal{M}_G^0 = j(L^\infty(X))$ invariant.

The second main result of the paper is Theorem 4.5. It states that for the considered class of partitions the above problem is solved in the affirmative.

Whether the same result holds in the general case is unknown to the authors.

Contents. In § 1, we give notation and terminology used hereafter.

In § 2, we study proper pairs of the form (θ, ξ) where $\theta = \theta(G)$ is the orbit partition of the group G and the partition ξ is measurable and properly located with respect to $[G]$. For such pairs, the factor partition θ/ξ is correctly defined in the factor space X/ξ and also the proper pairs $(\widetilde{\theta}, \widetilde{\xi})$ obtained by countable multiplication of the pair (θ, ξ) are studied.

Using the Connes–Feldman–Weiss theorem [3], it is shown that for a proper pair (θ, ξ), the approximate finiteness of the group $[\theta]$ (see § 1.1 for its definition) is equivalent to the approximate finiteness of the partitions $\theta \vee \xi$ and θ/ξ.

In § 3, we consider proper pairs (θ, ξ) where the group $[\theta]$ is an a. f. group of type II_1 and the partition ξ is continuous. The classification and existence theorems are proved.

In § 4, the problem of conjugacy of subalgebras of $\mathcal{M}_G(\xi)$ in the factor $\mathcal{M}_G$ is considered.

A detailed discussion of various issues concerning orbit theory, equivalence relations, and their connection with von Neumann factor theory is available in the reviews [7], [16].

1. Preliminary Information

1.1. Full Groups and Orbit Partitions. We use the notation and terminology from [17], [12].

Let $(X_i, \mathcal{F}_i, m_i)$, $i = 1, 2$, be Lebesgue spaces with finite or σ-finite measures. By an *isomorphism* $S \colon X_1 \to X_2$, we mean an isomorphism of measurable spaces such that the measure $Sm_1 = m_1 \circ S^{-1}$ is equivalent to the measure m_2. By $\mathcal{A}(X)$ we denote the group of all measurable reversible transformations of the measurable space $(X, \mathcal{F})$, and by $\mathcal{A}(X, m)$ its subgroup $\{S \in \mathcal{A}(X) \mid Sm = m\}$.

Let G be a countable subgroup in $\mathcal{A}(X)$ and let $\theta(G)$ denote the partition of X into its orbits $Gx = \{gx \mid g \in G\}$, $x \in X$. If θ is some partition of X, then we denote by $[\theta]$ the group of all elements of $\mathcal{A}(X)$ that leave θ fixed, i. e., $C \in \theta$, $S \in [\theta] \Longrightarrow S(C) = C$. If $\theta = \theta(G)$ for some countable subgroup in $\mathcal{A}(X)$, then we call θ an *orbit* partition. We call $[\theta(G)]$ the *full group* of the group G and denote it by $[G]$. The element of a partition θ containing an element $x \in X$ is denoted by $\theta(x)$; if $A \subset X$, then $\theta(A)$ is the smallest θ-set containing A. The relation $x \overset{\theta}{\sim} y$ means that $\theta(x) = \theta(y)$. If ξ, η are two measurable partitions, then their supremum $\xi \vee \eta$ is defined by the relation $x \overset{\xi\vee\eta}{\sim} y \Longleftrightarrow x \overset{\xi}{\sim} y$ and $x \overset{\eta}{\sim} y$, $x, y \in X$, and their intersection $\xi \cap \eta$ (non-measurable, in general) is the smallest enlargement of the partitions ξ and η.

If an ergodic group G admits a finite (respectively, infinite σ-finite) invariant measure, then G, as well as $[G]$ and $\theta(G)$, are said to be of type II_1 (type II_∞). If no finite G-invariant measure exists, then G is a group of type III.

By a *partial isomorphism* of X, we mean an isomorphism $V\colon A \to B$ where A and B are subsets in X of positive measure (more precisely, an isomorphism from $(A, \mathcal{F}\cap A, \mu|_A)$ to $(B, \mathcal{F}\cap B, \mu|_B)$). The sets A and B are called, respectively, the initial and the final domains of the partial isomorphism and they are denoted by $E(V)$ and $F(V)$. By $\mathcal{U}(X)$ we denote the set of all partial isomorphisms of X. For U and V from $\mathcal{U}(X)$, U^{-1} and $U \cdot V$ in the case where $m(F(V) \cap E(U)) > 0$ are defined in the obvious way.

For a partition θ of the space X, we introduce its *normalizer* $\mathcal{N}(\theta)$ as the set $\{S \in \mathcal{A}(X) \mid S\theta = \theta\}$ of transformations leaving the partition θ invariant. We also put

$$\mathcal{U}(\theta) = \big\{U \in \mathcal{U}(X) \mid Ux \overset{\theta}{\sim} x \text{ for almost all } x \in E(U)\big\},$$
$$\mathcal{UN}(\theta) = \big\{U \in \mathcal{U}(X) \mid U(\theta|_{E(U)}) = \theta|_{F(U)}\big\},$$

and for a countable subgroup $G \in \mathcal{A}(X)$, we set by definition:

$$\mathcal{N}[G] = \mathcal{N}(\theta(G)) = \{S \in \mathcal{A}(G) \mid S[G]S^{-1} = [G]\},$$
$$\mathcal{U}[G] = \mathcal{U}(\theta(G)), \qquad \mathcal{UN}[G] = \mathcal{UN}(\theta(G)).$$

For U and U_n, $n \in \mathbb{N}$, in $\mathcal{U}(X)$, we use the notation $U = \bigoplus_n U_n$ in the following situation:

$$E(U) = \bigcup_n E(U_n), \quad F(U) = \bigcup_n F(U_n), \qquad U|_{E(U_n)} = U_n,$$

where the sets $E(U_n), n \in \mathbb{N}$, and, respectively, $F(U_n), n \in \mathbb{N}$, are disjoint. Note that a transformation g is contained in $[G]$ if and only if it admits the representation $g = \oplus_n u_n$, where $u_n = g_n|_{E(U_n)}$ and $g_n \in G$.

If G is an ergodic type II_∞ group, m a G-invariant measure, and $S \in \mathcal{N}[G]$, then there exists a number $\operatorname{mod} S \in (0, +\infty)$ such that for almost all $x \in X$

$$\frac{dm(Sx)}{dm(x)} = \operatorname{mod} S.$$

By ε_X and ν_x we denote, respectively, the partition of X into distinct points and the trivial partition of X; e_X is the identity automorphism of X. The partition into ergodic components of the group G is denoted by $\omega(G)$; we also use the notation $\omega(\theta)$ if $\theta = \theta(G)$.

Let ξ be a measurable partition of X and let π_ξ be the canonical projection of X onto the factor space X/ξ supplied with the σ-algebra $\mathcal{F}/\xi = \{A \subset X/\xi \mid \pi_\xi^{-1} A \in \mathcal{F}\}$. Since the measure m on X is not assumed to be finite, the natural measure m_ξ^0 defined on the factor space $(X/\xi, \mathcal{F}/\xi)$ by the equality $m_\xi^0(A) = m(\pi_\xi^{-1} A), A \in \mathcal{F}/\xi$, is not σ-finite in general. Therefore, the factor measure m/ξ is hereafter understood as any σ-finite measure on $(X/\xi, \mathcal{F}/\xi)$ equivalent to the measure m_ξ^0.

For each factor measure m/ξ, there exists a unique $\operatorname{mod} 0$ system of conditional measures $C \mapsto m_C$, $C \in \xi$. This means that:

1) (C, m_C) are the Lebesgue spaces for almost all $C \in \xi$;
2) for any $A \in \mathcal{F}$
   (a) $(C \cap A, m_C)$ is m/ξ-measurable for almost all $C \in \xi$,
   (b) the function $C \mapsto m_C(A \cap C)$ is m_C-measurable for almost all $C \in \xi$,
   (c) $m(A) = \int_{X/\xi} m_C(A \cap C)\, dm/\xi(C)$.

In the case where the measure m is probabilistic, the measures m/ξ and m_C, $C \in \xi$, can also be chosen to be probabilistic, and in this case the factor measure m/ξ is uniquely defined: $m/\xi(A) = m(\pi_\xi^{-1}A)$, $A \in \mathcal{F}/\xi$.

The equivalence relation $\mathcal{R}_\theta$ with measure $\mu = \mu_\theta$ (measurable equivalence relation) corresponding to the orbit partition $\theta = \theta(G)$ of the countable subgroup $G = \mathcal{A}(X)$ is defined as follows. Let $\Gamma_g = \{(x, gx) \mid x \in X\}$ be the graph of $g \in G$ and $\mathcal{R}_\theta = \bigcup_g \Gamma_g$; the measure μ is uniquely defined by the property that for any $g \in G$, the set Γ_g is μ-measurable and the natural projection $(x, gx) \mapsto x$ of the space $(\Gamma_g, \mu|_{\Gamma_g})$ onto (X, μ) is a measure-preserving isomorphism. It is not difficult to check that such a definition $(\mathcal{R}_\theta, \mu_\theta)$ is mod 0 correct and does not depend on the choice of the countable group G having orbit partition θ; thus, $x \overset{\theta}{\sim} y \iff (x, y) \in \mathcal{R}_\theta$ for almost all x, y of X.

The canonical projections $r\colon \mathcal{R}_\theta \mapsto X$ and $s\colon \mathcal{R}_\theta \mapsto X$ defined by the equations

$$s(x, y) = x, \quad r(x, y) = y \qquad ((x, y) \in \mathcal{R}_\theta)$$

are measurable mappings. For the corresponding measurable partitions $\xi_{\mathrm{s}} = s^{-1}\varepsilon_X$ and $\xi_{\mathrm{r}} = r^{-1}\varepsilon_X$, the relations $\xi_{\mathrm{s}} \vee \xi_{\mathrm{r}} = \varepsilon_{\mathcal{R}_\theta}$ and, in the case where the group G is ergodic, $\xi_{\mathrm{s}} \wedge \xi_{\mathrm{r}} = \nu_{\mathcal{R}_\theta}$, hold.

If, under the natural identification of X with $\mathcal{R}_{\theta/\xi_\theta}$, we take m as a factor measure on $\mathcal{R}_{\theta/\xi_\theta}$, then for almost all $C \in \xi_{\mathrm{s}}$, the conditional measure in C is just the counting measure, i.e., for measurable subsets $A \in \mathcal{R}_\theta$,

$$\mu_\theta(A) = \int_X |A \cap (\{x\}, Gx)|\, dm(x).$$

1.2. Connected Pairs of Discrete Measurable Partitions. We will need the following definitions and results from [8], [21].

