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The Miracle of Integer Eigenvalues

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To the memory of Professor A. M. Vershik

ABSTRACT. For partially ordered sets $(X, \preccurlyeq)$, we consider the square matrices M^X with rows and columns indexed by linear extensions of the partial order on X. Each entry $(M^X)_{PQ}$ is a formal variable defined by a pedestal of the linear order Q with respect to linear order P. We show that all eigenvalues of any such matrix M^X are $\mathbb{Z}$-linear combinations of those variables.

KEY WORDS: partially ordered set (poset), pedestal, filter, Young diagram.

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1. The Statement of the Main Result

Let $X = \{\alpha_1, \dots, \alpha_n\}$ be a partially ordered set with the partial order $\preccurlyeq$. A linear extension P of $\preccurlyeq$ is a bijection $P\colon X \to [1, \dots, n]$ such that for any pair α_i, α_j, satisfying $\alpha_i \preccurlyeq \alpha_j$, we have $P(\alpha_i) \leqslant P(\alpha_j)$.

Let P, Q be two linear extensions of $\preccurlyeq$. We call the node $Q^{-1}(k) \in X$ a (P,Q)-*disagreement node* (or descent node, following [9]) iff

$$P\big(Q^{-1}(k-1)\big) > P\big(Q^{-1}(k)\big).$$

By definition, the node $Q^{-1}(1)$ is a (P,Q)-agreement node. With every pair P, Q, we associate the function $\varepsilon_{PQ}\colon \{1, \dots, n-1\} \to \{0,1\}$, given by

$$\varepsilon_{PQ}(k) = \begin{cases} 1, & \text{if } Q^{-1}(k+1) \text{ is a } (P,Q)\text{-descent}, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

Note that for some pairs $(P,Q) \neq (P,Q')$, the functions ε_{PQ}, $\varepsilon_{PQ'}$ can coincide (see § 4.3).

Let us denote by $\mathcal{E} = \{\varepsilon\colon \{1, \dots, n-1\} \to \{0,1\}\}$ the set of all 2^{n-1} different ε functions, and we associate with every ε a corresponding formal variable a_ε. For any poset X, consider the square matrix M^X, whose matrix elements are indexed by the pairs (P,Q), and are given by $(M^X)_{PQ} = a_{\varepsilon_{PQ}}$.

For example, the poset $(X, \preccurlyeq)$ with three elements and one relation, $X = \{\{u,v,w\}, u < v\}$, has three linear extensions of $\preccurlyeq$: $u < v < w$, $u < w < v$, and $w < u < v$. Let P be the linear extension $u < v < w$ and Q – the linear extension $u < w < v$. We have $\varepsilon_{PQ} = (0,1)$, since 2 is not a descent ($u < v$ in both Q and P) and 3 is a descent ($w < v$ in Q but not in P). The matrix M^X is

$$\begin{pmatrix} a_{00} & a_{01} & a_{10} \\ a_{01} & a_{00} & a_{10} \\ a_{01} & a_{10} & a_{00} \end{pmatrix}. \tag{2}$$

The eigenvalues of this matrix are $a_{00} - a_{01}$ (twice) and $a_{00} + a_{01} + a_{10}$, so they are $\mathbb{Z}$-linear combinations of the letters entering the matrix. One of us (O.O.) conjectured that this holds (the eigenvalues are $\mathbb{Z}$-linear combinations of the letters entering the matrix M^X) for every poset X. Below we present the proof of this conjecture.

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Theorem 1. *For every poset X, the matrix M^X is non-degenerate and all its eigenvalues are linear combinations of the variables a_ε with integer coefficients.*

Here "non-degenerate" means non-degenerate over the field of rational functions in the matrix elements.

The matrices M^X were introduced in paper [5]. It is proven there that the row sums $\sum_Q (M^X)_{PQ}$ do not depend on the row P, so the matrix M^X is "stochastic" (up to scale) and

$$\Pi_X(\{a_\varepsilon\}) := \sum_Q (M^X)_{PQ}$$

is its main eigenvalue. In [5], the corresponding sums are called the "pedestal polynomials". They enter into the expression for the generating functions of the monotone functions $f\colon X \to \{0,1,2,\dots\}$ (e. g., the generating function of the number of plane partitions, spacial partitions, etc.)

$$\sum_{\text{monotone } f\colon X\to\{0,1,2,\dots\}} t^{\sum_{x\in X} f(x)} = \Pi_X(t) \prod_{k=1}^{n} \frac{1}{1-t^k}, \tag{3}$$

where the polynomial $\Pi_X(t)$ is obtained from $\Pi_X(\{a_\varepsilon\})$ by the substitution

$$a_\varepsilon \rightsquigarrow t^{\sum_{k=1}^{n-1} k\varepsilon(k)}.$$

We put relevant combinatorial facts about pedestals and pedestal polynomials into § 4.

Our main tool is the filter semigroup of operators M_F^X, which we introduce in the next section. They appeared first in [1], [3], where their spectral properties were studied. In fact, part of the proof of Theorem 1 can be obtained by following the proof of Theorems 1, 2 in [1]. We give a shorter and more direct proof.

The next section contains some general facts about posets. It is followed by the section containing proofs.

2. The Filter Semigroup

At the end of this section we will introduce the filter semigroup. As it is easier to describe geometrically as the face semigroup of a hyperplane arrangement, we do this first.

