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# Approximation of Operator Semigroups Using Linear-Fractional Operator Functions and Weighted Averages

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Abstract. An analytic semigroup of operators on a Banach space is approximated by a sequence of positive integer powers of a linear-fractional operator function. It is proved that the order of the approximation error in the domain of the generating operator equals $O(n^{-2}\ln(n))$. For a self-adjoint positive definite operator $A$ decomposed into a sum of self-adjoint positive definite operators, an approximation of the semigroup $\exp(-tA)$ $(t \geqslant 0)$ by weighted averages is also considered. It is proved that the order of the approximation error in the operator norm equals $O(n^{-1/2}\ln(n))$.

Key words: approximation of semigroup, Trotter–Chernoff formula, analytic semigroup.

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## Introduction

It is well known that important results on approximation of semigroups were obtained by Trotter and Chernoff (see [1]–[3]). Approximation formulas based on results of these papers are called *Trotter–Chernoff formulas*. These formulas find applications in solving evolution problems. For example, given an evolution problem with operator $A$ which is the generating operator of a strongly continuous semigroup and equals the sum of operators $A_1, \dots, A_m$, also being generating operators of strongly continuous semigroups, we can construct evolution problems corresponding to the operators $A_1, \dots, A_m$ on the basis of Trotter–Chernoff formulas and use solutions of these problems to approximate a solution of the initial one (see, e.g., [4] and [5]).

Thus, obviously, in the case under consideration, the approximation error of a solution of an evolution problem is directly related to the approximation error of Trotter–Chernoff formulas.

In this paper, given a self-adjoint positive definite operator (SAPDO) $A$ represented as $A = A_1+\dots+A_m$, where $A_1, \dots, A_m$ are SAPDOs as well, we approximate the semigroup $\{\exp(-tA)\}_{t\geqslant 0}$ by weighted averages composed either of resolvents or of the semigroups corresponding to the operators $A_1, \dots, A_m$. In the author's opinion, approximation of an operator function by weighted averages has certain advantages. First, if the argument of an operator function if self-adjoint and equals a sum of self-adjoint operators, then the corresponding weighted average is self-adjoint as well, i. e., the approximating operator retains the self-adjointness property. Secondly, the summands of a weighted average are independent of each other and can be calculated in parallel, which is important for practical applications. In modern applications, the parallelization of algorithms for solving evolution problems, especially those multidimensional in spatial variables, is becoming increasingly urgent. It is easy to see that the approximation which we propose gives a parallel algorithm for solving the initial evolution problem.

In this paper we obtain error bounds for the operator-norm approximation of a semigroup by weighted averages.

A wide range of questions concerning the operator-norm convergence of Trotter–Chernoff formulas were considered in [6]–[11]. Error bounds for the operator-norm approximation of these formulas for self-adjoint positive definite operators were obtained in [6]. In the subsequent papers [7] and [8] the class of Trotter–Kato product formulas was significantly expanded; moreover, in [8] optimal approximation error bounds in a Hilbert space were obtained. It should also be mentioned that in [8] the symmetric Trotter–Kato product formula was generalized to the case where the main operator is a SAPDO decomposed into a finite sum of SAPDOs. In [9] an "almost" optimal bound in the operator norm for the Trotter–Kato product formula was obtained (we say "almost" because of the presence of a logarithmic factor in the bound). In [10] error bounds for symmetric and asymmetric Trotter–Kato product formulas were also obtained for the operator-norm approximation of initial operators depending on fractional powers. Note that such bounds are important for practical applications of the formulas. In [11] a Chernoff-type approximation theorem was proved for quasi-sectorial contractions in a Hilbert space in the operator norm. It should be mentioned that this theorem implies, as a consequence, the convergence of the approximation by the arithmetic means of resolvents or of semigroups.

The next question considered in the present paper is closely related to those mentioned above and again concerns approximation of semigroups, but this time, by means of an operator analogue of linear-fractional functions. To be more precise, we approximate an analytic operator semigroup on a Banach space by a sequence of positive integer powers of a linear-fractional operator function. Our purpose is to estimate the approximation error. Note that, in approximating a semigroup, it would be ideal to obtain an error bound in the operator norm, but this is a nontrivial problem. If such a bound cannot be obtained, then the next step is obtaining an error bound at least in the domain of the generating operator. Restricting the approximation domain further is inexpedient, because in applications initial data for many evolution problems (the solving operators of evolution problems are semigroups) are far from being smooth. We prove that in the domain of the generating operator the error bound for the proposed approximation is almost optimal.

Approximation of a semigroup by means of the operator analogue of a linear-fractional function has certain advantages. The linear-fractional operator function can be reduced to the sum of the identity operator and a resolvent with constant coefficients by a simple transformation. This is an important nuance, because we can compute the resolvent in practice by preliminarily approximating the generating operator in a finite-dimensional space approximating the initial space.

The reader may ask the quite natural question of why we use a linear-fractional approximation rather than a higher-order rational approximation. As mentioned, our purpose is to obtain a bound, as close to optimal as possible, for the norm of the approximation error in the domain of the generating operator. If estimating the approximation error requires the $D(A^2)$ smoothness ($-A$ is the generating operator of the semigroup), then, in applications, additional boundary conditions must be imposed, which is highly undesirable. We will show that, in the case under consideration, a bound is of order $O(n^{-2}\ln(n))$ in $D(A)$ (the optimal bound is of order $O(n^{-2})$). Obviously, it is possible to construct a rational approximation of higher order, e.g., a Padé approximant, but this requires the $D(A^2)$ or even higher smoothness (you win some, you lose some).

In relation to the question considered in this paper, we mention the papers [12]–[14]. In [12] a general method was proposed for constructing an approximation of a semigroup and estimating the approximation error in the domain of fractional powers of the generating operator, and the optimality of the obtained bounds was investigated. Both cases of $C_0$ semigroups and of analytic semigroups were considered, and a survey of results on semigroup approximation was presented. In [13] the class of quasi-sectorial operators was introduced and the properties of such operators were studied in detail. For semigroups generated by quasi-sectorial operators, an optimal bound for the Euler approximation was obtained. In [14] an optimal bound for the Euler approximation was also obtained in the case where the given operator generates an analytic semigroup. In the conclusion of the introduction, we cannot help but mention the well-known monograph [15] by Hille and Phillips. Probably, there is not a single mathematician working on the application of semigroup theory who has not read this book or has not found useful information in it.