Let H be a group of automorphisms of the space X. We call it *orbitally discrete* if there exists a countable subgroup $G \subset H$ such that $H \subset [G]$.

Proposition 1.1. *Every subgroup of an orbitally discrete group is itself orbitally discrete.*

If the group H is orbitally discrete, then we can define its orbit partition by assuming $\theta(H) = \theta(G)$ where the group G is countable and $G \subset H \subset [G]$. It is not hard to check that this definition is mod 0 correct and does not depend on the choice of the group G; by definition, $[H] = [G]$.

Proposition 1.2. *Let ξ be a measurable partition and $\theta = \theta(G)$ be an orbit partition of the countable group G. Then the group $[\xi] \cap [G]$ is orbitally discrete, and its orbit partition coincides with $\theta \vee \xi$.*

A measurable partition ξ is called *conditionally discrete* if almost all its elements have atomic conditional measures.

Proposition 1.3. *If ξ and η are two measurable conditionally discrete partitions, then the group $[\xi] \cap \mathcal{N}(\eta)$ is orbitally discrete.*

To shorten the notation, we denote the group $[\xi] \cap \mathcal{N}(\eta)$ by $\mathcal{G}(\xi, \eta)$ and its orbit partition by $\theta(\xi|\eta)$.

A pair of conditionally discrete measurable partitions (ξ, η) is called *connected* if $\xi \vee \eta = \varepsilon$ and $\theta(\xi|\eta) = \xi$.

Proposition 1.4. *For a connected pair of measurable conditionally discrete partitions ξ and η, their non-measurable intersection $\xi \cap \eta$, and hence the factor partitions $\xi \cap \eta/\xi$ and $\xi \cap \eta/\eta$, are correctly defined and these partitions are orbit partitions of suitable countable groups of automorphisms.*

Theorem 1.5. *If, for a connected pair of measurable conditionally discrete partitions ξ and η, the partitions $\xi \cap \eta/\xi$ and $\xi \cap \eta/\eta$ are of infinite type, then there exists a measurable subset $A \subset X$ such that $\xi(A) = \eta(A) = X$.*

We introduce some more concepts. A *polymorphism* (more precisely, a polymorphism with quasi-invariant measure) (see [6]) is a diagram $\Pi = \Pi(\mu)$ of the form

$$(X_1,\ m_1) \xleftarrow{\pi_{X_1}} (X_1 \times X_2,\ \mu) \xrightarrow{\pi_{X_2}} (X_2,\ \mu_2),$$

in which $(X_1 \times X_2, \mu)$ and (X_i, μ_i), $i = 1, 2$, are Lebesgue spaces and the natural projections $\pi_{X_i} \colon X_1 \times X_2 \to X_i$ are measurable. Thus, the partitions $\xi_i = \pi_{X_i}^{-1} \varepsilon_{X_i}$ of the space $X_1 \times X_2$ are measurable, $\xi_1 \vee \xi_2 = \varepsilon_{X_1 \times X_2}$, and the measures m_i are factor measures for μ when $(X_1 \times X_2)/\xi_i$ are naturally identified with X_i. The systems of conditional measures corresponding to these factor measures are denoted by $\{\mu_1^{x_1} \mid x_1 \in X_1\}$ and $\{\mu_2^{x_2} \mid x_2 \in X_2\}$.

It is clear that any pair of measurable partitions (ξ, η) of the Lebesgue space (X, m) for which $\xi \vee \eta = \varepsilon_X$ corresponds to a polymorphism

$$(X/\xi, m/\xi) \xleftarrow{\pi_\xi} (X, m) \xrightarrow{\pi_\eta} (X/\eta, \mu/\eta),$$

where X is identified with $(X/\xi \times X/\eta)$, since $\xi \vee \eta = \varepsilon_X$, and $m/\xi, m/\eta$ are some factor measures.

Let $\theta = \theta(G)$ be the orbit partition of the countable subgroup $G \subset \mathcal{A}(X)$ and $(\mathcal{R}_\theta, \mu_\theta)$ be the corresponding measurable equivalence relation. Then it can be considered as a polymorphism

$$\Pi_\theta \colon (X, m) \xleftarrow{\pi_{\mathrm{s}}} (X \times X, \mu_\theta) \xrightarrow{\pi_{\mathrm{r}}} (X, m),$$

where $\pi_{\mathrm{s}}(x, y) = x$ and $\pi_{\mathrm{r}}(x, y) = y$ for $(x, y) \in X \times X$ (the measure μ_θ is a continuation from $\mathcal{R}_\theta$ to $X \times X$ such that $\mu_\theta((X \times X) \setminus \mathcal{R}_\theta) = 0$). The partitions $\xi_{\mathrm{s}} = \pi_{\mathrm{s}}^{-1}(\varepsilon_X)$ and $\xi_{\mathrm{r}} = \pi_{\mathrm{r}}^{-1}(\varepsilon_X)$ form a connected pair of partitions of the space $(X \times X, \mu_\theta)$, and

$$\xi_{\mathrm{s}} \cap \xi_{\mathrm{r}}/\xi_{\mathrm{s}} = \theta, \qquad \xi_{\mathrm{s}} \cap \xi_{\mathrm{r}}/\xi_{\mathrm{r}} = \theta$$

under the natural identification of $(X \times X)/\xi_{\mathrm{s}}$ and $(X \times X)/\xi_{\mathrm{r}}$ with X.

1.3. Von Neumann Algebra of a Connected Pair. We denote by $\mathcal{B}(\mathcal{H}_m)$ the algebra of all bounded linear operators acting in the Hilbert space $\mathcal{H}_m = L^2(X, m)$. To each transformation $g \in \mathcal{A}(X)$, there corresponds a unitary operator U_g in $\mathcal{H}_m$ defined by the equality

$$(U_g f)(x) = f(g^{-1}x)\biggl(\frac{d(gm)}{dm}(x)\biggr)^{1/2}, \qquad x \in X, \quad f \in \mathcal{H}_m.$$

For $\varphi \in L^\infty(X, m)$, consider the multiplier A_φ defined by

$$(A_\varphi f)(x) = \varphi(x) f(x), \qquad x \in X, \quad f \in \mathcal{H}_m.$$

Then $g \mapsto U_g$, $g \in \mathcal{A}(X)$, is a unitary representation of the group $\mathcal{A}(X)$ in the Hilbert space $\mathcal{H}_m$ and $\varphi \mapsto A_\varphi$, $\varphi \in L^\infty(X, m)$, is an isomorphism of the algebra $L^\infty(X, m)$ into $\mathcal{B}(\mathcal{H}_m)$.

Now consider any pair of measurable partitions (ξ, η) in the space X. By $L^\infty(\xi)$ denote the subalgebra in $L^\infty(X, m)$ consisting of all ξ-measurable functions from $L^\infty(X, m)$, and let $\mathcal{G}(\xi, \eta) = [\xi] \cap \mathcal{N}(\eta)$. We denote by $\mathcal{M}(\xi, \eta)$ the von Neumann algebra in $\mathcal{B}(\mathcal{H}_m)$ generated by two families of operators:

$$U_g, \quad g \in \mathcal{G}(\xi, \eta), \qquad A_\varphi, \quad \varphi \in L^\infty(\eta).$$

Theorem 1.6. *If* (ξ,η) *is a connected pair of measurable conditionally discrete partitions, then* $\mathcal{M}(\xi,\eta)'=\mathcal{M}(\eta,\xi)$. *Moreover, if* $\xi\wedge\eta=\nu_X$, *then* $\mathcal{M}(\xi,\eta)$ *is a factor and* $\mathcal{M}_0(\xi,\eta)=\{A_\varphi\mid\varphi\in L^\infty(\xi)\}'$ *is a Cartan subalgebra in* $\mathcal{M}(\xi,\eta)$ (*that is, an abelian regular subalgebra which is the image of some normal conditional expectation*).

Now consider the polymorphism

$$\Pi(\mu):(X_1,m_1)\xleftarrow{\pi_{\mathrm{s}}}(X_1\times X_2,\mu)\xrightarrow{\pi_{\mathrm{r}}}(X_2,m_2),$$

for which $(\xi_{\mathrm{s}},\xi_{\mathrm{r}})$ is a connected pair of conditionally discrete partitions. In this situation, the algebra $\mathcal{M}(\xi_{\mathrm{s}},\xi_{\mathrm{r}})$ can be described as follows.

Due to the connectedness condition, the partitions $\theta_{\mathrm{s}}=(\xi_{\mathrm{s}}\cap\xi_{\mathrm{r}})/\xi_{\mathrm{s}}$ and $\theta_{\mathrm{r}}=(\xi_{\mathrm{s}}\cap\xi_{\mathrm{r}})/\xi_{\mathrm{r}}$ are orbit partitions of some countable automorphism groups. A partition ξ_{s} consists of elements of the form $X_2^x=\{x\}\times\theta_{\mathrm{r}}(x)$, $x\in X_1$. The space $\mathcal{H}_\mu=L^2(X_1\times X_2,\mu)$ decomposes into a direct integral

$$\mathcal{H}_\mu=\int_{X_1}^{\oplus}\mathcal{H}_x\,dm_1(x)$$

of Hilbert spaces $\mathcal{H}_x=L^2(X_2^x,\mu^x)$, $x\in X_1$, where $\{\mu^x,\,x\in X_1\}$ is a system of conditional measures of the partition ξ_{s}. If $x\overset{\theta_{\mathrm{s}}}{\sim}y$, then the equality

$$(V_{x,y}f)(z)=\left(\frac{d\mu^x}{d\mu^y}\right)^{1/2}(z)f(z),\qquad f\in\mathcal{H}_y,\quad z\in X_2^x,$$

defines the linear isometry $V_{x,y}\colon\mathcal{H}_x\to\mathcal{H}_y$.