2.1. Faces. Consider the central real hyperplane arrangement A_n consisting of hyperplanes

$$\{H_{ij}\colon 1 \leqslant i < j \leqslant n\}$$

in $\mathbb{R}^n$, defined by:

$$H_{ij} = \{(x_1,\dots,x_n)\colon x_i = x_j\}.$$

Every open connected component of the complement

$$\mathbb{R}^n \setminus \Big\{\bigcup H_{ij}\Big\}$$

is called a *chamber*. A *cone* is any union of closures of chambers which is *convex*. Let us introduce the (finite) set $\mathfrak{O}(n)$ of all different cones thus obtained.

Let X be a poset of n elements with a binary relation $\preccurlyeq$. To every pair

$$i,j \in X, \qquad i \preccurlyeq j,$$

there corresponds a half-space

$$K_{ij} = \{x_i \leqslant x_j\} \subset \mathbb{R}^n$$

(here we assume that X is identified with $\{1, 2, \dots, n\}$ as a plain set, ignoring the order). Consider the cone

$$A(X, \preccurlyeq) = \left\{ \bigcap_{i,j\colon i \preccurlyeq j} K_{ij} \right\} \in \mathfrak{O}(n),$$

where the intersection is taken over all pairs i, j such that $i \preccurlyeq j$.

The following statements are well known (and easy to prove), see [2], [4], [6], [9].

Claim 2. *The above defined correspondence*

$$(X, \preccurlyeq) \to A(X, \preccurlyeq)$$

is a one-to-one correspondence between the set of all partial orders on $\{1, 2, \dots, n\}$ *and the set of all cones* $\mathfrak{O}(n)$.

We present an illustration of this claim for $n = 4$ (see Fig. 1).

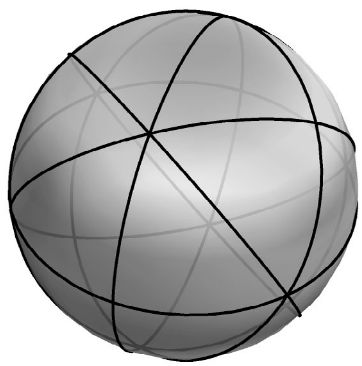

Fig. 1. The central real hyperplane arrangement A_4 in $\mathbb{R}^4$, projected to $\mathbb{R}^3$ along the line $x = y = z = t$ and intersected with the sphere $\mathbb{S}^2 \subset \mathbb{R}^3$. It is a partition of $\mathbb{S}^2$ into 24 equal triangles, each with angles $(\pi/2, \pi/3, \pi/3)$. The types of convex unions of the triangles are: the sphere, the hemisphere, the region between two great semicircles, an elementary triangle – or e-triangle, a pair of e-triangles with a common side, a triangle made from three e-triangles, a "square" formed by four e-triangles with a common $\pi/2$-vertex, a triangle made from a "square" and a fifth adjacent e-triangle, a triangle formed by six e-triangles with a common $\pi/3$-vertex. The number of the corresponding convex shapes are: 1, 12, 60, 24, 36, 48, 6, 24, 8, with the total being 219. This is precisely the number of partial orders on the set of four distinct elements (see sequence A001035 in OEIS [8]).

Let f', f'' be two faces in $A(X) = A(X, \preccurlyeq)$. (We allow that one or both of them are, in fact, chambers, i. e., faces of highest dimension). Define the face $f = f''(f') \in A(X)$ – or the *face-product* $f''f'$ – by the following procedure: choose points $x' \in f'$, $x'' \in f''$ in general position and let $s_{x'x''}\colon [0, 1] \to \mathbb{R}^n$ be a linear segment, $s_{x'x''}(0) = x'$, $s_{x'x''}(1) = x''$. Consider the face $f \in A(X)$ which contains all the points $s_{x'x''}(1 - \varepsilon)$ of our segment for $\varepsilon > 0$ small enough. Such a face does exist due to the convexity of $A(X)$. By definition, $f''(f') = f$. Note that if f'' is a chamber then $f''f' = f''$.

The face-product is associative. We mention for completeness that the semigroups $A(X, \preccurlyeq)$ are what have been called *left-regular bands* (see [6]).

Claim 3. *For every choice of faces $f, g, h \in A(X, \preccurlyeq)$, we have*

$$f(gh) = (fg)h,$$
$$ff = f, \qquad fgf = fg.$$

We do not give the proofs here as we are not using these relations.

2.2. Filters. Let F be a filter on X of rank k, i. e., a surjective map $F\colon X \to \{1, \dots, k\}$ preserving the partial order, and let

$$\{b_1, \dots, b_{j_1}\}, \{b_{j_1+1}, \dots, b_{j_2}\}, \dots, \{b_{j_{k-1}+1}, \dots, b_{j_k}\} \subset X$$

be its "floors":

$$\{b_{j_{r-1}+1}, \dots, b_{j_r}\} = F^{-1}(r), \qquad r = 1, \dots, k.$$

Consider the face $f_F \in A(X, \preccurlyeq)$ defined by the equations

$$x_{b_{j_{r-1}+1}} = \dots = x_{b_{j_r}}, \qquad r = 1, \dots, k,$$

and inequalities

$$x_{b_{j_1}} < x_{b_{j_2}} < \dots < x_{b_{j_k}}.$$

(More precisely, we write an equation for every floor of F which contains at least two elements of X.) This is a one-to-one correspondence between faces and filters. The filters of the highest rank n, i. e., the linear extensions of $\preccurlyeq$, correspond to the chambers.

The corresponding filter-product looks as follows. For F', F'' – two filters of X, the filter $F = F''F'$ on X is uniquely defined by the following properties:

- for u, v with $F''(u) < F''(v)$, we have $F(u) < F(v)$;
- for u, v with $F''(u) = F''(v)$, we have $F(u) < F(v)$ iff $F'(u) < F'(v)$.