## 1. Linear-Fractional Approximation

Consider the linear-fractional function $(1-x/2)(1+x/2)^{-1}$. It is easy to see that it approximates the function $e^{-x}$ in a neighborhood of the point $x=0$. In particular, $e^{-x}-(1-x/2)(1+x/2)^{-1} = O(x^3)$. It is also obvious that this function is a Padé approximant.

Our purpose is to approximate an analytic semigroup by using an operator analogue of powers of a linear-fractional function, namely, of the functions $((1-\frac{x}{2n})(1+\frac{x}{2n})^{-1})^n$.

The following theorem holds.

**Theorem 1.1.** *Let $A$ be a linear densely defined closed operator on a Banach space $X$. Suppose that the sector $\Theta = \{z : |\arg(z)| < \varphi_0,\ 0 < \varphi_0 < \pi/2\}$ contains entirely the spectrum of $A$ and, for any $z$ ($\neq 0$) not belonging to $\Theta$, $\|(zI-A)^{-1}\| \leqslant c_0|z|^{-1}$ ($c_0 = \mathrm{const} > 0$). Then*

$$\left\|\left[\exp(-tA) - \left(\left(I-\frac{t}{2n}A\right)\left(I+\frac{t}{2n}A\right)^{-1}\right)^n\right]u\right\| \leqslant \frac{ct}{n^2}\ln(ne)\|Au\|, \qquad u\in D(A), \tag{1.1}$$

*where $\exp(-tA)$ ($t\geqslant 0$) is the analytic semigroup generated by the operator $-A$, $n$ is a positive integer, $c$ is a positive constant not depending on $t$ and $n$, and $\|\cdot\|$ is the norm of $X$.*

Below we formulate auxiliary facts used in the proof of Theorem 1.1 as lemmas and remarks.

The following lemma is valid (throughout the paper $c$ denotes a positive constant).

**Lemma 1.2.** *Suppose that an operator $A$ satisfies the assumptions of Theorem 1.1. Then*

$$\|\tau A(I-\tau A)^k(I+\tau A)^{-(k+j+1)}\| \leqslant \frac{c_1(\lambda)}{k+j}, \tag{1.2}$$

*where $k$ and $j$ are positive integers, $\tau > 0$, $c_1(\lambda) = c/\lambda^3$, $\lambda = \cos\varphi$, $\varphi_0 \leqslant \varphi < \pi/2$, and $c =$* const $> 0$.

**Proof.** Applying the Dunford–Taylor integral (see [17; Ch. VII]), we obtain

$$\tau A(I-\tau A)^k(I+\tau A)^{-(k+j+1)} = \frac{1}{2\pi i}\int_\Gamma \frac{z(1-z)^k}{(1+z)^{k+j+1}}(zI-\tau A)^{-1}\,dz, \tag{1.3}$$

where $\Gamma$ is the boundary of the sector $|\arg(z)| < \varphi$, $\varphi_0 \leqslant \varphi < \pi/2$ (the integral is taken in the direction from infinity with $\arg(z) = \varphi$ to infinity with $\arg(z) = -\varphi$).

Passing to norms in (1.3) and taking into account the assumption of Theorem 1.1, we obtain

$$\|\tau A(I-\tau A)^k(I+\tau A)^{-(k+j+1)}\| \leqslant c\int_0^{+\infty}\frac{|1-z|^k}{|1+z|^{k+j+1}}\,d\rho = c\int_0^{+\infty}\psi(\rho)\,d\rho, \tag{1.4}$$

where $z = \rho(\cos\varphi + i\sin\varphi)$ and $\psi(\rho) = (1-2\lambda\rho+\rho^2)^{k/2}(1+2\lambda\rho+\rho^2)^{-(k+j+1)/2}$.

Let us estimate the improper integral in (1.4). We represent it as the sum of three integrals:

$$\int_0^{+\infty}\psi(\rho)\,d\rho = \int_0^{2\lambda} + \int_{2\lambda}^{k+1} + \int_{k+1}^{+\infty}. \tag{1.5}$$

Obviously, we have $1+2\lambda\rho+\rho^2 \geqslant (1+\lambda\rho)^2$ and $1-2\lambda\rho+\rho^2 \leqslant 1$ for $0\leqslant\rho\leqslant 2\lambda$. Taking into account these inequalities, we obtain the following estimate for the first integral on the right-hand side of (1.5):

$$\int_0^{2\lambda}\psi(\rho)\,d\rho \leqslant \int_0^{+\infty}\frac{d\rho}{(1+\lambda\rho)^{k+j+1}} = \frac{1}{\lambda(k+j)}. \tag{1.6}$$

Let us estimate the second integral on the right-hand side of (1.5). For any $j\geqslant 1$, we have

$$\int_{2\lambda}^{k+1}\psi(\rho)\,d\rho \leqslant \frac{1}{2\lambda}\int_{2\lambda}^{k+1}\zeta(\rho)\,d\rho, \tag{1.7}$$

where $\zeta(\rho) = \chi(\rho)(1 - \chi(\rho))^m(\rho(1 + \chi(\rho))^{m+1})^{-1}$, $m = k/2$, and $\chi(\rho) = 2\lambda\rho(1 + \rho^2)^{-1}$.

Since $0 \leqslant \chi(\rho) < 1$ for $2\lambda \leqslant \rho < +\infty$, it follows that

$$\chi(\rho)(1 - \chi(\rho))^m \leqslant \frac{1}{2(m+1)}\,. \tag{1.8}$$

According to Bernoulli's inequality, for $\rho \geqslant 2\lambda$, we have

$$\rho(1 + \chi(\rho))^{m+1} \geqslant \rho + \lambda\lambda_0(k+2), \qquad \lambda_0 = 4\lambda^2(1 + 4\lambda^2)^{-1}. \tag{1.9}$$

Therefore, for $j \geqslant 1$, we obtain the following inequality from (1.7) with (1.8) and (1.9) taken into account:

$$\int_{2\lambda}^{k+1} \psi(\rho)\,d\rho \leqslant \frac{1}{\lambda(k+2)} \ln\frac{9}{4\lambda^3}\,. \tag{1.10}$$

Let us show that, for any $j > 2$, we have

$$\int_{2\lambda}^{k+1} \psi(\rho)\,d\rho \leqslant \frac{1}{4\lambda^3(k+j+2)}\,. \tag{1.11}$$

By virtue of Bernoulli's inequality, for $j > 2$, we have $(1 + 2\lambda\rho + \rho^2)^{(j-1)/2} \geqslant 1 + (j-1)\rho^2/2$. This inequality, together with (1.8), implies

$$\int_{2\lambda}^{k+1} \psi(\rho)\,d\rho \leqslant \frac{2}{\lambda(k+2)} \int_{2\lambda}^{+\infty} \frac{d\rho}{\rho(2 + (j-1)\rho^2)} \leqslant \frac{1}{4\lambda^3(k+j+2)}\,.$$

This proves (1.11).