Theorem 1.7 (see [21]). *If* $(\xi_{\mathrm{s}},\xi_{\mathrm{r}})$ *is a connected pair of conditionally discrete measurable partitions of the space* $(X_1\times X_2,\,\mu)$ *defined by the polymorphism* $\Pi(\mu)$, *then the algebra* $\mathcal{M}(\xi_{\mathrm{s}},\xi_{\mathrm{r}})$ *coincides with the set of all decomposable operators* $A=\int_{X_1}A_x\,dm_1(x)$ *in* $\int_{X_1}^{\oplus}\mathcal{H}_x\,dm_1(x)$ *such that*

$$A_y=V(x,y)A_xV(y,x)\quad\text{if}\quad x\overset{\theta_{\mathrm{s}}}{\sim}y,\quad x,y\in X_1.$$

Let G be a countable ergodic subgroup in $\mathcal{A}(X)$ and let $\theta=\theta(G)$ be its orbit partition. The measurable equivalence relation $(\mathcal{R}_\theta,\mu_\theta)$ can be considered as a polymorphism

$$\Pi(\mu_\theta)\colon(X,m)\xleftarrow{\pi_{\mathrm{s}}}(X\times X,\mu_\theta)\xrightarrow{\pi_{\mathrm{r}}}(X,m).$$

The pair $(\xi_{\mathrm{s}},\xi_{\mathrm{r}})$ is a connected pair of measurable conditionally discrete partitions with $\xi_{\mathrm{s}}\vee\xi_{\mathrm{r}}=\varepsilon_{X\times X}$ and $\theta_{\mathrm{s}}=\theta_{\mathrm{r}}=\theta$. Thus, to each countable ergodic group G, there corresponds a factor $\mathcal{M}(\xi_{\mathrm{s}},\xi_{\mathrm{r}})$, which we will denote by $\mathcal{M}_G$. Since $\mathcal{M}_G$ is independent of the choice of group G with a given orbit partition θ, we also use the notation $\mathcal{M}_\theta$ for $\mathcal{M}_G$.

The above construction of the factor $\mathcal{M}_G$ is equivalent to Krieger's construction in [13], [11]. The factor $\mathcal{M}_G$ coincides with the crossed product of $L^\infty(X,m)$ and G in the case where G acts freely on X.

The canonical embedding $j\colon L^\infty(X,m)\to\mathcal{M}_G$ is defined by the equality $j(\varphi)=A_{\overline{\varphi}}$, where $\overline{\varphi}(x,y)=\varphi(y)$, $(x,y)\in X\times X$. We denote the image $j(L^\infty(X,m))$ by $\mathcal{M}_G^0$.

The measure $\mu=\mu_\theta$ was chosen such that the conditional measures $\mu^x, x\in X$, of the partition ξ_{s} corresponding to the factor measure $\mu/\xi_{\mathrm{s}}=m$ were counting measures in $X^x=\{x\}\times\theta(x)$. Therefore, in the Hilbert space $\mathcal{H}_x=L^2(X^x,\mu^x)$, we can define a natural orthonormalized basis $\{e_y^x,\,y\in\theta(x)\}$ where

$$e_y^x((x,z))=\begin{cases}1, & \text{if } y=z,\\ 0, & \text{if } y\neq z,\end{cases}\qquad (x,z)\in X^x.$$

Here we can assume that $\mathcal{H}_x=\mathcal{H}_y$ at $x\overset{\theta}{\sim}y$, i.e., the operators $V_{x,y}$ are identities.

For each $g \in [G]$, we define $\overline{g} \in \mathcal{A}(X \times X, \mu_\theta)$ by setting $\overline{g}(x, y) = (x, gy)$, $(x, y) \in X \times X$. Obviously, the mapping $g \mapsto T_g = U_{\overline{g}}, g \in [G]$, is an isomorphism of the group $[G]$ into the group of unitary operators from $\mathcal{M}_G$; moreover, the operators T_g are elements of the normalizer

$$\mathcal{N}_{\mathcal{M}_G}(\mathcal{M}_G^0) = \{U \in \mathcal{M}_G \mid U\mathcal{M}_G^0 U^* = \mathcal{M}_G^0,\, U \text{ unitary}\}$$

of the algebra $\mathcal{M}_G^0$; $((\mathcal{N}_{\mathcal{M}_G}(\mathcal{M}_G^0))'' = \mathcal{M}_G$ and

$$j(\varphi \circ g) = T_g^* j(\varphi) T_g, \qquad g \in [G], \quad \varphi \in L^\infty(X, m).$$

The next result, obtained earlier in [13], follows from Theorem 1.7.

Theorem 1.8. *For a countable ergodic group $G \in \mathcal{A}(X)$, the algebra $\mathcal{M}_G^0$ is a Cartan subalgebra in the factor $\mathcal{M}_G$. Given a natural representation of the Hilbert space $\mathcal{H}_\mu$ as a direct integral $\int_X^\oplus \mathcal{H}_x \, dm(x)$, the factor $\mathcal{M}_G$ consists exactly of all decomposable operators $A = \int A_x \, dm(x)$ such that $A_x = A_y$ if $x \overset{\theta}{\sim} y$, and the subalgebra $\mathcal{M}_G^0$ consists of all operators of the above form for which the operators A_x are diagonal in the basis $\{e_y^x \mid y \in \theta(x)\}$ for almost all x.*

The type of factor $\mathcal{M}_G$ is the same as the type of the group G.

2. Proper Pairs

2.1. Orbit Partition of a System of Partial Isometries.

Lemma 2.1. *Let $\mathcal{U}$ be a countable subset of $\mathcal{U}(X)$ and $\mathcal{R}_0 \subset X \times X$ be a binary relation defined by the relation $(x, y) \in \mathcal{R}_0 \iff ux = vy$ for some $u, v \in \mathcal{U} \cup \{e_X\}$. Further, let θ be the finest partition of the space X for which $\mathcal{R}_\theta \supset \mathcal{R}_0$. Then there exists a countable subgroup $G \subset \mathcal{A}(X)$ such that $\mathcal{R}_\theta = \mathcal{R}_G$ ($[\theta] = [G]$).*

Proof. Consider the space $X \times \mathcal{U}'$ with measure $m \times \lambda$, where $\mathcal{U}' = \mathcal{U} \cup \{e_X\}$ and λ is the counting measure on $\mathcal{U}'$. On the set $A = \{(x, u) \in X \times \mathcal{U}' \mid x \in E(\mathcal{U})\}$, we define the partitions ξ and η by the relations

$$(x, u) \overset{\xi}{\sim} (y, v) \iff x = y,$$
$$(x, u) \overset{\eta}{\sim} (y, v) \iff ux = vy$$

for (x, u), (y, v) in A. These partitions are measurable, since the mappings $(x, u) \mapsto x$ and $(x, u) \mapsto ux$ from A to X are measurable. Then the partition θ under consideration coincides with the partition $\xi \cap \eta|_{X \times \{e_X\}}$ where $X \times \{e_X\}$ is naturally identified with X. The partition $\xi \cap \eta$ is an orbit partition of a countable group of automorphisms, so the partition $\xi \cap \eta|_{X \times \{e_X\}} = \theta$ also has this property.

□

The partition θ described in Lemma 2.1 will be called the orbit partition of a countable system of partial isomorphisms of U and $[\theta] = [G]$ its full group. We denote them by $\theta(U)$ and $[U]$, respectively.

2.2. Piecewise invariant partitions. A measurable partition ξ of space X is called *piecewise invariant* with respect to the group $G \subset \mathcal{A}(X)$ if every transformation g of G admits a decomposition $g = \bigoplus_n u_n$ where $u_n \in \mathcal{UN}(\xi)$, i. e.,

$$u_n = g|_{E(u_n)}, \qquad \bigcup_n E(u_n) = \bigcup_n gE(u_n) = X, \qquad u_n(\xi|_{E(u_n)}) = \xi|_{F(u_n)}.$$

Any invariant partition is obviously piecewise invariant. If ξ is piecewise invariant with respect to G, then it is piecewise invariant with respect to $[G]$.

If G is a countable subgroup in $\mathcal{A}(X), \theta = \theta(G)$, and the measurable partition ξ is piecewise invariant with respect to $[G]$, then we can correctly define the factor partition θ_ξ in the factor space X_ξ as follows. For each $g \in G$, we choose some its representation in the form $g = \bigoplus_n u_{g,n}$, where $u_n \in \mathcal{UN}(\xi)$, and consider a countable family of partial isomorphisms $\mathcal{U} = \{u_{g,n} \mid g \in G,\, n \in \mathbb{N}\}$. Then, the orbit partition $\theta(\mathcal{U})$ coincides with θ and for each $u \in U$, a factor isomorphism $u_\xi \in \mathcal{U}(X_\xi)$ is defined. We will call the orbit partition θ_ξ of the countable system of partial isomorphisms $\mathcal{U}_\xi = \{u_\xi \mid u \in \mathcal{U}\}$ the factor partition of θ over ξ. It is clear that this definition is mod 0 correct and does not depend on the choice of the countable group G for which $\theta = \theta(G)$, as well as on the choice of the countable set of partial isomorphisms corresponding to G. Moreover, for any $g \in [G]$ and for almost all points x of X

$$\xi(gx) \overset{\theta_\xi}{\sim} \xi(x).$$

If the partition θ is ergodic, then θ_ξ is also ergodic.

Piecewise invariant partitions usually occur in the following situation. Let $\widetilde{\xi}$ be a measurable partition of the space $\widetilde{X}$ and $X \subset \widetilde{X}$ be a subset of positive measure such that the smallest measurable $\widetilde{\xi}$-set $\widetilde{\xi}(X)$ containing X coincides with $\widetilde{X}$. Consider the restriction $\xi = \widetilde{\xi}_X$ of the partition $\widetilde{\xi}$ of X and let $\widetilde{G}$ be a countable subgroup in $\mathcal{N}(\widetilde{\xi})$, and $\theta = \widetilde{\theta}|_X$ be the restriction of its orbit partition $\widetilde{\theta} = \theta(\widetilde{G})$ on X. Then the partition ξ is piecewise invariant with respect to $[\theta]$ and the factor partition θ_ξ coincides with the orbit partition $\theta(\widetilde{G}_{\widetilde{\xi}})$, where $\widetilde{G}_{\widetilde{\xi}} = \{\widetilde{g}_{\widetilde{\xi}} \mid \widetilde{g} \in \widetilde{G}\}$, with the natural identification of $\widetilde{X}/\widetilde{\xi}$ and X/ξ.