Indeed, let f', f'' be the two faces corresponding to the filters F', F'', and let the general position points x', x'' belong to corresponding faces.

The fact that $F''(u) < F''(v)$ means that $x''_u < x''_v$. But the point $s_{x'x''}(1-\varepsilon)$ is close to the point x''; therefore, $[s_{x'x''}(1-\varepsilon)]_u < [s_{x'x''}(1-\varepsilon)]_v$ for all ε small enough.

If $F''(u) = F''(v)$ and $F'(u) < F'(v)$, then $x''_u = x''_v$, $x'_u < x'_v$. Since the map $s_{x'x''}\colon [0,1] \to \mathbb{R}^n$ is linear, for all $t < 1$, we have $[s_{x'x''}(t)]_u < [s_{x'x''}(t)]_v$.

Let F be a filter on X and P be some filter of rank n, i.e., a linear order on X. Then the filter FP is again a filter of rank n. Consider the square matrix $M_F^X = \|(M_F^X)_{P,Q}\|$ where P, Q are linear orders on X:

$$(M_F^X)_{P,Q} = \begin{cases} 1 & \text{if } Q = FP, \\ 0 & \text{if } Q \neq FP. \end{cases}$$

The operators M_F^X play a central role in our proof.

Examples of the operators M_F^X are given in § 4.3.

3. Proof of the Main Result

The plan of the proof is the following.

1) We will show that the matrix M^X can be written as a linear combination of M_F^X-s with integer monomial coefficients.
2) We will show that all M_F^X-s can be made upper-triangular via conjugation with the same matrix, and the resulting upper-triangular matrices have integer entries on the diagonal.

3.1. The Filter Decomposition. Let us rewrite M^X as the sum over all 2^{n-1} functions $\varepsilon\colon \{1,\dots,n-1\} \to \{0,1\}$:

$$M^X = \sum_{\varepsilon} a_\varepsilon B_{X,\varepsilon}, \tag{4}$$

where the entries of each matrix $B_{X,\varepsilon}$ are 0 or 1.

For every function ε, we define the number $r(\varepsilon) = 1 + \sum_{j=1}^{n-1} \varepsilon(j)$ and we partition the segment $\{1,\dots,n\}$ into $r(\varepsilon)$ consecutive segments

$$\{1,\dots,n\} = \{1,\dots,c_1\} \cup \{c_1+1,\dots,c_1+c_2\} \\ \cup \{c_1+c_2+1,\dots,c_1+c_2+c_3\} \cup \dots \cup \{c_1+\dots+c_{r(\varepsilon)}+1,\dots,n\},$$

where the values c_1+1, c_1+c_2+1, $\dots$, $c_1+\dots+c_{r(\varepsilon)}+1$ – are all the points where the function ε takes the value 1.

For $c_1,\dots,c_r$ – integers summing up to n, we denote by $\mathcal{F}_{c_1,\dots,c_r}$ the set of all filters $F\colon X \to [1,2,\dots,r]$ such that $|F^{-1}(i)| = c_i$ for all $i = 1,\dots,r$.

Lemma 4. *Suppose that the matrix $B_{X,\varepsilon}$ and the function ε have the parameters r and c_1, $\dots, c_r$. Then the following inclusion-exclusion identity holds*:

$$B_{X,\varepsilon} = \sum_{F\in\mathcal{F}_{c_1,\dots,c_r}} M_F^X - \Bigg[\sum_{\substack{F\in\mathcal{F}_{c_1+c_2,c_3,\dots,c_r} \\ \cup\mathcal{F}_{c_1,c_2+c_3,\dots,c_r}\cup\cdots}} M_F^X \Bigg] \\ + \Bigg[\sum_{\substack{F\in\mathcal{F}_{c_1+c_2+c_3,c_4,\dots,c_r} \\ \cup\mathcal{F}_{c_1+c_2,c_3+c_4,\dots,c_r}\cup\cdots}} M_F^X \Bigg] - \cdots, \tag{5}$$

where the sums are taken over all possible mergers of neighboring indices c_i, and the signs are $(-1)^{\#mergers}$.

Proof. Indeed, if we take an order Q from the row P which appears on the lhs, then it agrees with P over the first c_1-1 locations, then it disagrees once, then it agrees again over next c_2-1 locations, then disagrees once again, etc. But an order Q from the row P which appears on the rhs and corresponds to the first sum in (5) agrees with P over the first c_1-1 locations, then it agrees or disagrees once, then it agrees again over next c_2-1 locations, then agrees or disagrees once again, etc. Therefore we have to remove all these Q-s which agree with P over the first c_1-1 locations, then agree once again, then agree also over next c_2-1 locations, etc.

See § 4.3 for some M_F^X operators. □

3.2. Conjugation of M_F^X-s to Upper-Triangular. Let $X = \{\alpha_1,\dots,\alpha_n\}$ be a poset with the partial order $\preccurlyeq$. We denote by Tot_X the set of all total orders extending $\preccurlyeq$. Our matrices M_F^X are of the size $|\mathrm{Tot}_X| \times |\mathrm{Tot}_X|$. Let us now abolish all order relations on X, getting the poset $\overline{X}$ with $|\mathrm{Tot}_{\overline{X}}| = n!$. Of course, M_F^X is a submatrix of $M_F^{\overline{X}}$. Imagine (after reindexing) that it is an upper-left submatrix. We claim that to the right of this submatrix all matrix elements of $M_F^{\overline{X}}$ are zero, and so M_F^X is a block of $M_F^{\overline{X}}$. Indeed, each row of $M_F^{\overline{X}}$ has exactly one 1, and the rest are 0-s. But each row of M_F^X already has one 1. So it is sufficient to know that the spectrum of $M_F^{\overline{X}}$ consists of integers.