We proceed to the third integral on the right-hand side of (1.5). As in the case of the second integral, we have

$$\int_{k+1}^{+\infty} \psi(\rho)\,d\rho \leqslant \frac{1}{2\sqrt{\lambda(k+1)}} \int_{k+1}^{+\infty} \frac{1}{\rho^{(2j+1)/2}}\,d\rho \leqslant \frac{1}{\sqrt{\lambda}(k+j)}\,. \tag{1.12}$$

The desired estimate follows from (1.5) and (1.4) with (1.6) and (1.10)–(1.12) taken into account. ☐

**Remark 1.3.** If an operator $A$ satisfies the assumptions of Theorem 1.1, then

$$\|(\tau A)(I + \tau A)^{-k}\| \leqslant c/k, \tag{1.13}$$

where $\tau > 0$, $k$ is a positive integer, and $c = \text{const} > 0$.

For $k = 1$, estimate (1.13) is obvious, and for $k > 1$, it is easily proved by using the Dunford–Taylor integral.

From the point of view of the practical application of Theorem 1.1, it is important to estimate the norms of the positive integer powers of the operator analogue of a linear-fractional approximation.

The following lemma is valid.

**Lemma 1.4.** *Suppose that an operator $A$ satisfies the assumptions of Theorem 1.1. Then, for any positive integer $k$, the inequality $\|L^k\| \leqslant c\ln(ke)$ holds, where $L = (I - \frac{\tau}{2}A)S$ for $S = (I + \frac{\tau}{2}A)^{-1}$ and $\tau > 0$ and $c = \text{const} > 0$.*

**Proof.** Note that the operator $L$ admits the representation $L = 2S - I$. It follows that $L^2 = 2S(L - I) + I$ and $L^3 = 2S(L^2 - L + I) - I$. Obviously, for any positive integer $k$, we have $L^k = 2S(L^{k-1} - L^{k-2} + \cdots + (-1)^{k+1}I) + (-1)^k I$, which implies $L^{2m} = -2\tau AS^2(L^{2m-2} + L^{2m-4} + \cdots + I) + I$. Therefore, according to Lemma 1.2 and Remark 1.3, we have $\|L^k\| \leqslant c\ln(ke)$ for an even power. For an odd power, the estimate $\|L^k\| \leqslant c\ln(ke)$ is proved in a similar way. ☐

**Remark 1.5.** As is known, under the assumptions of Theorem 1.1, the operator $-A$ generates the analytic semigroup $U(t) = \exp(-tA)$ $(t \geqslant 0)$, which is defined by using the Dunford–Taylor integral (see, e.g., [16; Ch. IX]), and the function $u(t) = U(t)u_0$ is a solution of the Cauchy problem

$$u'(t) + Au(t) = 0, \qquad t > 0,\ u(0) = u_0$$

for $u_0 \in D(A)$.

**Remark 1.6.** If an operator $A$ satisfies the assumptions of Theorem 1.1, then (see, e.g., [16])

$$\|A^k U(t)\| \leqslant \frac{c(k)}{t^k}, \tag{1.14}$$

where $t > 0$, $k$ is a positive integer, and $c(k) = \text{const} > 0$.

**Remark 1.7.** It is easy to note that, for $t \geqslant \tau > 0$, estimate (1.14) implies

$$\|(I + \tau A)A^k U(t)\| \leqslant \frac{c}{t^k}, \qquad c = \text{const} > 0. \tag{1.15}$$

We have prepared the ground for the proof of Theorem 1.1, to which we now proceed.

**Proof of Theorem 1.1.** We introduce the following notation: $\tau = t/n$, $t_k = k\tau$, $t_{k-1/2} = t_k - \tau/2$ (where $k$ and $n$ are positive integers),

$$L = \Big(I - \frac{\tau}{2}A\Big)\Big(I + \frac{\tau}{2}A\Big)^{-1}, \quad \text{and} \quad S = \Big(I + \frac{\tau}{2}A\Big)^{-1}.$$

Since the function $u(t) = U(t)u_0$, where $u_0 \in D(A)$, satisfies the equation $u'(t) + Au(t) = 0$ (see Remark 1.5), it follows that

$$u(t_k) = Lu(t_{k-1}) + \tau S\varphi_k, \qquad k = 1, \dots, n, \tag{1.16}$$

where

$$\varphi_k = \Big(\frac{1}{\tau}(u(t_k) - u(t_{k-1})) - u'(t_{k-1/2})\Big) + \Big(\frac{1}{2}A(u(t_k) + u(t_{k-1})) - Au(t_{k-1/2})\Big).$$

The recurrence relation (1.16) implies

$$u(t_k) = L^k u_0 + \tau \sum_{i=1}^{k} SL^{k-i}\varphi_i. \tag{1.17}$$

Substituting $u(t_k) = U(t_k)u_0$ into (1.17), replacing the vector $\varphi_i$ by $A^{-1}A\varphi_i$, and taking $L^k u_0$ to the left-hand side, we obtain

$$(U(t_k) - L^k)u_0 = \tau \sum_{i=1}^{k} G_{k-i}(A^{-1}\varphi_i), \qquad G_{k-i} = ASL^{k-i}. \tag{1.18}$$

According to Remark 1.5, we have $u'(t) - u'(t_{i-1/2}) = A(U(t_{i-1/2}) - U(t))u_0$. This equation and the formula (see, e.g., [16; Ch. IX])

$$A\int_r^t U(s)\,ds = U(r) - U(t), \qquad 0 \leqslant r \leqslant t, \tag{1.19}$$

imply

$$\tau G_{k-i}(A^{-1}\varphi_i) = g_i^{(1)} + g_i^{(2)} + g_i^{(3)} + g_i^{(4)}, \tag{1.20}$$

where

$$g_i^{(1)} = -G_{k-i}\int_{t_{i-1/2}}^{t_i}\int_{t_{i-1/2}}^{t} Z_i(s)Au_0\,ds\,dt,$$

$$g_i^{(2)} = G_{k-i}\int_{t_{i-1}}^{t_{i-1/2}}\int_{t}^{t_{i-1/2}} Z_i(s)Au_0\,ds\,dt, \qquad g_i^{(3)} = \frac{\tau}{2}G_{k-i}\int_{t_{i-1/2}}^{t_i} Z_i(t)Au_0\,dt,$$

$$g_i^{(4)} = -\frac{\tau}{2}G_{k-i}\int_{t_{i-1}}^{t_{i-1/2}} Z_i(t)Au_0\,dt, \qquad Z_i(t) = U(t_{i-1/2}) - U(t).$$