Simple examples of piecewise invariant partitions arise when one considers measurable subpartitions. Namely, every measurable subpartition ξ of an orbit partition $\theta(G)$ of a countable group of automorphisms of G is piecewise invariant with respect to G. Indeed, in this case, for each $g \in [G]$, one can choose a decomposition $g = \bigoplus_n u_n$ such that $E(u_n)$ and $F(u_n)$ are one-layer with respect to ξ. Then $\xi|_{E(u_n)} = \varepsilon|_{E(u_n)}$ and $\xi|_{F(u_n)} = \varepsilon|_{F(u_n)}$, and hence $u_n \in \mathcal{UN}(\xi)$. The factor partition θ_ξ defined above coincides in this case with the usual factor partition.

2.3. Proper Pairs. Let $\theta = \theta(G)$ be the orbit partition of a countable group of automorphisms $G \subset \mathcal{A}(X)$. We will say that a measurable partition ξ is *properly located* with respect to θ if the measure m/ξ is continuous, $\omega[\theta \vee \xi] = \xi$, and ξ is piecewise invariant with respect to $[\theta]$. The pair (θ, ξ) will be called *a proper pair*.

We will show that every proper pair (θ, ξ) can be obtained in the way described in Subsection 2.2 from a pair $(\widetilde{\theta}, \widetilde{\xi})$ by taking some embedding X in $\widetilde{X}$ where $\widetilde{\theta} = \theta(\widetilde{G})$, G is a countable subgroup of $\mathcal{N}(\widetilde{\xi})$, $\theta = \widetilde{\theta}|_X$, and $\xi = \widetilde{\xi}|_X$.

For $\widetilde{X}, \widetilde{\theta}, \widetilde{\xi}$ we take the countable multiplications of X, θ, ξ. Namely, let I be a countable set and let λ be the counting measure on I. On the space $\widetilde{X} = X \times I$, consider the measure $m \times \lambda$ and the partitions $\widetilde{\theta} = \theta \times \nu_I$ and $\widetilde{\xi} = \xi \times \nu_\xi$ defined by the relations

$$(x,i) \overset{\widetilde{\theta}}{\sim} (y,j) \quad \Longleftrightarrow \quad x \overset{\theta}{\sim} y,$$
$$(x,i) \overset{\widetilde{\xi}}{\sim} (y,j) \quad \Longleftrightarrow \quad x \overset{\xi}{\sim} y$$

for $x, y \in X$ and $i, j \in I$.

We fix some element $i_0 \in I$ and define an embedding φ_0 of the space X into $\widetilde{X}$ as follows: $x \mapsto (x, i_0)$. Let $\xi_0 = \varphi_0(\xi)$ and $\theta_0 = \varphi_0(\theta)$. Then $\xi_0 = \widetilde{\xi}|_{X_0}$ and $\theta_0 = \widetilde{\theta}|_{X_0}$, where $X_0 = \varphi_0(X) = X \times \{i_0\}$. The factor spaces $\widetilde{X}/\widetilde{\xi}$ and X_0/ξ_0 are identified.

Since ξ is piecewise invariant with respect to $[\theta]$, the partitions ξ_0 and $\widetilde{\xi}$ are piecewise invariant with respect to $[\theta_0]$ and $[\widetilde{\theta}]$, respectively, and the factor partitions $\widetilde{\theta}/\widetilde{\xi}$ and θ_0/ξ_0 coincide.

The condition $\omega[\theta \vee \xi] = \xi$ implies that $\omega[\theta_0 \vee \xi_0] = \xi_0$ and $\omega[\widetilde{\theta} \vee \widetilde{\xi}] = \widetilde{\xi}$, so the pairs $(\widetilde{\theta}, \widetilde{\xi})$ and (θ_0, ξ_0) are proper pairs.

Proposition 2.2. *Let (θ,ξ) be a proper pair, $(\widetilde{\theta},\widetilde{\xi})$ be its countable multiplication, and H be a countable subgroup of $\mathcal{A}(\widetilde{X}/\widetilde{\xi})$ such that $\theta(H)=\widetilde{\theta}/\widetilde{\xi}$. Then, for every automorphism $h\in H$, there exists an automorphism $\widetilde{h}\in[\widetilde{\theta}]\cap\mathcal{N}(\widetilde{\xi})$ such that h coincides with the factor automorphism $\widetilde{h}/\widetilde{\xi}$. Thus, if for some countable set $\widetilde{F}$ the relation $[\widetilde{F}]=[\widetilde{\theta}\vee\widetilde{\xi}]$ holds, then $[\widetilde{F}\cup\{\widetilde{h}\mid h\in H\}]=[\widetilde{\theta}]$.*

Proof. Since the pair (θ,ξ) is proper, the pairs (θ_0,ξ_0) and $(\widetilde{\theta},\widetilde{\xi})$ are also proper. Every partial isomorphism u of the set X_0 from $\mathcal{U}(\theta_0)\cap\mathcal{UN}(\xi_0)$ admits an extension $\widetilde{u}\in\mathcal{U}(\widetilde{X})$ such that $\widetilde{u}\in\mathcal{U}(\widetilde{\theta})\cap\mathcal{UN}(\xi_0)$, $E(\widetilde{u})=\widetilde{\xi}(E(u))$, and $F(\widetilde{u})=\widetilde{\xi}(F(u))$. Indeed, due to the condition $\omega[\widetilde{\theta}\vee\widetilde{\xi}]=\widetilde{\xi}$, we can find a sequence of partial isomorphisms $\{g_n\}_{n=0}^{\infty}$ and $\{g_n'\}_{n=0}^{\infty}$ of $\mathcal{U}[\widetilde{\theta}\vee\widetilde{\xi}]$ such that

$$E(h_n)=E(u),\quad E(h_n')=F(u),\qquad n\in\mathbb{N},$$
$$\bigcup_{n=0}^{\infty}E(h_n)=\widetilde{\xi}(E(u)),\quad \bigcup_{n=0}^{\infty}F(h_n')=\widetilde{\xi}(F(u)),\qquad h_0=e_{E(u)},\quad h_0'=e_{F(u)},$$

where $\{F(h_n),\ n=0,1,2,\dots\}$ and $\{F(h_n'),\ n=0,1,2,\dots\}$ form systems of pairwise non-intersecting sets. Then, taking $\widetilde{u}=\bigoplus_{n=0}^{\infty}h_n'uh_n$, we obtain the required extension of the partial isomorphism u.

Now take any $h\in H$. By definition of the partition $\theta(H)=\widetilde{\theta}/\widetilde{\xi}$, there exist a decomposition $h=\bigoplus_n v_n$ and partial isomorphisms $u_n\in\mathcal{U}[\widetilde{\theta}]\cap\mathcal{UN}[\widetilde{\theta}]$ such that $v_n=u_n/\widetilde{\xi}$. Moreover, without loss of generality, one can assume that $E(u_n)$ and $F(u_n)$ are contained in X_0. We extend, as above, each u_n to a partial isometry $\widetilde{u}_n\in\mathcal{U}(\widetilde{\theta})\cap\mathcal{UN}(\widetilde{\xi})$ in such a way that

$$E(\widetilde{u}_n)=\widetilde{\xi}(E(u_n))=\pi_{\widetilde{\xi}}^{-1}E(v_n),\qquad F(\widetilde{u}_n)=\widetilde{\xi}(F(u_n))=\pi_{\widetilde{\xi}}^{-1}(v_n).$$

Then, taking $\widetilde{h}=\bigoplus_n\widetilde{u}_n$, we get an element of $[\widetilde{\theta}]\cap\mathcal{N}(\widetilde{\xi})$ for which $\widetilde{h}/\widetilde{\xi}=h$.

We put $\widetilde{H}=\{\widetilde{h}\mid h\in H\}$ and check that $[\widetilde{F}\cup\widetilde{H}]=[\widetilde{\theta}]$. The inclusion $[\widetilde{F}\cup\widetilde{H}]\subset[\widetilde{\theta}]$ is true by construction. Let $\widetilde{g}\in[\widetilde{\theta}]$. It follows from the definition of the partition $\widetilde{\theta}/\widetilde{\xi}$ that $\widetilde{\xi}(\widetilde{x})\overset{H}{\sim}\widetilde{\xi}(\widetilde{g}\widetilde{x})$ for almost all $\widetilde{x}\in\widetilde{X}$. Therefore, there exist elements $h_n\in H$ and a decomposition $\widetilde{g}=\bigoplus_n\widetilde{u}_n$ such that $\widetilde{u}_n\in\mathcal{UN}[\widetilde{\xi}]$ and $\widetilde{u}_n/\widetilde{\xi}=h_n|_{E(\widetilde{u}_n/\tilde{\xi})}$. Since the partial isomorphisms $\widetilde{h}_n^{-1}g|_{E(u_n)}$ are in $\mathcal{U}[\widetilde{\theta}],\widetilde{\xi}]$ and $[\widetilde{\theta}\vee\widetilde{\xi}]=[\widetilde{F}]$, it follows that $\widetilde{g}$ is in $[\widetilde{F}\cup\widetilde{H}]$. □

Thus, for every proper pair (θ,ξ) in X, there exist such a proper pair $(\widetilde{\theta},\widetilde{\xi})$ in the space $\widetilde{X}$ and an embedding $\varphi_0\colon X\to X_0\subset\widetilde{X}$ such that $\widetilde{\xi}(X_0)=\widetilde{X}$, $\widetilde{\theta}|_{X_0}=\varphi_0(\theta)$, $\widetilde{\xi}|_{X_0}=\varphi_0(\xi)$, and we can choose countable transformation groups $\widetilde{F}$ and $\widetilde{H}$ for which $[\widetilde{F}]=[\widetilde{\theta}\vee\widetilde{\xi}]$, $\widetilde{H}\subset\mathcal{N}(\widetilde{\xi})\cap[\widetilde{\theta}]$, $[\widetilde{F}\cup\widetilde{H}]=[\widetilde{\theta}]$, and the factor partition $\widetilde{\theta}/\widetilde{\xi}$ coincides with the orbit partition of the group $\widetilde{H}/\widetilde{\xi}=\{\widetilde{h}/\widetilde{\xi}\mid\widetilde{h}\in\widetilde{H}\}$.

If (θ,ξ) is a proper pair, $\theta=\theta(G)$, and the group G is ergodic, then there are only the following two possibilities for the partition ξ:

(a) for almost all elements of $C\in\xi$, the conditional measures of m are atomic;

(b) for almost all elements of $C\in\xi$, the conditional measures of m are continuous.

Indeed, consider a measurable set $A=\{x\in X\mid m_{\xi(x)}(\{x\})=0\}$. If the set A has positive measure, then, due to the piecewise invariance of ξ with respect to G, it follows that A is G-invariant and hence $m(X\setminus A)=0$, since the group G is ergodic.