In what follows, the initial poset X will not appear any more and we will deal only with the "totally unordered" poset $\bar{X}$. The fact that the matrices $M_F^{\bar{X}}$ can be conjugated simultaneously to upper-triangular ones can be deduced from the results of papers [1], [3]. We give a shorter and more direct proof.

Let us consider an even bigger matrix $N_F^{\overline{X}}$ of size $2^{n(n-1)/2}$. Here F is a filter on $\overline{X}$, while the rows and columns of $N_F^{\overline{X}}$ are indexed by the *tournaments* between the n entries of $\overline{X}$. A *tournament* is an assignment of an order $\preccurlyeq$ to each pair $i \neq j$ of elements of the set $\overline{X}$, independently for each pair.

If we have a tournament $\preccurlyeq$ and a filter F on $\overline{X}$, then we define a new tournament $\preccurlyeq_F$ by the rule:

1) if $F(i) = F(j)$, then $i \preccurlyeq_F j$ iff $i \preccurlyeq j$;

2) if $F(i) < F(j)$, then $i \preccurlyeq_F j$.

We define $N_F^{\overline{X}}$ by

$$(N_F^{\overline{X}})_{\preccurlyeq\preccurlyeq'} = \begin{cases} 1 & \text{if } \preccurlyeq' = \preccurlyeq_F, \\ 0 & \text{if } \preccurlyeq' \neq \preccurlyeq_F. \end{cases}$$

Any linear order defines a tournament in an obvious way, so our matrices $M_F^{\overline{X}}$ are blocks of $N_F^{\overline{X}}$, and it is sufficient to study only them.

The key observation now is the fact that $N_F^{\overline{X}}$ is a tensor product of $n(n-1)/2$ two-by-two matrices, corresponding to all pairs (i, j), since the tournament orders $\preccurlyeq$ can be assigned to the pairs independently. And since the tensor product of upper-triangular matrices is upper-triangular, it is sufficient to check our claim just for the filters and tournaments in the case $n = |\overline{X}| = 2$.

The two-element no-order set $\overline{X} = \{1, 2\}$ carries three different filters and has two possible tournaments. The three two-by-two matrices $N_F^{\overline{X}}$-s are

$$N_1 := \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \qquad N_2 := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad N_3 := \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}.$$

Conjugating them by the discrete Fourier transform matrix

$$U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix},$$

brings them to the triple of upper-triangular matrices:

$$UN_1U^{-1} = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \qquad UN_2U^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad UN_3U^{-1} = \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}.$$

Extending the conjugation through the tensor product finishes the proof. □

Remark 5. Recall the definition (4) of matrices $B_{X,\varepsilon}$: for a poset X, matrices $\{B_{X,\varepsilon}\}$, where $\varepsilon \in \{0, 1\}^{\{1,\dots,n-1\}}$, are defined by $M^X = \sum_\varepsilon a_\varepsilon B_{X,\varepsilon}$. Let $\mathcal{L}(X)$ be the Lie algebra generated by the matrices $\{B_{X,\varepsilon}\}$. The proof shows that the Lie algebra $\mathcal{L}(X)$ is solvable.

Remark 6. Let us denote by Φ_T the algebra of functions on the set $\mathrm{Tour}_{\overline{X}}$ of tournaments considered as the set of vertices of the $n(n-1)/2$-dimensional cube in $\mathbb{R}^{n(n-1)/2}$. This algebra carries an increasing filtration by subspaces

$$0 \subset \Phi_T^{\leqslant 0} \subset \Phi_T^{\leqslant 1} \subset \cdots \subset \Phi_T^{\leqslant n(n-1)/2} = \Phi_T$$

consisting of restrictions of polynomials of degree $\leqslant 0, \leqslant 1, \dots$ to the vertices of the cube. This filtration is strictly multiplicative in the sense that

$$\Phi_T^{\leqslant k} = \underbrace{\Phi_T^{\leqslant 1} \cdots \Phi_T^{\leqslant 1}}_{k}.$$

Our considerations imply that all operators $N_F^{\overline{X}}$ preserve this filtration and commute with each other on the associated graded space $\bigoplus_k \Phi_T^{\leqslant k}/\Phi_T^{\leqslant k-1}$.

Restricting functions from Φ_T to the subset $\mathrm{Tot}_X \subset \mathrm{Tour}_{\overline{X}}$, we again obtain a strictly multiplicative filtration on the algebra $\Phi_X := \mathbb{R}^{\mathrm{Tot}_X}$ of functions on Tot_X, preserved by all operators M_F^X where F runs through filters on the poset X.

4. Appendices

4.1. Pedestals. Let X again be a finite poset with the partial order $\preccurlyeq$, and let P, Q be a pair of linear orders on X consistent with $\preccurlyeq$. We define the function q_{PQ} on X by

$$q_{PQ}(Q^{-1}(k)) = \#\{l\colon l \leqslant k,\, Q^{-1}(l) \text{ is a } (P,Q)\text{-descent node}\}. \tag{6}$$

Clearly, the function q_{PQ} is non-decreasing on X and $q_{PQ}(Q^{-1}(1)) = 0$. It is called the pedestal of Q with respect to P.