Next, we show that

$$\sum_{i=1}^{k}\|g_i^{(j)}\| \leqslant \frac{c\tau}{k+1}\ln(ek)\|Au_0\|, \qquad j = 1,2,3,4. \tag{1.21}$$

Let us estimate the term $g_i^{(1)}$ on the right-hand side of (1.20). Transforming the operator $Z_i(s)$ in the expression for the vector $g_i^{(1)}$ by formula (1.19), we obtain

$$g_i^{(1)} = -G_{k-i}S\int_{t_{i-1/2}}^{t_i}\int_{t_{i-1/2}}^{t}\int_{t_{i-1/2}}^{s} V(\xi)Au_0\,d\xi\,ds\,dt, \tag{1.22}$$

where $V(\xi) = (I + \frac{\tau}{2}A)AU(\xi)$.

According to Lemma 1.2, we have

$$\|G_{k-i}S\| = \|\tau AS^2L^{k-i}\| \leqslant \frac{c}{k-i+1}\,. \tag{1.23}$$

Passing to norms and taking into account estimates (1.15) and (1.23) in the expression (1.22), we obtain the inequality

$$\|g_i^{(1)}\| \leqslant \|Au_0\|\frac{c}{(k-i+1)\tau}\int_{t_{i-1/2}}^{t_i}\int_{t_{i-1/2}}^{t}\int_{t_{i-1/2}}^{s}\frac{1}{\xi}\,d\xi\,ds\,dt \leqslant \frac{c\tau}{(k-i+1)i}\|Au_0\|. \tag{1.24}$$

This yields (1.21) for $j = 1$.

The proof of estimate (1.21) for $j = 2,3$ is similar.

Let us estimate the vector $g_i^{(4)}$. For $i > 1$, as in the case of $g_i^{(1)}$, we easily obtain

$$\|g_i^{(4)}\| \leqslant \frac{c\tau}{(k-i+1)i}\|Au_0\|. \tag{1.25}$$

Consider the case of $i = 1$ separately. Let us represent $g_1^{(4)}$ in the form $g_1^{(4)} = g + \widetilde{g}$, where

$$g = -\frac{\tau}{2}G_{k-1}\int_0^{\tau_k} Z_1(t)Au_0\,dt,$$

$$\widetilde{g} = \frac{1}{2}G_{k-1}S\int_{\tau_k}^{\tau/2}\int_t^{\tau/2} V(s)Au_0\,ds\,dt, \quad \tau_k = \frac{\tau}{2(k+1)}.$$

In the expression for $\widetilde{g}$ we have transformed the operator $Z_1(t)$ by formula (1.19).

Estimate (1.23) and Remark 1.6 imply the inequality

$$\|\widetilde{g}\| \leqslant \frac{c}{k}\left(\int_0^{\tau/2}\int_t^{\tau/2}\frac{1}{s}\,ds\,dt + \tau\int_{\tau_k}^{\tau/2}\int_t^{\tau/2}\frac{1}{s^2}\,ds\,dt\right)\|Au_0\| \leqslant \frac{c\tau}{k+1}\ln(ek)\|Au_0\|. \tag{1.26}$$

Since $U(t)$ is uniformly bounded (see, e.g., [16]) and $\|\tau G_{k-1}\| = \|\tau AS\|\,\|L^k\| \leqslant c\ln(ke)$ (see Lemma 1.4), it follows that

$$\|g\| \leqslant \|\tau G_{k-1}\| \int_0^{\tau_k} \|Z_1(t)\|\,dt\,\|Au_0\| \leqslant \frac{c\tau}{k+1}\ln(ek)\|Au_0\|. \tag{1.27}$$

Now it is clear that, in view of (1.26) and (1.27), the following estimate for the vector $g_1^{(4)}$ holds:

$$\|g_1^{(4)}\| \leqslant \frac{c\tau}{k+1}\ln(ek)\|Au_0\|\,. \tag{1.28}$$

Therefore, by virtue of (1.28) and (1.25), inequality (1.21) is valid for $j = 4$.

Finally, from (1.18) with (1.21) taken into account we obtain

$$\|(U(t_k) - L^k)u_0\| \leqslant \frac{ct}{n(k+1)}\ln(ek)\|Au_0\|, \qquad u_0 \in D(A). \tag{1.29}$$

Inequality (1.29) with $k = n$ implies estimate (1.1). $\square$

## 2. Semigroup Approximation Using Weighted Averages

In this section our purpose is to obtain an operator-norm estimate for the error of an approximation of the semigroup $\exp(-tA)$ $(t \geqslant 0)$ by weighted averages in the case where $A$ is a SAPDO represented as a finite sum of SAPDOs (the case of two terms was considered in the author's paper [6]).

The following theorem holds.

**Theorem 2.1.** *Let $A_1, \dots, A_m$ $(m > 1)$ be SAPDOs on a Hilbert space $H$, i.e., $A_j = A_j^* \geqslant \alpha_j I$, where $\alpha_j = \mathrm{const} > 0$ (here $j = 1, \dots, m$ and $I$ is the identity operator). Suppose that $A = A_1 + \dots + A_m$ is a self-adjoint operator with domain $D(A) = \bigcap_{j=1}^m D(A_j)$ and $\eta_j > 0$, $j = 1, \dots, m$, are numbers satisfying the condition $\eta_1 + \dots + \eta_m = 1$. Then*

$$\|\exp(-tA) - (S(t/n))^n\| \leqslant (c_2 + c_1\ln(n))\frac{1}{\sqrt{2n+1}}, \tag{2.1}$$

*where $S(t) = \eta_1 S_1(t) + \dots + \eta_m S_m(t)$, $S_j(t) = (I + t\eta_j^{-1}A_j)^{-1}$ for $j = 1, \dots, m$ and $t > 0$, $n$ is a positive integer, and $c_2$ and $c_1$ are positive constants.*

For practical applications of Theorem 2.1, it is important to know precise expressions for the constants $c_2$ and $c_1$. They are $c_2 = 3(c_1 + 2c_4) + 4/3\sqrt{3}c_4$, where $c_4 = m + c_3$, $c_3 = \eta_1^{-1/2}a_1 + \dots + \eta_m^{-1/2}a_m$, and $c_1 = \eta_1^{-3/2}(\eta_1^{-1}a_1 + 1) + \dots + \eta_m^{-3/2}(\eta_m^{-1}a_m + 1)$ with $a_j = \|A_jA^{-1}\|$ for $j = 1, \dots, m$.