In the case (a), it follows from the condition $\omega[\theta\vee\xi]=\xi$ that $\theta\vee\xi=\theta[\theta\vee\xi]=\xi$, i.e, $\xi\geqslant\theta$ (the latter means that ξ is a measurable subpartition of θ).

2.4. Approximate Finiteness Conditions. Recall that a group G is called *approximately finite* (a.f.), or *hyperfinite*, if there exists an element $g\in\mathcal{A}(X)$ such that $[G]=[g]$. It is known that $[G]$ is a.f. if and only if the partition $\theta=\theta(G)$ is tame, i.e., there exists a decreasing sequence of measurable partitions θ_n whose non-measurable intersection $\bigcap_n\theta_n$ coincides with θ (i.e., $\theta(x)=\bigcup_n\theta_n(x)$ for almost all $x\in X$).

Another condition equivalent to hyperfiniteness, the amenability of the equivalence relation $\mathcal{R}_\theta$, is obtained by A. Connes, J. Feldman and B. Weiss [3] as follows.

An *invariant* (more precisely, *left invariant*) *mean* on $(\mathcal{R}_\theta, m_\theta)$ is a positive mapping

$$P\colon L^\infty(\mathcal{R}_\theta, m_\theta) \to L^\infty(X, m)$$

such that

$$P(1) = 1 \qquad \text{and} \qquad P(f \circ u) = (Pf) \circ u, \quad u \in \mathcal{U}(X),$$

where for $f \in L^\infty(\mathcal{R}_\theta, m_\theta)$,

$$(f \circ u)(x, y) = \begin{cases} f(u^{-1}x, y), & (x, y) \in \mathcal{R}_\theta \cap (F(u) \times X), \\ 0 & \text{otherwise} \end{cases}$$

and for $f \in L^\infty(X, m)$,

$$(f \circ u)(x, y) = \begin{cases} f(u^{-1}x), & x \in F(u), \\ 0 & \text{otherwise.} \end{cases}$$

An equivalence relation $\mathcal{R}_\theta$ is called *amenable* if it admits an invariant mean.

Theorem 2.3 (see [3]). *The group $[\theta] = [G]$ is approximately finite if and only if the relation $\mathcal{R}_\theta$ is amenable.*

Corollary 2.4 (see [3]). *If the group $[G]$ is approximately finite and $h \in \mathcal{N}[G]$, then the group $[G \cup \{h\}]$ is also approximately finite.*

We use these important results to prove the following theorem.

Theorem 2.5. *Let (θ, ξ) be a proper pair. Then the group $[\theta]$ is approximately finite if and only if the groups $[\theta \vee \xi]$ and $[\theta/\xi]$ are approximately finite.*

Lemma 2.6. *Let H and G be countable subgroups in $\mathcal{A}(X)$, $\theta = \theta[G \cup H]$, $\xi = \omega(G)$ and $H \subset \mathcal{N}(\xi)$. If $[\theta]$ is a.f., then $[H_\xi]$ is a.f., where $H_\xi = \{h/\xi, h \in H\}$.*

Proof. Without loss of generality, we can assume that $mX = 1$ and consider a factor measure m/ξ on X/ξ. Let $\theta_0 = \theta(H_\xi)$ and m_θ, m_{θ_0} be the measures on the equivalence relations $\mathcal{R}_\theta$ and $\mathcal{R}_{\theta_0}$ corresponding to the measures m and m/ξ.

Since $[\theta]$ is a. f., the relation $\mathcal{R}_\theta$ is amenable, so there exists an invariant mean $P\colon L^\infty(\mathcal{R}_\theta, m_\theta) \to L^\infty(X, m)$. The natural projection $\pi_\xi\colon X \to X/\xi$ defines an embedding $\alpha_\xi\colon \varphi \mapsto \varphi \circ \pi_\xi$, $\varphi \in L^\infty(X/\xi)$, of the space $L^\infty(X/\xi, m/\xi)$ into $L^\infty(X, m)$. As for almost all (x, y) it follows from $(x, y) \in \mathcal{R}_\theta$ that $(\pi_\xi x, \pi_\xi y) \in \mathcal{R}_{\theta_0}$, the equality $(\beta_\xi\psi)(x, y) = \psi(\pi_\xi x, \pi_\xi y)$ defines an embedding

$$\beta_\xi\colon L^\infty(\mathcal{R}_{\theta_0}, m_{\theta_0}) \to L^\infty(\mathcal{R}_\theta, m_\theta).$$

Since $G \subset [\xi]$ and P is an invariant mean on $\mathcal{R}_\theta$, we have for any $g \in H$ and $\psi \in L^\infty(\mathcal{R}_{\theta_0}, m_{\theta_0})$ that $(\beta_\xi\psi) \circ g = \beta_\xi\psi$ and $P(\beta_\xi\psi) \circ g = P((\beta_\xi\psi) \circ g) = P(\beta_\xi\psi)$, i. e., the function $P(\beta_\xi\psi)$ is G-invariant. Since $\omega(G) = \xi$, this function is ξ-measurable and hence it belongs to $\alpha_\xi(L^\infty(x/\xi, m/\xi)$.

Let $\overline{P}\psi = \alpha_\xi^{-1}(P(\beta_\xi\psi))$. Then the mapping

$$\overline{P}\colon L^\infty(\mathcal{R}_{\theta_0}, m_{\theta_0}) \to L^\infty(x/\xi, m/\xi)$$

is an invariant mean on $\mathcal{R}_{\theta_0}$. Indeed, $\overline{P}$ is positive and $\overline{P}(1) = 1$. We represent every partial isomorphism $u^0 \in \mathcal{U}(\theta_0)$ as $u^0 = \bigoplus_n u_n^0$, where $u_n^0 = (g_n/\xi)|_{E(u_n^0)}$ and $g_n \in G$. The operator $u = \bigoplus_n g_n|_{\pi\xi^{-1}E(u_n^0)}$ is a partial isomorphism from $\mathcal{U}(\theta) \cap \mathcal{UN}(\xi)$, $u/\xi = u^0$, and $E(u)$ and $F(u)$ are ξ-measurable sets. If $\psi \in L^\infty(\mathcal{R}_{\theta_0}, m_{\theta_0})$, then

$$\begin{aligned}(\overline{P}\psi) \circ u_0 &= \alpha_\xi^{-1}(P(\beta_\xi\psi)) \circ u_0 = \alpha_\xi^{-1}((P(\beta_\xi\psi) \circ u) \\ &= \alpha_\xi^{-1}(P((\beta_\xi\psi) \circ u)) = \alpha_\xi^{-1}(P(\beta_\xi(\psi \circ u^0))) = \overline{P}(\psi \circ u^0).\end{aligned}$$

Thus, $\overline{P}$ is an invariant mean, that is, the relation $\mathcal{R}_{\theta_0}$ is amenable and the group $[\theta_0]$ is approximately finite. $\square$

Note that under the conditions of Lemma 2.6, the pair (θ, ξ) is proper and $\theta/\xi = \theta_0$.

Proof of Theorem 2.5. Consider a countable multiplication $(\widetilde{\theta}, \widetilde{\xi})$ of a pair (θ, ξ). Then

$$\begin{aligned} [\theta] \text{ is a. f.} \quad &\Longleftrightarrow \quad [\widetilde{\theta}] \text{ is a. f.}, \\ [\theta \vee \xi] \text{ is a. f.} \quad &\Longleftrightarrow \quad [\widetilde{\theta} \vee \widetilde{\xi}] \text{ is a. f.}, \\ [\theta/\xi] \text{ is a. f.} \quad &\Longleftrightarrow \quad [\widetilde{\theta}/\widetilde{\xi}] \text{ is a. f.} \end{aligned}$$

So we can, without loss of generality, consider the pair $(\widetilde{\theta}, \widetilde{\xi})$ instead of (θ, ξ).

Due to Proposition 2.2, we can choose countable subgroups G and H in $\widetilde{\theta}$ such that $[G] = [\widetilde{\theta} \vee \widetilde{\xi}]$, $H \subset \mathcal{N}[\widetilde{\xi}]$, $[G \cup H] = [\widetilde{\theta}]$, and $[H_{\widetilde{\theta}/\tilde{\xi}}] = [\tilde{\theta}/\widetilde{\xi}]$, where $H_{\tilde{\xi}} = \{h/\widetilde{\xi} \mid h \in H\}$. Thus, the conditions of Lemma 2.6 are satisfied and hence $[\widetilde{\theta}/\widetilde{\xi}]$ is approximately finite if $\widetilde{\theta}$ is a. f. Thus, $[\widetilde{\theta} \vee \widetilde{\xi}]$ is a. f. as a subgroup of the a. f. group $[\widetilde{\theta}]$.

Conversely, let $[\widetilde{\theta} \vee \widetilde{\xi}]$ and $[\widetilde{\theta}/\widetilde{\xi}]$ be a. f. groups. Then there exists $h_0 \in [\widetilde{\theta}/\widetilde{\xi}]$ for which $[h_0] = [\widetilde{\theta}/\widetilde{\xi}]$. By Proposition 2.2, there is an $h \in \mathcal{N}[\widetilde{\xi}] \cap [\widetilde{\theta}] \subset \mathcal{N}[\widetilde{\theta} \vee \widetilde{\xi}]$ for which $h/\widetilde{\xi} = h_0$ and $[G \cup \{h\}] = \widetilde{\theta}$, where G is a countable group such that $[G] = [\widetilde{\theta} \vee \widetilde{\xi}]$. Applying Corollary 2.4, we obtain that $[\widetilde{\theta}]$ is a. f. $\square$

3. Conjugacy of Measurable Partitions

In this section, we consider proper pairs (θ, ξ) in the case where $\theta = \theta(G)$ is the orbit partition of an ergodic a. f. type II_1 group G and the measurable partition ξ has continuous conditional measures.

As noted above, if ξ has discrete conditional measures, then it follows from the properness condition that $\xi \geqslant \theta$. The description of measurable subpartitions of orbit partitions is straightforward and we omit it.

3.1. Classification Theorem. Let G be a countable ergodic type II_1 automorphism group of the space X and m be a G-invariant measure on X, $mX = 1, \theta = \theta(G)$.

If ξ is properly located with respect to $[G]$, i. e., (θ, ξ) is a proper pair, then there is a correctly defined factor partition θ/ξ on the factor space X/ξ. Due to the ergodicity of G, the invariant probability measure m is unique and hence the factor measure m/ξ is uniquely determined by (θ, ξ). Thus, in this case, isomorphic pairs (θ, ξ) correspond to isomorphic triples $(X/\xi, \theta/\xi, m/\xi)$.