For example, let X be a 3×2 Young diagram and

$$P = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}, \qquad Q = \begin{bmatrix} 1 & 2 & 5 \\ 3 & 4 & 6 \end{bmatrix}$$

be the two standard tableaux. Then,

$$q_{PQ} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

Let $\mathcal{E}_P$ denote the set of all pedestals q_{PQ}. The correspondence

$$Q \to q_{PQ} \in \mathcal{E}_P$$

is a one-to-one map, as explained below.

Clearly, there is a map $\mathcal{E}_P \to \mathcal{E}$, where every pedestal q_{PQ} corresponds to its "discrete derivative" ε_{PQ}.

Pedestals were introduced in [7] in the following context. Consider the set $\mathcal{P} = \mathcal{P}_X$ of all non-negative integer-valued non-decreasing functions p on X. Denote by $v(p)$ the "volume" of p

$$v(p) = \sum_{\alpha \in X} p(\alpha),$$

and let G be the following generating function:

$$G_X(t) = \sum_{k \geqslant 0} g_k t^k = \sum_{p \in \mathcal{P}_X} t^{v(p)},$$

i. e., g_k is the number of non-decreasing p-s with $v(p) = k$. For example, if the poset X is, in fact, the set $X_n = [1, 2, \dots, n]$ ordered linearly, then

$$G_{X_n}(t) = \prod_{l=1}^{n} \frac{1}{1 - t^l}$$

is the generating function of the sequence g_k of the number of partitions π of the integer k into at most n parts:

$$k = \pi(1) + \pi(2) + \cdots + \pi(n),$$

with $\pi(i) \geqslant 0$, $\pi(i) \leqslant \pi(i+1)$. Let $\mathcal{Y}_n$ denote the set of all such partitions π (i.e. Young diagrams).

In order to write a formula for G_X for an arbitrary poset X, one needs pedestals. Namely, let us fix some ordering P of X, consider all pedestals q_{PQ}, and let

$$\Pi_P(t) = \sum_Q t^{v(q_{PQ})} \tag{7}$$

be the generating function (in fact, generating polynomial) of the sequence of the number of pedestals with a given volume. Then we have the following identity:

$$G_X(t) = \Pi_P(t) G_{X_n}(t) \equiv \Pi_P(t) \prod_{l=1}^{n} \frac{1}{1-t^l} \tag{8}$$

(compare with (3)). In particular, it follows from (8) that the polynomial $\Pi_P(t)$ does not depend on P, and thus can be denoted by $\Pi_X(t)$. The reason that (8) holds is the existence of the bijection $b\colon \mathcal{P}_X \to \mathcal{E}_P \times \mathcal{Y}_n$ between the set $\mathcal{P}_X$ of non-decreasing functions and the direct product $\mathcal{E}_P \times \mathcal{Y}_n$, respecting the volumes. Namely, to each pedestal q_{PQ} and each partition π, it associates the following function p on X:

$$p(Q^{-1}(k)) = q_{PQ}(Q^{-1}(k)) + \pi(k), \qquad k = 1, \dots, n.$$

Clearly, the function thus defined is non-decreasing on X. To check that b is a one-to-one correspondence, see [7], relation (46) and the construction of the inverse map b^{-1} given there. The bijectivity of b implies, in particular, that for each P, all the pedestals q_{PQ} are distinct.

In the case when X is a (2D) Young diagram, the functions $p \in \mathcal{P}_X$ are called "reverse plane partitions". The generating function G_X for these is also given by the famous Stanley [9] formula

$$G_X(t) = \prod_{\alpha \in X} \frac{1}{1-t^{h(\alpha)}},$$

where $h(\alpha)$ is the hook length of the cell $\alpha \in X$. When X is a rectangle, this is the MacMahon formula. That means that in the case where X is a Young diagram, nice cancellations happen on the rhs of (8). One can check that for some 3D Young diagram X, no cancellations happen in (8), and this is the reason why the analog of the MacMahon formula in the 3D case does not exist.

4.2. Pedestal Polynomials. That the function $\Pi_P(t)$ (see (7)) does not depend on the order P on X, but only on X, has the following generalization. Instead of characterizing the pedestal q_{PQ} just by its volume, let us associate with it the monomial

$$m_{PQ}(x_1, x_2, x_3, \dots) = x_1^{l_1 - 1} x_2^{l_2 - l_1} \cdots x_r^{l_r - l_{r-1}} x_{r+1}^{n - l_r + 1},$$

where r is the number of (P,Q)-descent nodes, and $l_1, \dots, l_r$ are their locations (see (6)). Note that

$$m_{PQ}(1, t, t^2, \dots) = t^{v(q_{PQ})}.$$

It was shown in [5] that the polynomial

$$\mathfrak{h}_P(x_1, x_2, x_3, \dots) = \sum_{Q \in \mathrm{Tot}_X} m_{PQ}(x_1, x_2, x_3, \dots)$$

is also independent of P, so it can be denoted by $\mathfrak{h}_X(x_1, x_2, x_3, \dots)$. Another way of expressing this is to say that the matrix $\widetilde{M}^X$ (of size $|\mathrm{Tot}_X| \times |\mathrm{Tot}_X|$) with entries

$$(\widetilde{M}^X)_{PQ} = m_{PQ}(x_1, x_2, x_3, \dots)$$

is *stochastic*: the vector $(1, 1, \dots, 1)^\top$ is the right eigenvector with the eigenvalue $\mathfrak{h}_X(x_1, x_2, x_3, \dots)$.