Below we give auxiliary facts used to prove Theorems 2.1 and 2.9 (the latter is similar to Theorem 2.1).

**Remark 2.2.** If the assumptions of Theorem 2.1 hold, then the $A_jA^{-1}$ $(j = 1, \dots, m)$ are closed operators (this is easy to prove) which are defined on the whole space and bounded (by the closed graph theorem), i.e., $a_j = \|A_jA^{-1}\| < \infty$.

**Remark 2.3.** Let $A$ and $B$ be SAPDOs such that $D(A) \subset D(B)$ and $B \leqslant A$ ($(Bu, u) \leqslant (Au, u)$ for any $u \in D(A)$). Then

(a) $0 \leqslant A^{-1} \leqslant B^{-1}$, i.e., $0 \leqslant (A^{-1}u, u) \leqslant (B^{-1}u, u)$ for any $u \in H$ (see [16; Theorem VI.2.21]);

(b) $D(A^{1/2}) \subset D(B^{1/2})$ and $\|B^{1/2}u\| \leqslant \|A^{1/2}u\|$ for any $u \in D(A^{1/2})$.

Note that (b) follows from (a) (see [16; Theorem VI.2.30]).

**Remark 2.4.** Let $A$ and $B$ be SAPDOs such that $D(A) \subset D(B)$ and $\|Bu\| \leqslant \|Au\|$ for $u \in D(A)$. Then $\|A^{-1}u\| \leqslant \|B^{-1}u\|$ for $u \in H$.

Indeed, since $A^{-1}B \subset (BA^{-1})^*$ and $\|BA^{-1}\| \leqslant 1$, it follows that $\|A^{-1}Bu\| = \|(BA^{-1})^*u\| \leqslant \|u\|$ for $u \in D(B)$. Therefore, $\|A^{-1}u\| = \|(A^{-1}B)B^{-1}u\| \leqslant \|B^{-1}u\|$.

**Remark 2.5.** It is well known that if an operator if bounded and self-adjoint, then the norm of the corresponding operator function is less than or equal to the norm of the corresponding scalar function on the spectrum (see, e.g., [18; Ch. IX, Sec. 5]).

**Lemma 2.6.** *Let $A$ be a SAPDO on a Hilbert space $H$. Then*

$$\|A^\alpha(I+sA)^{-1}\exp(-tA)\| \leqslant (s+t)^{-\alpha}, \qquad 0<\alpha\leqslant 1, \tag{2.2}$$

$$\|((I+tA)^{-1}-\exp(-tA))\varphi\| \leqslant c_1(\alpha)t^\alpha\|A^\alpha(I+tA)^{-1}\varphi\|, \qquad 0<\alpha\leqslant 2, \tag{2.3}$$

*where $\varphi \in D(A)$, $s \geqslant 0$, $t>0$, $c_1(\alpha)=2/\alpha$ for $0<\alpha<1$, and $c_1(\alpha)=1$ for $1\leqslant\alpha\leqslant 2$.*

**Proof.** Let us prove estimate (2.2). We have (see, e.g., [16]) $\lim_{n\to\infty} V_n(t)\varphi = \exp(-tA)\varphi$ for $\varphi \in H$, where $V_n(t) = (I+\frac{t}{n}A)^{-n}$; therefore, it suffices to prove the inequality

$$\|A^\alpha(I+sA)^{-1}V_n(t)\| \leqslant (s+t)^{-\alpha}.$$

By the definition of a fractional power of an operator, we have (see, e.g., [16])

$$A^\alpha u = \frac{\sin(\pi\alpha)}{\pi}\int_0^\infty \lambda^{\alpha-1}(A+\lambda I)^{-1}Au\,d\lambda, \qquad u\in D(A),\ 0<\alpha<1. \tag{2.4}$$

Substituting the vector $u=(I+sA)^{-1}V_n(t)\varphi$ ($\varphi\in H$) into (2.4), passing to norms, and taking into account Remark 2.5, we obtain

$$\begin{aligned}
&\|A^\alpha(I+sA)^{-1}V_n(t)\varphi\|\\
&\quad\leqslant \max_{x\geqslant 0}\left(\frac{\sin(\pi\alpha)}{\pi}\int_0^\infty \lambda^{\alpha-1}(x+\lambda)^{-1}x(1+sx)^{-1}\left(1+\frac{t}{n}x\right)^{-n}d\lambda\right)\|\varphi\|\\
&\quad= \max_{x\geqslant 0}\left(x^\alpha(1+sx)^{-1}\left(1+\frac{t}{n}x\right)^{-n}\right)\|\varphi\|\\
&\quad\leqslant \max_{x\geqslant 0}(x^\alpha(1+sx)^{-1}(1+tx)^{-1})\|\varphi\| \leqslant (s+t)^{-\alpha}\|\varphi\|.
\end{aligned} \tag{2.5}$$

Obviously, for $\alpha=1$, relation (2.5) without the integral term holds. Therefore,

$$\|A^\alpha(I+sA)^{-1}V_n(t)\varphi\| \leqslant (s+t)^{-\alpha}\|\varphi\|, \qquad 0<\alpha\leqslant 1. \tag{2.6}$$

Inequality (2.6) implies (2.2).

Let us prove estimate (2.3). For an integer $k$, we obviously have

$$\|A^kV_n(t)\| \leqslant \frac{1}{t^k}(k^k)\left(1+\frac{k}{n-k}\right)^{k-n}, \qquad n>k. \tag{2.7}$$

For a noninteger $k$, separating out the fractional part ($k=[k]+\alpha$, $0<\alpha<1$) and applying representation (2.4), we obtain the same relation as for an integer $k$, i. e., (2.7).

By virtue of (2.7), for $n\geqslant 4$, we have

$$\|A^\alpha V_n(t)\| \leqslant \frac{1}{t^\alpha}, \qquad 1<\alpha<2. \tag{2.8}$$

Taking into account (2.8), we obtain

$$\begin{aligned}
\|A^\alpha(I+sA)^{-1}V_n(t)\| &\leqslant \|A(I+sA)^{-1}(I+tA)^{-1}\|\|(I+tA)A^{\alpha-1}V_n(t)\|\\
&\leqslant (s+t)^{-1}(\|A^{\alpha-1}V_n(t)\|+t\|A^\alpha V_n(t)\|)\\
&\leqslant 2t^{1-\alpha}(s+t)^{-1}, \qquad 1<\alpha<2.
\end{aligned} \tag{2.9}$$

The following representation holds (see [16; Ch. IX]):

$$(L(t) - V_n(t))u = \int_0^t \left(s - \frac{t-s}{n}\right)\left(I + \frac{t-s}{n}A\right)^{-1} V_n(t-s)AL^2(s)A\,ds, \qquad u \in D(A), \quad (2.10)$$

where $L(t) = (I + tA)^{-1}$.