Under the approximate finiteness condition, all countable ergodic groups of type II_1 are orbitally isomorphic to each other. Therefore, the problem of classifying proper pairs in this case is reduced to the problem of conjugacy of measurable partitions with respect to the normalizer $\mathcal{N}(\theta)$.

Theorem 3.1. *Let G be an ergodic a.f. group of measure-preserving automorphisms of the space (X, m), $mX = 1$, and ξ_1, ξ_2 be properly located with respect to $[G]$ measurable partitions with continuous conditional measures. Then the following conditions are equivalent*:

1) *there exists an element $g \in \mathcal{N}[G]$ such that $g\xi_1 = \xi_2$*;

2) *there exists an isomorphism $g_0 \colon X/\xi_1 \to X/\xi_2$ such that $g_0(m/\xi_1) = m/\xi_2$ and $g_0(\theta/\xi_1) = \theta/\xi_2$, where $\theta = \theta(G)$.*

Proof. Suppose condition 1) is satisfied. It follows from the uniqueness of the invariant measure for an ergodic group of type II_1 that $gm = m$. Therefore, if we take the factor isomorphism $X/\xi_1 \to X/\xi_2$ as g_0, then $g_0(\theta/\xi_1) = \theta/\xi_2$.

Now we check the implication 2) $\Longrightarrow$ 1).

Since almost all conditional measures of partitions ξ_i $(i = 1, 2)$ are continuous, there exist isomorphisms $\varphi_i \colon (X, m) \to (X/\xi_i \times Y,\, m/\xi_i \times \mu)$ such that $\varphi_i(\xi_i) = \varepsilon_i \times \nu_Y$, where (Y, μ) is some

Lebesgue space with continuous probability measure μ and ε_i are the partitions X/ξ_i into points. Then the automorphism $\varphi = \varphi_2 \circ (g_0 \times e_Y) \circ \varphi_1^{-1}$ translates ξ_1 onto ξ_2, preserves the measure m, and the factor isomorphism $(X/\xi_1, m/\xi_1) \to (X/\xi_2, m/\xi_2)$ induced by φ translates θ/ξ_1 onto θ/ξ_2. Let $\theta_1 = \varphi^{-1}(\theta)$, then $\theta_1/\xi_1 = \theta/\xi_1$ and the pair (θ_1, ξ_1) is isomorphic to the pair (θ, ξ_2).

We can assume that $(X, m) = (X/\xi_1 \times Y,\ m/\xi_1 \times \mu)$ and $\xi_1 = \varepsilon_1 \times \nu_Y$. Consider the countable multiplications $(\widetilde{\theta}, \widetilde{\xi}_1)$ and $(\widetilde{\theta}_1, \widetilde{\xi}_1)$ of the pairs (θ, ξ_1) and (θ_1, ξ_1) in the space $(\widetilde{X}, \widetilde{m}) = (X \times Y \times I,$ $m \times \mu \times \lambda)$ where (I, λ) is a countable set with the counting measure. Here the space X is identified with the subset $X/\xi_1 \times A$, where A is a subset of full measure of the space $(\widetilde{Y}, \widetilde{\mu}) = (Y \times I,\ \mu \times \lambda)$, $\xi = \widetilde{\xi}|_X$, and $\theta_1 = \widetilde{\theta}_1|X$, with $\widetilde{\xi}_1 = \varepsilon_1 \times \nu_{\widetilde{Y}}$.

Let S be an ergodic measure-preserving automorphism of the space $(\widetilde{Y}, \widetilde{\mu})$, then $\widetilde{S} = S \times e_{\widetilde{Y}} \in \mathcal{A}(\widetilde{X}, \widetilde{m})$ and $\omega(\widetilde{S}) = \widetilde{\xi}_1$. The subgroups $[\widetilde{\theta} \vee \widetilde{\xi}_1]$ and $[\widetilde{\theta}_1 \vee \widetilde{\xi}_1]$ of the group $\mathcal{A}(\widetilde{X}, \widetilde{m})$ are approximately finite and have the same ergodic decomposition as $[\widetilde{S}]$ with ergodic components $\omega[\widetilde{\theta} \vee \widetilde{\xi}_1] = \omega[\widetilde{\theta}_1 \vee \widetilde{\xi}_1] = \widetilde{\xi}_1$, since the pairs $(\widetilde{\theta}, \widetilde{\xi}_1)$ and $(\widetilde{\theta}_1, \widetilde{\xi}_1)$ are proper. From the results of part 2 of W. Krieger's paper [14], it follows that there exist automorphisms ψ and ψ_1 of $\mathcal{A}(\widetilde{X}, \widetilde{m})$ leaving the partition $\widetilde{\xi}$ fixed and such that $\psi(\widetilde{\theta} \vee \widetilde{\xi}_1) = \theta(\widetilde{S})$, $\psi_1(\widetilde{\theta}_1 \vee \widetilde{\xi}_1) = \theta(\widetilde{S})$, and $\psi(X) = \psi_1(X) = X$. Therefore, we can, without loss of generality, assume that $[\widetilde{\theta} \vee \widetilde{\xi}_1] = [\widetilde{\theta}_1 \vee \widetilde{\xi}_1] = [\widetilde{S}]$.

Since the groups $[\widetilde{\theta}]$ and $[\widetilde{\theta}_1]$ are a. f., then, by Theorem 2 of [12], the group $[\widetilde{\theta}/\widetilde{\xi}_1] = [\widetilde{\theta}_1/\widetilde{\xi}_1]$ is also a. f. Choose an automorphism $R \in [\widetilde{\theta}/\widetilde{\xi}_1]$ such that $\theta(R) = \widetilde{\theta}/\widetilde{\xi}_1$. Using Proposition 1.4, we see that $[\widetilde{R}, \widetilde{S}] = [\widetilde{\theta}]$ and $[\widetilde{R}_1, \widetilde{S}] = [\widetilde{\theta}_1]$. Since $\widetilde{R}$ and $\widetilde{R}_1$ are in $\mathcal{N}[\widetilde{S}]$ and $\widetilde{R}/\widetilde{\xi}_1 = \widetilde{R}_1/\widetilde{\xi}_1 = R$, they are of the form

$$\widetilde{R}(x_0, \widetilde{y}) = (Rx_0, V(x_0)\widetilde{y}), \qquad \widetilde{R}_1(x_0, \widetilde{y}) = (Rx_0, V_1(x_0)\widetilde{y}),$$
$$(x_0, \widetilde{y}) \in X/\widetilde{\xi}_1 \times \widetilde{Y} = \widetilde{X},$$

where $x_0 \mapsto V(x_0) \in \mathcal{N}[S]$ and $x_0 \mapsto V_1(x_0) \in \mathcal{N}[S]$ are measurable fields of automorphisms on X/ξ_1.

Since the group $[\theta]$ preserves the measure $\widetilde{m}$,

$$L = \frac{d\widetilde{m}(\widetilde{R}(x_0, \widetilde{y}))}{d\widetilde{m}(x_0, \widetilde{y})} = \frac{dm/\xi_1(Rx_0)}{dm/\xi_1(x_0)} \cdot \operatorname{mod} V(x_0)$$

for almost all $(x_0, \widetilde{y}) \in \widetilde{X}$ and exactly the same equality is true for $\widetilde{R}_1$ and V_1. Hence, $\operatorname{mod} V(x_0) = \operatorname{mod} V_1(x_0)$ a. e. in X/ξ_1. From the results of Part 4 of W. Krieger's paper [14], it follows that there exists an automorphism $\widetilde{P} \in \mathcal{N}[\widetilde{S}]$ such that $\widetilde{P}\widetilde{R}\widetilde{P}^{-1} \in [\widetilde{S}]$ and $\widetilde{P}$ preserves the measure $\widetilde{m}$ and leaves the partition $\widetilde{\xi}$ fixed. The relations $\widetilde{P}[\widetilde{\theta}]\widetilde{P}^{-1} = \widetilde{P}[\widetilde{R}, \widetilde{S}]\widetilde{P}^{-1} = [\widetilde{R}_1, \widetilde{S}] = [\widetilde{\theta}_1]$ are valid, i.e., $\widetilde{P}\widetilde{\theta} = \widetilde{\theta}_1$.

The sets X and $\widetilde{P}X$ in the space $\widetilde{X}$ have the same conditional measure equal to 1 in almost all elements of the partition $\widetilde{\xi}$, so there exists an automorphism $\widetilde{S}_1 \in [S]$ such that $\widetilde{S}_1\widetilde{P}X = X$. So $(\widetilde{S}_1\widetilde{P})|_X\theta = \theta_1$ and $\widetilde{S}\widetilde{P}|_X\xi_1 = \xi_1$, that is, the pairs (θ, ξ_1) and (θ_1, ξ_1), and hence the original pairs (θ, ξ_1) and (θ_1, ξ_2), are isomorphic. Thus, condition 2) is satisfied. □

3.2. Existence Theorem.

Theorem 3.2. *Let (X_0, m_0) be a Lebesgue space with continuous measure and $m(X_0) = 1$. For every ergodic a.f. group $[H]$ of automorphisms of the space (X_0, m_0), there exist an ergodic a.f. group G of measure-preserving automorphisms of the space (X, m), $mX = 1$, and a measurable partition ξ of X with continuous conditional measures such that the pair (θ, ξ), with $\theta = \theta(G)$, is proper, $X_0 = X/\xi$, $m_0 = m/\xi$, and $\theta/\xi = \theta(H)$.*

Proof. Let (X_0, m_0) be a Lebesgue space with continuous measure, $m_0X_0 = 1$, and (Y, ν) be a Lebesgue space with infinite continuous measure. We take arbitrary ergodic automorphisms

$Q \in \mathcal{A}(X_0)$ and $S \in \mathcal{A}(Y, \lambda)$. Again using Part 4 of paper [14], we can find a measurable field $V\colon x_0 \mapsto V(x_0) \in \mathcal{N}[S]$ such that

$$\frac{dm_0(Qx_0)}{dm_0(x_0)} = (\operatorname{mod} V(x_0))^{-1} \quad \text{for almost all} \ \ x_0 \in X_0.$$

Consider the automorphisms $\widetilde{S}$ and $\widetilde{Q}$ of the space $(X_0 \times Y, m_0 \times \lambda)$ defined by the equations

$$\widetilde{S}(x_0, y) = (x_0, Sy), \qquad \widetilde{Q}(x_0, y) = (Qx_0, V(x_0)y),$$
$$(x_0, y) \in X_0 \times Y,$$

and the full group $[\widetilde{S}, \widetilde{Q}] \subset \mathcal{A}(X_0 \times Y)$ generated by the automorphisms $\widetilde{S}$ and $\widetilde{Q}$. We denote by $\widetilde{\theta}$ the orbit partition of this group, choose some subset $A \subset Y$ of measure 1 and put

$$X = X_0 \times A, \qquad m = m_0 \times \lambda|_A, \qquad \xi = \widetilde{\xi}|_X,$$

where $\xi = \pi_{X_0}^{-1} \varepsilon_{X_0}$.