By replacing the monomials $m_{PQ}(x_1, x_2, x_3, \dots)$ with variables $a_{\varepsilon_{PQ}}$, one obtains from $\widetilde{M}^X$ our matrix M^X.

Remark 7. As we just said, we know from [5] that the rows of the matrix M^X consist of the same matrix elements permuted. So it is tempting to consider the set of permutations $\pi_{PP'} \in S_{|\mathrm{Tot}_X|}$ which permute the elements of row P to those of row P'. Unfortunately, rows of the matrix M^X can contain repeated elements, so the permutations $\pi_{PP'}$ are not uniquely defined.

4.3. Examples. Here we present several examples in which our posets X correspond to partitions; we first list the linear orders, that is, the standard Young tableaux of a given shape, and then present the pedestal matrix with lines and columns labelled by the standard Young tableaux in the listed order.

In all examples we considered, the pedestal matrix is diagonalisable in the generic point. However, for special values of variables, the pedestal matrix might have non-trivial Jordan blocks. We give a minimal example - partition $(3, 1)$. It is essentially the same example as the one before the main theorem, with the pedestal matrix (2), because the box $(1, 1)$ comes first in any linear order and can be omitted.

Here it is enough to take a partial evaluation $a_{10} \mapsto -2a_{01}$. Then the Jordan form is

$$\begin{pmatrix} a_{00} - a_{01} & 1 & 0 \\ 0 & a_{00} - a_{01} & 0 \\ 0 & 0 & a_{00} + 2a_{01} \end{pmatrix}.$$

It would be interesting to understand the regimes in which the pedestal matrix is not diagonalisable.

A. *Partition* $(3, 2)$. The standard tableaux are:

$$\begin{array}{|c|c|c|}\hline 1&3&5\\ \hline 2&4\\ \cline{1-2}\end{array}\,,\quad \begin{array}{|c|c|c|}\hline 1&2&5\\ \hline 3&4\\ \cline{1-2}\end{array}\,,\quad \begin{array}{|c|c|c|}\hline 1&3&4\\ \hline 2&5\\ \cline{1-2}\end{array}\,,\quad \begin{array}{|c|c|c|}\hline 1&2&4\\ \hline 3&5\\ \cline{1-2}\end{array}\,,\quad \begin{array}{|c|c|c|}\hline 1&2&3\\ \hline 4&5\\ \cline{1-2}\end{array}\,.$$

The pedestal matrix $\widetilde{M}^X$ is $x_1^2 A_{(3,2)}$, where

$$A_{(3,2)} = \begin{pmatrix} x_1^3 & x_2^3 & x_1^2x_2 & x_2^2x_3 & x_1x_2^2 \\ x_2^3 & x_1^3 & x_2^2x_3 & x_1^2x_2 & x_1x_2^2 \\ x_1^2x_2 & x_2^2x_3 & x_1^3 & x_2^3 & x_1x_2^2 \\ x_2^2x_3 & x_1^2x_2 & x_2^3 & x_1^3 & x_1x_2^2 \\ x_2^2x_3 & x_1^2x_2 & x_2^3 & x_1x_2^2 & x_1^3 \end{pmatrix}.$$

After a replacement

$$\phi\colon (x_1^3, x_1^2x_2, x_1x_2^2, x_2^3, x_2^2x_3) \to (a_1, a_2, a_3, a_4, a_5), \tag{9}$$

we have

$$A^\phi_{(3,2)} = \begin{pmatrix} a_1 & a_4 & a_2 & a_5 & a_3 \\ a_4 & a_1 & a_5 & a_2 & a_3 \\ a_2 & a_5 & a_1 & a_4 & a_3 \\ a_5 & a_2 & a_4 & a_1 & a_3 \\ a_5 & a_2 & a_4 & a_3 & a_1 \end{pmatrix}.$$

The eigenvalues of $A^{\phi}_{(3,2)}$ are:

$$a_1 - a_3, \qquad a_1 + a_2 - a_4 - a_5, \qquad a_1 - a_2 + a_4 - a_5,$$
$$a_1 - a_2 - a_4 + a_5, \qquad a_1 + a_2 + a_3 + a_4 + a_5.$$

B. *Partition* $(3,1,1)$. The standard tableaux are:

$$\begin{array}{|c|c|c|}\hline 1&4&5\\\hline 2\\\cline{1-1} 3\\\cline{1-1}\end{array}, \quad \begin{array}{|c|c|c|}\hline 1&3&5\\\hline 2\\\cline{1-1} 4\\\cline{1-1}\end{array}, \quad \begin{array}{|c|c|c|}\hline 1&2&5\\\hline 3\\\cline{1-1} 4\\\cline{1-1}\end{array}, \quad \begin{array}{|c|c|c|}\hline 1&3&4\\\hline 2\\\cline{1-1} 5\\\cline{1-1}\end{array}, \quad \begin{array}{|c|c|c|}\hline 1&2&4\\\hline 3\\\cline{1-1} 5\\\cline{1-1}\end{array}, \quad \begin{array}{|c|c|c|}\hline 1&2&3\\\hline 4\\\cline{1-1} 5\\\cline{1-1}\end{array}.$$