Let us reduce (2.10) to the form

$$(L(t) - V_n(t))u = \int_0^t (A^{2-\alpha}L(s)V_n(t-s))K(s,t)(A^\alpha L(t)u)\,ds,$$

where

$$K(s,t) = \left(s - \frac{t-s}{n}\right)(I + tA)\left(I + \frac{t-s}{n}A\right)^{-1} L(s).$$

Since

$$(s_1 + s_2)\|(I + tA)(I + s_1A)^{-1}(I + s_2A)^{-1}\| \leqslant t$$

for $s_1 \geqslant 0$, $s_2 \geqslant 0$, $s_1 + s_2 > 0$, and $t \geqslant s_1 + s_2$, it follows that $\|K(s,t)\| \leqslant t$, and (2.10) implies

$$\|(L(t) - V_n(t))u\| \leqslant t \int_0^t \|A^{2-\alpha}L(s)V_n(t-s)\|\,\|A^\alpha L(t)u\|\,ds, \qquad 0 < \alpha \leqslant 2. \quad (2.11)$$

Using estimate (2.6) for $1 \leqslant \alpha < 2$ and estimate (2.9) for $0 < \alpha < 1$, we obtain the desired bound from inequality (2.11). In view of the inequality $\|L(s)V_n(t-s)\| \leqslant 1$, for $\alpha = 2$, (2.3) follows directly from (2.11). □

**Lemma 2.7.** *Let $A$ and $B$ be SAPDOs with domains $D(A) \subset D(B)$. If $\|Bu\| \leqslant \|Au\|$ for $u \in D(A)$, then*

$$\|B(I + sB)^{-1}u\| \leqslant \|A(I + qtA)^{-1}u\|, \qquad u \in H, \quad (2.12)$$

*where $0 < q \leqslant 1$ and $s \geqslant t > 0$.*

**Proof.** By Remark 2.4 we have $\|A^{-1}u\| \leqslant \|B^{-1}u\|$ for $u \in H$. This, together with Theorem V.4.12 of [16], implies $(A^{-1}u, u) \leqslant (B^{-1}u, u)$. Taking into account these inequalities, we obtain

$$\begin{aligned}\|(qtI + A^{-1})u\|^2 &= q^2t^2\|u\|^2 + 2qt(A^{-1}u, u) + \|A^{-1}u\|^2 \\ &\leqslant s^2\|u\|^2 + 2s(B^{-1}u, u) + \|B^{-1}u\|^2 = \|(sI + B^{-1})u\|^2.\end{aligned}$$

Again applying Remark 2.4, we see that

$$\|(sI + B^{-1})^{-1}u\| \leqslant \|(qtI + A^{-1})^{-1}u\|, \qquad u \in H.$$

This is the required inequality. □

**Corollary 2.8.** *Let $A$ and $B$ be SAPDOs with domains $D(A) \subset D(B)$. If $\|Bu\| \leqslant b_0\|Au\|$ for $u \in D(A)$, then*

$$\|B(I + sB)^{-1}u\| \leqslant b_1\|A(I + tA)^{-1}u\|, \qquad u \in H,$$

*where $s \geqslant t > 0$ and $b_1 = \max(1, b_0)$.*

For $b_0 \leqslant 1$, it suffices to substitute $q = 1$ into inequality (2.12), and for $b_0 > 1$, to substitute $q = 1/b_0$ and replace the operator $A$ by $b_0A$.

Now we can proceed to the proof of Theorem 2.1.

**Proof of Theorem 2.1.** We introduce the following notation: $U(t) = \exp(-tA)$, $u(t) = U(t)u_0$ with $u_0 \in D(A)$, $\tau = t/n$, and $t_k = k\tau$, where $k$ and $n$ are positive integers. Since the function $u(t) = U(t)u_0$ with $u_0 \in D(A)$ satisfies the equation $u'(t) + Au(t) = 0$ (see Remark 1.5), we have

$$u(t_k) = L(\tau)u(t_{k-1}) + \tau L(\tau)\varphi_k, \qquad k = 1, \dots, n, \tag{2.13}$$

where $L(\tau) = (I + \tau A)^{-1}$ and $\varphi_k = (u(t_k) - u(t_{k-1}))/\tau - u'(t_k)$.

Equation (2.13) gives (for brevity, we will write $S$ and $L$ instead of $S(\tau)$ and $L(\tau)$, respectively) $u(t_k) = Su(t_{k-1}) + g_k$, where $g_k = (L - S)u(t_{k-1}) + \tau L\varphi_k$. This implies

$$u(t_k) = S^k u_0 + \sum_{i=1}^{k} S^{k-i} g_i. \tag{2.14}$$

Substituting $u(t_k) = U(t_k)u_0$ into (2.14), replacing the vector $\varphi_i$ by $(I - S)^{1/2}(I - S)^{-1/2}\varphi_i$, and taking $S^k u_0$ to the left-hand side, we obtain

$$(U(t_k) - S^k)u_0 = \sum_{i=1}^{k} S^{k-i}(I - S)^{1/2} S_0 g_i, \qquad S_0 = (I - S)^{-1/2}. \tag{2.15}$$

Since $S(t)$ is a self-adjoint operator and the spectrum $\sigma(S(t))$ lies in the interval $[0, 1]$ for any $t \geqslant 0$, it follows from Remark 2.5 that

$$\|S^k(I - S)^{1/2}\| \leqslant \max_{0 \leqslant \lambda \leqslant 1} [\lambda^k (1 - \lambda)^{1/2}] \leqslant (2k + 1)^{-1/2}. \tag{2.16}$$

Equation (2.15), together with (2.16), yields the inequality

$$\|(U(t_k) - S^k)u_0\| \leqslant \sum_{i=1}^{k} (2(k - i) + 1)^{-1/2} (\|S_0(L - S)u(t_{i-1})\| + \tau\|S_0 L\varphi_i\|). \tag{2.17}$$

In what follows, we need the estimates

$$\|S_0(\tau)AL(\tau)f\| \leqslant \tau^{-\alpha}(m\|A^\alpha Lf\| + c_3\|A^\alpha L^\alpha f\|), \tag{2.18}$$

$$\|S_0(\tau)L(\tau)A\varphi\| \leqslant \tau^{-\alpha} c_4 \|A^\alpha \varphi\|, \tag{2.19}$$

$$\|S_0(\tau)(S(\tau) - L(\tau))f\| \leqslant \tau c_1 \|ALf\|, \qquad S_0(\tau) = (I - S(\tau))^{-1/2}, \tag{2.20}$$

where $\alpha = 1/2$, $\varphi \in D(A)$, $f \in H$, and $c_4 = m + c_3$.