The automorphism $\widetilde{Q}$ preserves the measure $m_0 \times \lambda$, since

$$\frac{d(m_0 \times \lambda)(\widetilde{Q}(x_0, y))}{d(m_0 \times \lambda)(x_0, y_0)} = \frac{dm_0(Qx_0)}{dm_0(x_0)} \cdot \operatorname{mod} V(x_0) = 1.$$

The automorphism $\widetilde{S}$ also preserves measure, so $[\widetilde{Q}, \widetilde{S}] \subset \mathcal{A}(X_0 \times Y, m_0 \times \lambda)$ and hence $[\theta]$ preserves the measure m.

Since S is ergodic, $\omega(\widetilde{S}) = \varepsilon_{X_0} \times \lambda_Y = \widetilde{\xi}$. On the other hand, $\widetilde{\theta} \in \mathcal{N}(\widetilde{\xi})$ and the factor automorphism $\widetilde{Q}/\widetilde{\xi} = Q$ is ergodic. Hence, the groups $[\widetilde{\theta}]$ and $[\theta] = [\widetilde{\theta}]|_X$ are ergodic.

Let $\widetilde{G}_0$ be the group of automorphisms generated by $\widetilde{S}$ and $\widetilde{Q}$. For any $g \in [\widetilde{\theta}]$, there is a $\widetilde{g} \in [\widetilde{\theta}]$ such that $\widetilde{g}|_X = g$. Since $[\widetilde{G}_0] = [\widetilde{\theta}]$, the automorphism $\widetilde{g}$ admits a representation of the form $\widetilde{g} = \bigoplus_n \widetilde{g}_n|_{\widetilde{A}_n}$, $\widetilde{g}_n \in \widetilde{G}_0$, $\widetilde{A}_n \subset X_0 \times Y$, and therefore $g = \bigoplus_n \widetilde{g}_n|_{\widetilde{A}_n \cap X}$. Since $\widetilde{g}_n \in \widetilde{G}_0 \subset \mathcal{N}(\widetilde{\xi})$, the partial isomorphisms $\widetilde{g}_n|_{\widetilde{A}_n \cap X}$ are in $\mathcal{UN}(\xi)$. Thus, the partition ξ is piecewise invariant with respect to $[\theta]$. Furthermore,

$$\omega[\theta \vee \xi] = \omega[\widetilde{\theta} \vee \widetilde{\xi}]|_X = \omega(\widetilde{S})|_X = \widetilde{\xi}|_X = \xi.$$

So the pair (θ, ξ) is a proper one.

By construction,

$$X/\xi = X_0 \times Y/\widetilde{\xi} = X_0, \qquad m/\xi = m, \qquad \theta/\xi = \widetilde{\theta}/\widetilde{\xi} = \theta(Q),$$

and since the measure λ is continuous, the conditional measures of the partition ξ are also continuous.

The group $[\widetilde{S}]$ is approximately finite and $\widetilde{Q} \in \mathcal{N}[\widetilde{S}]$. By Corollary 2.4, the group $[\widetilde{\theta}] = [\widetilde{S}, \widetilde{Q}]$ is also approximately finite. □

Corollary 3.3. *From Theorems* 3.1 and 3.2, *it follows that the next two problems are equivalent.*

1) *The conjugacy problem of measurable partitions* ξ *with continuous conditional measures with respect to* $\mathcal{N}[G]$, *where* G *is an ergodic a.f. group of type* II_1 *and* ξ *is properly located with respect to* $[G]$.

2) *The classification problem of ergodic a.f. groups with respect to a measure-preserving orbit isomorphism.*

Note that the second problem was considered in [19], [20].

4. Classification of Subalgebras $\mathcal{M}_G(\xi)$

4.1. Calculation of Relative Commutants. Let G be a countable ergodic subgroup in $\mathcal{A}(X), \theta = \theta(G)$, and $\mathcal{M}_G$ be the corresponding factor in the Hilbert space $\mathcal{H}_\mu = L^2(X \times X)$, $\mu = \mu_\theta$. To every measurable partition ξ of the space X, there corresponds a commutative subalgebra $\mathcal{M}_G(\xi) = j(L^\infty(\xi))$, where j is an isomorphism of $L^\infty(X, m)$ onto the Cartan subalgebra $\mathcal{M}_G^0$, and $L^\infty(\xi)$ is the subalgebra of all ξ-measurable functions from $L^\infty(X, m)$.

For each subgroup $H \subset [G]$, consider the subalgebra $\mathcal{M}_{G,H}$ of $\mathcal{M}_G$ generated by $\mathcal{M}_G^0$ and the operators T_g, $g \in H$, where $g \mapsto T_g$ is the canonical isomorphism of the group $[G]$ to the normalizer $\mathcal{N}_{\mathcal{M}_G}(\mathcal{M}_G^0)$ of the subalgebra $\mathcal{M}_G^0$.

By Proposition 1.1, the group H is orbitally discrete. Let $\theta_1 = \theta(H)$ be its orbit partition, which is a subpartition of $\theta = \theta(G)$.

Consider the space $\mathcal{H}$ as a direct integral $\mathcal{H} = \int_X^\oplus \mathcal{H}_x \, dm(x)$, where $\mathcal{H}_x = L^2(X^x, \mu^x)$, $X^x = \{x\} \times \theta(x)$, and $\{\mu^x \mid x \in X\}$ is a system of conditional measures of the partition $\pi_{\mathrm{s}}^{-1}\varepsilon_X$ (cf. § 1).

The group H corresponds to the group $\overline{H} = \{\overline{h} \mid h \in H\} \subset \mathcal{A}(X \times X, \mu)$, where $\overline{h}(x, y) = (x, hy)$. The partition $\zeta_1 = \theta(\overline{H})$ is measurable, discrete, and is a subpartition of $\zeta_{\mathrm{s}} = \pi_{s_X}^{-1}\varepsilon_X = \theta(\overline{G})$:

$$(x, y) \overset{\zeta_1}{\sim} (x, z) \quad \Longleftrightarrow \quad y \overset{\theta_1}{\sim} z, \quad (x, y), (x, z) \in X^x.$$

For each $x \in X$, denote by $\mathcal{K}_x = \mathcal{K}_x^H$ the subalgebra in $\mathcal{B}(\mathcal{H}_x)$ that consists of all operators $A_x \in \mathcal{B}(\mathcal{H}_x)$ satisfying the relation

$$(A_x e_y, e_z) = 0 \quad \text{if} \quad y \overset{\theta_1}{\not\sim} z.$$

Let $\mathcal{B}_{G.H}$ be the subalgebra of $\mathcal{B}(\mathcal{H})$ consisting of all decomposable operators $A = \int A_x \, dm(x)$ from $\mathcal{M}_G$ for which $A_x \in \mathcal{K}_x^H$ for almost all $x \in X$.

Lemma 4.1. $\mathcal{M}_{G,H} = \mathcal{B}_{G,H}$.

Proof. The partitions ζ_1 and ζ_{r}, where $\zeta_{\mathrm{r}} = \pi_{\mathrm{r}}^{-1}\varepsilon_X$, form a connected pair of measurable conditionally discrete partitions of the space $(X \times X, \widetilde{m})$. Let $\{\widetilde{m}_C \mid C \in \zeta_1\}$ be the system of conditional measures of partition ζ_1 corresponding to some factor measure $\widetilde{m}/\zeta_1$ on $X \times X/\zeta_1$ and $\mathcal{H}_C = L^2(C, \widetilde{m}_C)$, $C \in \zeta_1$. Then

$$\mathcal{H} = \int_X^\oplus \mathcal{H}_x \, dm(x) = \int_X^\oplus \Bigg(\bigoplus_{C \in \zeta_1,\, C \in X^x} \mathcal{H}_C \Bigg) dm(x) = \int_{X \times X/\zeta_1} \mathcal{H}_C \, d\widetilde{m}/\zeta_1(C).$$

Applying Theorem 1.7 to the polymorphism

$$\big(X \times X/\zeta_1,\, \widetilde{m}/\zeta_1\big) \xleftarrow{\pi_{\zeta_1}} \big(X \times X,\, \widetilde{m}\big) \xrightarrow{\pi_{\zeta_{\mathrm{r}}}} (X, m),$$

we get the required result. □

Lemma 4.2. *The relative commutant $\mathcal{M}_{G,H}^{\mathrm{c}} = \mathcal{M}_G \cap \mathcal{M}'_{G,H}$ of the algebra $\mathcal{M}_{G,H}$ in the factor $\mathcal{M}_G$ coincides with $\mathcal{M}_G(\omega(H))$.*

Proof. The equality follows from the maximality of the subalgebra $\mathcal{M}_G^0$ in $\mathcal{M}_G$ and the relation $j(\varphi \circ g) = T_g^*(j(\varphi))T_g$, $g \in [G]$, $\varphi \in L^\infty(X, m)$. □

Lemma 4.3. *For any measurable partition ξ of the space X,*

$$(\mathcal{M}_G(\xi))^{\mathrm{c}} = \mathcal{M}_{G,[\xi \vee \theta]}.$$

Proof. By Proposition 1.2, the group $[\xi]\cap[G]$ is orbitally discrete and $\theta([\xi]\cap[G])=\xi\vee\theta$. Let φ be the function from $L^\infty(\xi)$ that separates the elements of the partition ξ. It corresponds to the operator $j(\varphi)=B=\int B_x\,dm(x)$, where for almost all $x\in X$, the operators $B_x\in\mathcal{B}(\mathcal{H}_x)$ are diagonal in the basis $\{e_y^x\mid y\in\theta(x)\}$, with $B_xe_y^x\neq B_xe_z^x$ if $y\overset{\xi}{\nsim}z$. If $A\in(\mathcal{M}_G(\xi))^{\mathrm{c}}$, then $A=\int A_x\,dm(x)$ and for almost all x, the operators A_x and B_x commute. Hence, $A_x\in\mathcal{K}_x^{[\xi\vee\theta]}$ for almost all x and, by Lemma 4.1, $A\in\mathcal{M}_{G,[\xi\vee\theta]}$, i.e., $(\mathcal{M}_G(\xi))^{\mathrm{c}}\subset\mathcal{M}_{G,[\xi\vee\theta]}$. □