The pedestal matrix is $x_1^2 A_{(3,1,1)}$, where

$$A_{(3,1,1)} = \begin{pmatrix} x_1^3 & x_1x_2^2 & x_2^3 & x_1^2x_2 & x_2^2x_3 & x_1x_2^2 \\ x_1x_2^2 & x_1^3 & x_2^3 & x_1^2x_2 & x_2^2x_3 & x_1x_2^2 \\ x_1x_2^2 & x_2^3 & x_1^3 & x_2^2x_3 & x_1^2x_2^2 & x_1x_2^2 \\ x_1x_2^2 & x_1^2x_2 & x_2^2x_3 & x_1^3 & x_2^3 & x_1x_2^2 \\ x_1x_2^2 & x_2^2x_3 & x_1^2x_2 & x_2^3 & x_1^3 & x_1x_2^2 \\ x_1x_2^2 & x_2^2x_3 & x_1^2x_2 & x_2^3 & x_1x_2^2 & x_1^3 \end{pmatrix}.$$

After the same replacement (9) (the matrices $A_{(3,1,1)}$ and $A_{(3,2)}$ contain the same monomials), we have

$$A^{\phi}_{(3,1,1)} = \begin{pmatrix} a_1 & a_3 & a_4 & a_2 & a_5 & a_3 \\ a_3 & a_1 & a_4 & a_2 & a_5 & a_3 \\ a_3 & a_4 & a_1 & a_5 & a_2 & a_3 \\ a_3 & a_2 & a_5 & a_1 & a_4 & a_3 \\ a_3 & a_5 & a_2 & a_4 & a_1 & a_3 \\ a_3 & a_5 & a_2 & a_4 & a_3 & a_1 \end{pmatrix}.$$

The eigenvalues of $A^{\phi}_{(3,1,1)}$ are:

$$(a_1 - a_3)_2, \qquad a_1 + a_2 - a_4 - a_5, \qquad a_1 - a_2 + a_4 - a_5,$$
$$a_1 - a_2 - a_4 + a_5, \qquad a_1 + a_2 + 2a_3 + a_4 + a_5,$$

where the notation $(y)_k$ indicates that the multiplicity of the eigenvalue y is k.

Example $(3,1,1)$ shows degeneration: the letter a_3 appears twice in every row of $A^{\phi}_{(3,1,1)}$. The corresponding monomial is $x_1^3x_2^2$, so when writing down the decomposition of the matrix B_{a_3}, we need filters from $\mathcal{F}_{3,2}$. There are three of them in $\mathcal{F}_{3,2}$ (as with matrix notation, element (i,j) is in the intersection of row i and column j):

- F_1 – Floor 1 contains cells $(1,1)$, $(1,2)$ and $(2,1)$;
- F_2 – Floor 1 contains cells $(1,1)$, $(1,2)$ and $(1,3)$;
- F_3 – Floor 1 contains cells $(1,1)$, $(2,1)$ and $(3,1)$.

The matrices of the action of these filters on the linear orders are

$$M_{F_1} = \begin{pmatrix} 0&1&0&0&0&0 \\ 0&1&0&0&0&0 \\ 0&0&1&0&0&0 \\ 0&0&0&1&0&0 \\ 0&0&0&0&1&0 \\ 0&0&0&0&1&0 \end{pmatrix}, \qquad M_{F_2} = \begin{pmatrix} 0&0&0&0&0&1 \\ 0&0&0&0&0&1 \\ 0&0&0&0&0&1 \\ 0&0&0&0&0&1 \\ 0&0&0&0&0&1 \\ 0&0&0&0&0&1 \end{pmatrix},$$

$$M_{F_3} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

The family $\mathcal{F}_5$ contains one filter, which acts as the identity I. The matrix B_{a_3} is thus

$$B_{a_3} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} = M_{F_1} + M_{F_2} + M_{F_3} - I$$

as dictated by the inclusion-exclusion formula.

C. *Partition* $(3, 2, 1)$. In this example, to save space, we write down the pedestal matrix in which the replacement

$$\begin{aligned} &(x_1^6, x_1^5x_2, x_1^4x_2^2, x_1^4x_2x_3, x_1^3x_2^3, x_1^3x_2^2x_3, x_1^2x_2^4, x_1^2x_2^3x_3, x_1^2x_2^2x_3^2, x_1^2x_2^2x_3x_4) \\ &\quad \to (a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9, a_{10}) \end{aligned}$$

is already made.

The standard tableaux are:

$$\begin{array}{|c|c|c|}\hline 1&4&6\\\hline 2&5\\\cline{1-2} 3\\\cline{1-1}\end{array}, \quad \begin{array}{|c|c|c|}\hline 1&3&6\\\hline 2&5\\\cline{1-2} 4\\\cline{1-1}\end{array}, \quad \begin{array}{|c|c|c|}\hline 1&2&6\\\hline 3&5\\\cline{1-2} 4\\\cline{1-1}\end{array}, \quad \begin{array}{|c|c|c|}\hline 1&3&6\\\hline 2&4\\\cline{1-2} 5\\\cline{1-1}\end{array},$$

$$\begin{array}{|c|c|c|}\hline 1&2&6\\\hline 3&4\\\cline{1-2} 5\\\cline{1-1}\end{array}, \quad \begin{array}{|c|c|c|}\hline 1&4&5\\\hline 2&6\\\cline{1-2} 3\\\cline{1-1}\end{array}, \quad \begin{array}{|c|c|c|}\hline 1&3&5\\\hline 2&6\\\cline{1-2} 4\\\cline{1-1}\end{array}, \quad \begin{array}{|c|c|c|}\hline 1&2&5\\\hline 3&6\\\cline{1-2} 4\\\cline{1-1}\end{array},$$