Let us prove (2.18). Since $I - S \geqslant \eta_j(I - S_j) = \tau A_j S_j > 0$, we have

$$(I - S)^{-1} \leqslant \eta_j^{-1}(I - S_j)^{-1} = \tau^{-1}A_j^{-1} + \eta_j^{-1} I \leqslant (\tau^{-\alpha}A_j^{-\alpha} + \eta_j^{-\alpha} I)^2, \qquad \alpha = 1/2 \tag{2.21}$$

(see Remark 2.3(a)). Therefore,

$$\|S_0 f\| = \|(I - S)^{-\alpha} f\| \leqslant \tau^{-\alpha}\|A_j^{-\alpha} f\| + \eta_j^{-\alpha}\|f\|, \qquad f \in H. \tag{2.22}$$

Applying inequality (2.22), we obtain (in what follows, it is always assumed that $\alpha = 1/2$)

$$\|S_0 ALf\| = \|S_0(A_1 + \dots + A_m)Lf\| \leqslant \tau^{-\alpha} \sum_{j=1}^{m} (\|A_j^\alpha Lf\| + \tau^\alpha \eta_j^{-\alpha}\|A_j Lf\|). \tag{2.23}$$

According to Remark 2.3(b), we have $\|A_j^\alpha Lf\| \leqslant \|A^\alpha Lf\|$, and according to Remark 2.2, we have $\|A_j Lf\| \leqslant \|A_j A^{-1}\|\|ALf\| = a_j\|ALf\|$. Taking into account these inequalities, we see that

(2.23) implies $\|S_0ALf\| \leqslant m\tau^{-\alpha}\|A^\alpha Lf\| + c_3\|ALf\|$. This inequality, together with $\|ALf\| \leqslant \tau^{-\alpha}\|A^\alpha L^\alpha f\|$, gives (2.18).

Let us prove (2.20). By virtue of (2.21) we have $\|S_0h\| \leqslant \eta_j^{-\alpha}\|(I-S_j)^{-\alpha}h\|$ for $h \in H$. Substituting the vector $h = (I-S_j)f$, we obtain

$$\|S_0(I-S_j)f\| \leqslant \eta_j^{-\alpha}\|(I-S_j)^\alpha f\| \leqslant \eta_j^{-\alpha}\|f\|. \tag{2.24}$$

We have

$$S - L = \tau\sum_{j=1}^{m}\eta_j^{-1}(I-S_j)(\eta_j^{-1}A_jA^{-1} - I)AL. \tag{2.25}$$

Relations (2.24) and (2.25) imply the inequality

$$\|S_0(S-L)f\| \leqslant \tau\sum_{j=1}^{m}\eta_j^{-3/2}\|(\eta_j^{-1}A_jA^{-1}-I)ALf\| \leqslant \tau c_1\|ALf\|.$$

Inequality (2.19) follows from (2.18).

Using (2.20) and the equation $u(t) = U(t)u_0$, we obtain

$$\|S_0(L-S)u(t_i)\| \leqslant \tau c_4\|ALU(t_i)\|\|u_0\|.$$

By virtue of (2.2), we also have $\|AL(\tau)U(t)\| \leqslant (t+\tau)^{-1}$. Therefore,

$$\|S_0(L-S)u(t_{i-1})\| \leqslant \frac{c_1}{i}\|u_0\|. \tag{2.26}$$

Representing $\tau\varphi_i$ as the integral of $u'(t) - u'(t_i)$ from $t_{i-1}$ to $t_i$, transforming the difference by the formula $u'(t) = -Au(t) = -AU(t)u_0$ (see Remark 1.5), and substituting the result for $f$ in the inequality $\|S_0Lf\| \leqslant \tau^{-1/2}c_5\|A^{-1/2}f\|$ (this inequality follows from (2.19)), we obtain

$$\tau\|S_0L\varphi_i\| \leqslant \tau^{-1/2}c_4\int_{t_{i-1}}^{t_i}\|A^{1/2}(U(t)-U(t_i))\|\,dt\|u_0\|. \tag{2.27}$$

By virtue of (2.7) we have

$$\|A^{3/2}U(t)\| \leqslant \frac{1}{t^{3/2}}, \qquad t > 0. \tag{2.28}$$

Let us transform the integrand in (2.27) by formula (1.19) and take into account (2.28). As a result, we obtain

$$\begin{aligned}\tau\|S_0L\varphi_i\| &\leqslant c_4\tau^{-1/2}\int_{t_{i-1}}^{t_i}\int_t^{t_i}\|A^{3/2}U(s)\|\,ds\,dt\|u_0\| \\ &\leqslant 2c_5\tau^{-1/2}[2(t_i^{1/2} - t_{i-1}^{1/2}) - \tau t_i^{-1/2}]\|u_0\|\frac{2c_4}{\sqrt{i}(\sqrt{i}+\sqrt{i-1})^2}\|u_0\|.\end{aligned} \tag{2.29}$$

The substitution of (2.26) and (2.29) into (2.17) and simple calculations yield

$$\|(U(t_k) - S^k)u_0\| \leqslant (c_2 + c_1\ln k)\frac{1}{\sqrt{2k+1}}\|u_0\|. \tag{2.30}$$

Obviously, (2.30) implies (2.1). □

The following theorem is proved in a similar way.

**Theorem 2.9.** *Suppose that the assumptions of Theorem* 2.1 *hold. Then*

$$\left\| \exp(-tA) - \left(U_0\left(\frac{t}{n}\right)\right)^n \right\| \leqslant (c_6 + c_5 \ln n)\frac{1}{\sqrt{2n+1}}, \tag{2.31}$$

*where* $U_0(t) = \eta_1 U_1(t) + \cdots + \eta_m U_m(t)$ *and* $U_j(t) = \exp(-t\eta_j^{-1}A_j)$ $(j = 1, \ldots, m)$ *for* $t > 0$, $n$ *is a positive integer*, $c_6$ *and* $c_5$ *are positive constants*, $c_5 = m + c_3 + c_1 + c_0$, $c_0 = b_1\eta_1^{1/2}(\eta_1^{1/2} + 1) + \cdots + b_m\eta_m^{1/2}(\eta_m^{1/2} + 1)$, $b_j = \max(1, a_j)$, *and* $c_6 = 3c_5$ (*the remaining constants* $c_1$, $c_3$, *and* $a_j$ *are the same as in Theorem* 2.1).