Corollary 4.4. *If $\omega[\theta\vee\xi]=\xi$, then $\mathcal{M}_G(\xi)$ and $\mathcal{M}_{G,[\theta\vee\xi]}$ are relative commutants of each other in $\mathcal{M}_G$ and $\mathcal{M}_G(\xi)$ is the center of $\mathcal{M}_{G,[\theta\vee\xi]}$.*

4.2. Classification Theorem.

Theorem 4.5. *Let G be an ergodic a.f. group of type II_1 and let ξ_i, $i=1,2$, be measurable partitions with continuous conditional measures, properly located with respect to $[G]$. Then the following conditions are equivalent.*

1) *The partitions ξ_1 and ξ_2 are conjugate with respect to $\mathcal{N}[G]$.*
2) *The subalgebras $\mathcal{M}_G(\xi_1)$ and $\mathcal{M}_G(\xi_2)$ are conjugate in $\mathcal{M}_G$.*

Lemma 4.6. *Let G be a countable ergodic subgroup of $\mathcal{A}(X)$ and let H be a countable subgroup in $[G]$ of infinite type. Then for any U in the normalizer $\mathcal{N}_{\mathcal{M}_G}(\mathcal{M}_{G,H})$, there exists an automorphism $g\in[G]\cap\mathcal{N}[H]$ such that $UT_g^*\in\mathcal{M}_{G,H}$.*

Proof. Let $\varphi\in L^\infty(X,m)$ be a function separating the points of X, and let $B=j(\varphi)\in\mathcal{M}_G^0$. Then $UBU^*\in\mathcal{M}_{G,H}$ and $UT_gU^*\in\mathcal{M}_{G,H}$, $g\in H$. By Lemma 4.1, $\mathcal{M}_{G,H}$ consists of all operators $A=\int A_x\,dm(x)$ of $\mathcal{M}_G$ for which $A_x\in\mathcal{K}_x^H$ for almost all x. For the decompositions $B=\int B_x\,dm(x)$, $T_g=\int(T_g)_x\,dm(x)$ and $U=\int U_x\,dm(x)$, the inclusions $U_x^*B_xU_x\in\mathcal{K}_x^H$ and $U_x^*(T_g)_xU_x\in\mathcal{K}_x^H$ are valid for almost all x. Since the countable family $\{B,T_g\mid g\in H\}$ generates the algebra $\mathcal{M}_{G,H}$, we conclude that $U_x^*\mathcal{K}_x^HU_x=\mathcal{K}_x^H$ for almost all x.

The center Z_x^H of the algebra $\mathcal{K}_x^H$ obviously consists of all operators $C_x\in\mathcal{B}(\mathcal{H}_x)$ that are diagonal in the basis $\{e_y^x\mid y\in\theta(x)\}$ and such that $C_xe_y^x=C_xe_z^x$ if $y\overset{\theta_1}{\sim}z$ (here $\theta_1=\theta(H)$). Since $U_x^*Z_x^HU_x=Z_x^H$, for almost all x, a permutation γ_x of the set $\{C\mid C\subset\theta(x),C\in\theta_1\}$ is defined such that $(U_xe_y,e_z)=0$ if $z\notin\gamma_x\theta_1(y)$. Since $U\in\mathcal{M}_G$, it follows that $U_x=U_y$ and hence $\gamma_x=\gamma_y$ if $x\overset{\theta}{\sim}y$.

Consider the subset $\mathcal{R}_0=\bigcup_x\{x\}\times\gamma_x\theta_1(x)\subset\mathcal{R}_\theta\subset X\times X$. This subset is $\widetilde{m}$-measurable. Indeed, the functions φ_g, $g\in H$, defined by the equations

$$\varphi_g(x,y)=(U_x(T_g)_xe_x,e_y),\qquad(x,y)\in\mathcal{R}_\theta,$$

are measurable and $\mathcal{R}_0=\bigcup_{g\in H}\{\varphi_g>0\}$.

Consider the restrictions $\eta_{\mathrm{s}}=\xi_{\mathrm{s}}|_{\mathcal{R}_0}$ and $\eta_{\mathrm{r}}=\xi_{\mathrm{r}}|_{\mathcal{R}_0}$ to $\mathcal{R}_0$ of the measurable partitions $\xi_{\mathrm{s}}=\pi_{\mathrm{s}}^{-1}\varepsilon_X$ and $\xi_{\mathrm{r}}=\pi_{\mathrm{r}}^{-1}\varepsilon_X$. The partitions η_{s} and η_{r} are measurable, conditionally discrete and, as can be seen from the definition of the set $\mathcal{R}_0$, form a connected pair. In addition, by construction, $\eta_{\mathrm{s}}\cap\eta_{\mathrm{r}}/\eta_{\mathrm{s}}=\theta_1$ and $\eta_{\mathrm{s}}\cap\eta_{\mathrm{r}}/\eta_{\mathrm{r}}=\theta_1$. Since $[\theta_1]=[H]$ is a group of infinite type, Theorem 1.5 can be applied to the pair $(\eta_{\mathrm{s}},\eta_{\mathrm{r}})$. Choose a measurable subset $A\subset\mathcal{R}_0$ such that $\eta_{\mathrm{s}}|_A=\eta_{\mathrm{r}}|_A=\varepsilon_A$ and $\eta_{\mathrm{s}}(A)=\eta_{\mathrm{r}}(A)=\mathcal{R}_0$. These conditions mean that A is the graph of some automorphism $g_0\in\mathcal{A}(X)$. It follows from the inclusion $A\subset\mathcal{R}_\theta$ that $g_0\in[G]$, and since $A\subset\mathcal{R}_0$, we see that $g_0\in\mathcal{N}[H]$ and

$$(x,g_0\theta_1(x))=(x,\gamma_x\theta_1(x))\quad\text{for almost all}\quad x\in X.$$

Hence $U_x(T_{g_0}^*)_x\in\mathcal{K}_x^H$ for almost all x, and so $UT_{g_0}^*\in\mathcal{M}_{G,H}$. □

Corollary 4.7. *Let $(\widetilde{\theta},\widetilde{\xi})$ be a proper pair in the space $\widetilde{X}$, and let $\widetilde{G}$ and $\widetilde{H}$ be countable subgroups of $\mathcal{A}(\widetilde{X})$ such that $\widetilde{\theta}=\theta(\widetilde{G})$, $\theta(\widetilde{H})=\widetilde{\theta}\vee\widetilde{\xi}$, and $[G]$ is generated by the groups $[\widetilde{H}]$ and*

$[\widetilde{G}] \cap \mathcal{N}(\widetilde{\xi})$, while the group $[\widetilde{H}]$ is of infinite type. Then the action of the group $[\widetilde{G}] \cap \mathcal{N}(\widetilde{\xi})$ on $L^\infty(\widetilde{\xi})$is translated, under the canonical isomorphism $j\colon L^\infty(\widetilde{\xi}) \to \mathcal{M}_{\widetilde{G}}(\widetilde{\xi}) \subset \mathcal{M}_{\widetilde{G}}$, into an action induced by the normalizer $\mathcal{N}_{\mathcal{M}_{\widetilde{G}}}(\mathcal{M}_{\widetilde{G}}(\widetilde{\xi}))$ in $\mathcal{M}_{\widetilde{G}}(\widetilde{\xi})$.

Proof. By Corollary 4.4, the algebras $\mathcal{M}_{\widetilde{G},\widetilde{H}}$ and $\mathcal{M}_{\widetilde{G}}(\widetilde{\xi})$ are relative commutants of each other; therefore, $\mathcal{N}_{\mathcal{M}_{\widetilde{G}}}(\mathcal{M}_{\widetilde{G},\widetilde{H}}) = \mathcal{N}_{\mathcal{M}_{\widetilde{G}}}(\mathcal{M}_{\widetilde{G}}(\widetilde{\xi}))$. Applying lemma 4.3, we obtain the required result. □

Proof of Theorem 4.5. If (θ, ξ), with $\theta = \theta(G)$, is a proper pair, then, applying Corollary 4.7 to the countable multiplication $(\widetilde{\theta}, \widetilde{\xi})$ of the pair (θ, ξ), we see that the factor partition $\theta/\xi = \widetilde{\theta}/\widetilde{\xi}$ is uniquely determined by the pair $(\mathcal{M}_G, \mathcal{M}_G(\xi))$. If G is a type II_1 group and m is its invariant measure, then $\mathcal{M}_G$ is a type II_1 factor and the restriction of the trace on $\mathcal{M}_G$ to $\mathcal{M}_G(\xi)$ determines the factor measure m/ξ in X/ξ. Thus, from the conjugacy of $\mathcal{M}_G(\xi_1)$ and $\mathcal{M}_G(\xi_2)$ in $\mathcal{M}_G$, the isomorphism of the triples $(X/\xi_i,\ \theta/\xi_i,\ m/\xi_i)$, $i = 1, 2$, follows. If G is approximately finite, then we obtain from Theorem 3.1 that 2) $\Longrightarrow$ 1). The converse is obvious. □

Remark 4.8. It is easy to verify that the statement of Theorem 4.5 is also true for partitions ξ_i with discrete conditional measures.

Finally, we conclude with a simple statement that follows directly from Corollary 4.4.

Proposition 4.9. *Let θ be an orbit partition of the group G and let ξ_i be such measurable partitions that $\theta \vee \xi_i = \varepsilon$. Then ξ_1 and ξ_2 are conjugate with respect to $\mathcal{N}(G)$ if and only if $\mathcal{M}_G(\xi_1)$ and $\mathcal{M}_G(\xi_2)$ are conjugate in $\mathcal{M}_G$.*

It suffices to notice that $(\mathcal{M}_G(\xi_i))^{\mathrm{c}} = \mathcal{M}^0_G$. □

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Conflict of Interest

The authors of this work declare that they have no conflicts of interest.

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