$$\begin{array}{|c|c|c|}\hline 1&3&4\\\hline 2&6\\\cline{1-2} 5\\\cline{1-1}\end{array}, \quad \begin{array}{|c|c|c|}\hline 1&2&4\\\hline 3&6\\\cline{1-2} 5\\\cline{1-1}\end{array}, \quad \begin{array}{|c|c|c|}\hline 1&2&3\\\hline 4&6\\\cline{1-2} 5\\\cline{1-1}\end{array}, \quad \begin{array}{|c|c|c|}\hline 1&3&5\\\hline 2&4\\\cline{1-2} 6\\\cline{1-1}\end{array},$$

$$\begin{array}{|c|c|c|}\hline 1&2&5\\\hline 3&4\\\cline{1-2} 6\\\cline{1-1}\end{array}, \quad \begin{array}{|c|c|c|}\hline 1&3&4\\\hline 2&5\\\cline{1-2} 6\\\cline{1-1}\end{array}, \quad \begin{array}{|c|c|c|}\hline 1&2&4\\\hline 3&5\\\cline{1-2} 6\\\cline{1-1}\end{array}, \quad \begin{array}{|c|c|c|}\hline 1&2&3\\\hline 4&5\\\cline{1-2} 6\\\cline{1-1}\end{array}.$$

The matrix $A^{\phi}_{(3,2,1)}$ is

$$A^{\phi}_{(3,2,1)} = \begin{pmatrix}
a_1 & a_5 & a_7 & a_3 & a_9 & a_2 & a_6 & a_8 & a_3 & a_9 & a_5 & a_2 & a_8 & a_4 & a_{10} & a_6 \\
a_5 & a_1 & a_7 & a_3 & a_9 & a_6 & a_2 & a_8 & a_3 & a_9 & a_5 & a_2 & a_8 & a_4 & a_{10} & a_6 \\
a_5 & a_7 & a_1 & a_9 & a_3 & a_6 & a_8 & a_2 & a_9 & a_3 & a_5 & a_8 & a_2 & a_{10} & a_4 & a_6 \\
a_5 & a_3 & a_9 & a_1 & a_7 & a_6 & a_2 & a_8 & a_4 & a_{10} & a_6 & a_2 & a_8 & a_3 & a_9 & a_5 \\
a_5 & a_9 & a_3 & a_7 & a_1 & a_6 & a_8 & a_2 & a_{10} & a_4 & a_6 & a_8 & a_2 & a_9 & a_3 & a_5 \\
a_2 & a_6 & a_8 & a_3 & a_9 & a_1 & a_5 & a_7 & a_3 & a_9 & a_5 & a_4 & a_{10} & a_2 & a_8 & a_6 \\
a_6 & a_2 & a_8 & a_3 & a_9 & a_5 & a_1 & a_7 & a_3 & a_9 & a_5 & a_4 & a_{10} & a_2 & a_8 & a_6 \\
a_6 & a_8 & a_2 & a_9 & a_3 & a_5 & a_7 & a_1 & a_9 & a_3 & a_5 & a_{10} & a_4 & a_8 & a_2 & a_6 \\
a_6 & a_2 & a_8 & a_4 & a_{10} & a_5 & a_3 & a_9 & a_1 & a_7 & a_5 & a_3 & a_9 & a_2 & a_8 & a_6 \\
a_6 & a_8 & a_2 & a_{10} & a_4 & a_5 & a_9 & a_3 & a_7 & a_1 & a_5 & a_9 & a_3 & a_8 & a_2 & a_6 \\
a_6 & a_8 & a_2 & a_{10} & a_4 & a_5 & a_9 & a_3 & a_7 & a_5 & a_1 & a_9 & a_3 & a_8 & a_6 & a_2 \\
a_5 & a_3 & a_9 & a_2 & a_8 & a_6 & a_4 & a_{10} & a_2 & a_8 & a_6 & a_1 & a_7 & a_3 & a_9 & a_5 \\
a_5 & a_9 & a_3 & a_8 & a_2 & a_6 & a_{10} & a_4 & a_8 & a_2 & a_6 & a_7 & a_1 & a_9 & a_3 & a_5 \\
a_6 & a_4 & a_{10} & a_2 & a_8 & a_5 & a_3 & a_9 & a_2 & a_8 & a_6 & a_3 & a_9 & a_1 & a_7 & a_5 \\
a_6 & a_{10} & a_4 & a_8 & a_2 & a_5 & a_9 & a_3 & a_9 & a_2 & a_6 & a_9 & a_3 & a_7 & a_1 & a_5 \\
a_6 & a_{10} & a_4 & a_8 & a_2 & a_5 & a_9 & a_3 & a_8 & a_6 & a_2 & a_9 & a_3 & a_7 & a_5 & a_1
\end{pmatrix}.$$

The eigenvalues of $A^{\phi}_{(3,2,1)}$ are:

$$(a_1 - a_4 - a_7 + a_{10})_3, \qquad a_1 - a_4 + a_7 - a_{10}, \qquad (a_1 + a_2 - a_5 - a_6)_2,$$
$$(a_1 - a_2 - a_5 + a_6)_2, \qquad (a_1 - a_2 - a_3 + a_4 + a_7 - a_8 - a_9 + a_{10})_2,$$
$$(a_1 - a_2 - a_3 + a_4 - a_7 + a_8 + a_9 - a_{10})_2, \qquad (a_1 - a_4 + a_5 - a_6 + a_7 - a_{10})_2,$$
$$a_1 + 2a_2 + 2a_3 + a_4 - a_7 - 2a_8 - 2a_9 - a_{10},$$
$$a_1 + 2a_2 + 2a_3 + 2a_5 + 2a_6 + a_7 + 2a_8 + 2a_9 + a_{10}.$$

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Conflict of Interest

The authors of this work declare that they have no conflicts of interest.

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