**Proof.** We use the same notation as in the proof of Theorem 2.1. Obviously, $u(t_k) = U(\tau)u(t_{k-1})$, or, equivalently,

$$u(t_k) = U_0(\tau)u(t_{k-1}) + \psi_k, \qquad \psi_k = (U(\tau) - U_0(\tau))u(t_{k-1}). \tag{2.32}$$

From (2.32) it follows that

$$u(t_k) = (U_0(\tau))^k u_0 + \sum_{i=1}^{k} (U_0(\tau))^{k-i}\psi_i. \tag{2.33}$$

Substituting $u(t_k) = U(t_k)u_0$ into (2.33), replacing the vector $\psi_i$ by

$$(I - U_0(\tau))^{1/2}(I - U_0(\tau))^{-1/2}\psi_i,$$

and taking $(U_0(\tau))^k u_0$ to the left-hand side, we obtain

$$(U(t_k) - (U_0(\tau))^k)u_0 = \sum_{i=1}^{k} (U_0(\tau))^{k-i}(I - U_0(\tau))^{1/2}T_0(\tau)\psi_i, \tag{2.34}$$

where $T_0(\tau) = (I - U_0(\tau))^{-1/2}$.

Since the operator $U_0(t)$ is self-adjoint and the spectrum $\sigma(U_0(t))$ lies in the interval $[0, 1]$ for any $t \geqslant 0$, it follows from (2.34) that (see (2.17))

$$\|(U(t_k) - (U_0(\tau))^k)u_0\| \leqslant \sum_{i=1}^{k} (2(k-i)+1)^{-1/2}\|T_0(\tau)\psi_i\|. \tag{2.35}$$

It is easy to show that

$$\|T_0(t)(I - S(t))^{1/2}\| \leqslant 1, \qquad t > 0. \tag{2.36}$$

Indeed, since

$$((I + (t/n)\eta_j^{-1}A_j)^n u, u) \geqslant ((I + t\eta_j^{-1}A_j)u, u), \qquad u \in D(A_j^n),$$

it follows that $(S_j^n(t/n)u, u) \leqslant (S_j(t)u, u)$ for $u \in H$ (see Remark 2.3(a)). Letting $n$ tend to infinity, we see that $(U_j(t)u, u) \leqslant (S_j(t)u, u)$ for $j = 1, \ldots, m$. Multiplying these inequalities by $\eta_j$ and adding them together, we obtain $(U_0(t)u, u) \leqslant (S(t)u, u)$. This gives $((I - U_0(t))u, u) \geqslant ((I - S(t))u, u)$ for $t > 0$. This inequality, again by virtue of Remark 2.3(a), implies $((I - U_0(t))^{-1}u, u) \leqslant ((I - S(t))^{-1}u, u)$, or $\|T_0(t)u\| \leqslant \|S_0(t)\|$, where $S_0(t) = (I - S(t))^{-1/2}$. Substituting the vector $u = (I - S(t))^{1/2}\varphi$ with $\varphi \in H$ into this estimate, we arrive at estimate (2.36).

Obviously, we have

$$U_0(t) - U(t) = \sum_{j=1}^{m} \eta_j(U_j(t) - S_j(t)) + (S(t) - L(t)) + (L(t) - U(t)). \tag{2.37}$$

Equation (2.37) and estimate (2.36) imply the inequality

$$\|T_0(\tau)\psi_k\| \leqslant \|S_0(\tau)\zeta_k\| + \|S_0(\tau)\zeta_k^0\| + \sum_{j=1}^{m} \eta_j \|S_0(\tau)\zeta_k^j\|, \tag{2.38}$$

where

$$\zeta_k = (S(\tau) - L(\tau))u(t_{k-1}), \quad \zeta_k^0 = (L(\tau) - U(\tau))u(t_{k-1}),$$
$$\zeta_k^j = (U_j(\tau) - S_j(\tau))u(t_{k-1}).$$

Using (2.22) and (2.3), we obtain

$$\|S_0\zeta_k^j\| \leqslant \tau^{-1/2}\|A_j^{-1/2}\zeta_k^j\| + \eta_j^{-1/2}\|\zeta_k^j\| \leqslant (1+\eta_j^{-1/2})\tau\|A_jS_j(\tau)u(t_{k-1})\|.$$

Since $\|A_ju\| \leqslant a_j\|Au\|$ for $u \in D(A)$ (see Remark 2.2), it follows from Corollary 2.8 that $\|S_0\zeta_k^j\| \leqslant b_j(1+\eta_j^{-1/2})\tau\|AL(\tau)u(t_{k-1})\|$, where $b_j = \max(1, a_j)$. This inequality implies

$$\sum_{j=1}^{m} \eta_j\|S_0(\tau)\zeta_k^j\| \leqslant \tau c_0\|AL(\tau)u(t_{k-1})\|.$$

The inequality $\|S_0ALf\| \leqslant m\tau^{-1/2}\|A^{1/2}Lf\| + c_3\|ALf\|$ (see the proof of Theorem 2.1) yields $\|S_0f\| \leqslant m\tau^{-1/2}\|A^{-1/2}f\| + c_3\|f\|$ for $f \in H$. Substituting the vector $\zeta_k^0$ in place of $f$ and applying (2.3), we obtain $\|S_0\zeta_k^0\| \leqslant (m+c_3)\tau\|AL(\tau)u(t_{k-1})\|$. By virtue of (2.20), for the vector $S_0(\tau)\zeta_k$, we have $\|S_0(\tau)\zeta_k\| \leqslant \tau c_1\|ALu(t_{k-1})\|$. Substituting these estimates into (2.38), we arrive at the inequality $\|T_0(\tau)\psi_k\| \leqslant c_5\tau\|AL(\tau)u(t_{k-1})\|$. Substituting the vector $u(t_{k-1}) = U(t_{k-1})u_0$ (see Remark 1.5) and using (2.2), we obtain $\|T_0(\tau)\psi_k\| \leqslant c_4/k$. The substitution of this estimate into (2.35) and simple calculations yield

$$\|(U(t_k) - (U_0(\tau))^k)u_0\| \leqslant (c_6 + c_5\ln k)\frac{1}{\sqrt{2k+1}}\,. \tag{2.39}$$

Obviously, (2.39) implies (2.31). □

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