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A Remark on the Interpolation Inequality between Sobolev Spaces and Morrey Spaces

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Abstract. Interpolation inequalities play an important role in the study of PDEs and their applications. There are still some interesting open questions and problems related to integral estimates and regularity of solutions to elliptic and/or parabolic equations. The main purpose of our work is to provide an important observation concerning the L^p-boundedness property in the context of interpolation inequalities between Sobolev and Morrey spaces, which may be useful for those working in this domain. We also construct a nontrivial counterexample, which shows that the range of admissible values of p is optimal in a certain sense. Our proofs rely on integral representations and on the theory of maximal and sharp maximal functions.

Key words: interpolation inequality, L_p-boundedness, Sobolev spaces, Morrey spaces, Hardy–Littlewood maximal operator.

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1. Introduction

In the theory of partial differential equations, the study of interpolation inequalities in Lebesgue, Sobolev, and Morrey spaces play an important role. There is still much to be done in the investigation of interesting inequalities arising from L^p-estimates or regularity results for L^p-solutions of PDEs. In recent years, the number of publications devoted to the study of these inequalities and their improvements has increased; see, e.g., [6], [11], [5], [8] and references therein. In this paper we are interested in an interpolation inequality between the Sobolev and Morrey spaces obtained in [8]. We present an interesting independent proof of this inequality, which uses sharp maximal functions $\mathbf{M}_{\#}$, and apply it to prove the boundedness of the L^r-norm. The main result is the counterexample in Theorem 2.4, which shows the optimality of the admissible range of r values in the inequality. We believe that this work opens new perspectives in the study of the regularity, existence, and uniqueness of solutions and weak solutions of partial differential equations, as well as Sobolev and Morrey spaces, which play a very important role in PDE theory and its applications.

We consider Euclidean space $\mathbb{R}^d$, $d \geqslant 2$. For certain parameters $1 \leqslant p < d$ and $1 < q < pd/(d-p)$, we therein focus on a brief proof of the following L^r-boundedness property of the set of functions $u \in C_c^1(\mathbb{R}^d)$ with uniformly bounded norms in certain Sobolev and Morrey spaces:

$$\int_{\mathbb{R}^d} |u|^r d\mathbf{x} \leqslant C, \qquad r \in \left[p\left(\frac{q}{d}+1\right), \frac{pd}{d-p}\right]. \tag{1.1}$$

This property is connected with solution regularity of PDEs, especially nonlinear, and plays a significant role in analysis. In particular, it makes sense to study solution properties in the specified range of the parameter r. As we will show in the present work, there is a counterexample for which inequality in (1.1) does not hold for some values of r.

In order to state our result, we first recall some related notation and definitions. Here we only recall the basic notions that we use in statements and proofs. The notation used throughout the paper is standard; some conventions will be introduced in what follows. For definitions and properties of spaces and operators under consideration we refer the reader to any of the many textbooks and references in [1]–[4].

1. *Open ball*: If $\mathbf{x} \in \mathbb{R}^d$ and r is a positive real number, by $B_r(\mathbf{x})$ we denote the open ball $\{\mathbf{y} \in \mathbb{R}^d : |\mathbf{y} - \mathbf{x}| < r\}$ in $\mathbb{R}^d$.

2. *Mean value*: the notation $\fint_{B_r(\mathbf{x})} f(\mathbf{y})d\mathbf{y}$ is used for the integral average, or mean value, of f in the variable y over the ball $B_r(\mathbf{x})$, i.e.,

$$\fint_{B_r(\mathbf{x})} f(\mathbf{y})\, d\mathbf{y} = \frac{1}{|B_\rho(\mathbf{x})|} \int_{B_r(\mathbf{x})} f(\mathbf{y})\, d\mathbf{y},$$

where $|B|$ denotes the d-dimensional Lebesgue measure of a set $B \subset \mathbb{R}^d$.

3. For any set $I \subset \mathbb{R}$, by $C_c^k(\mathbb{R}^m, I)$ (by $C_c^\infty(\mathbb{R}^m, I)$) we denote the set of k-times differentiable (infinitely differentiable, respectively) functions f on $\mathbb{R}^m$ with compact support and values in I, i.e., functions for which there exists a compact set $K \subset \mathbb{R}^m$ such that

$$\operatorname{supp}(f) = \text{cl.}\{\mathbf{x} \in \mathbb{R}^m : f(\mathbf{x}) \neq 0\} \subset K.$$

4. *The Hardy–Littlewood maximal function*: For $\mathbf{x} \in \mathbb{R}^d$, the Hardy–Littlewood maximal function is defined for each locally integrable function f in $\mathbb{R}^d$ by

$$\mathbf{M}(f)(\mathbf{x}) = \sup_{\rho>0} \fint_{B_\rho(\mathbf{x})} |f|\, d\mathbf{y}.$$

5. *Sharp maximal function*: For $\mathbf{x} \in \mathbb{R}^d$, the sharp maximal function of f is defined as

$$\mathbf{M}_\#(f)(\mathbf{x}) = \sup_{\rho>0} \fint_{B_\rho(\mathbf{x})} \left| f(\mathbf{y}) - \fint_{B_\rho(\mathbf{x})} f(\mathbf{z})\, d\mathbf{z} \right| d\mathbf{y}. \tag{1.2}$$

6. By $a \sim b$ we mean that there exist some positive real constants $C_1, C_2 > 0$ such that $C_1 a \leqslant b \leqslant C_2 a$.

The rest of this paper is structured as follows. In the next section we state the main result and some preparatory lemmas. In Section 3 we give a detailed proof of the main theorem and some other necessary proofs.

2. Statement of Main Results

In this section we list all lemmas, theorems, and corollaries that will be discussed and proved. The first integral inequality is given in Lemma 2.1. It is not new (see [8]), but for the reader's convenience we give its self-contained proof in Section 3.

Lemma 2.1. *Let $1 < p < d$, and let $1 < q < pd/(d-p)$. Then the following integral inequality holds*:

$$\int_{\mathbb{R}^d} |u|^{p(q/d+1)} d\mathbf{x} \leqslant C \int_{\mathbb{R}^d} |\nabla u|^p d\mathbf{x} \left(\sup_{B_\rho(\mathbf{z})} \rho^{d/q} \fint_{B_\rho(\mathbf{z})} |u|\, d\mathbf{x} \right)^{qp/d}, \qquad u \in C_c^1(\mathbb{R}^d). \tag{2.1}$$

The following theorem contains an interpolation inequality in Lebesgue spaces. This theorem is an easy corollary of inequality (2.1), Sobolev's inequality, and Hölder's inequality (see [4]).

Theorem 2.2. *Let $1 < p < d$, and let $1 < q < pd/(d-p)$. If $u \in C_c^1(\mathbb{R}^d)$,*

$$\int_{\mathbb{R}^d} |\nabla u|^p d\mathbf{x} \leqslant 1, \tag{2.2}$$

and

$$\sup_{B_\rho(\mathbf{z})} \rho^{d(q_1-q)/q} \int_{B_\rho(\mathbf{z})} |u|^{q_1} d\mathbf{x} \leqslant 1 \tag{2.3}$$

for some $1 \leqslant q_1 \leqslant q$, *then*

$$\int_{\mathbb{R}^d} |u|^r d\mathbf{x} \leqslant C \quad \textit{for any } r \in \left[p\left(\frac{q}{d}+1\right), \frac{pd}{d-p}\right]. \tag{2.4}$$

The proof of Theorem 2.2 given in the next Section 3 brings up an interesting problem, namely, the following question concerning the range of r in (2.4): Is it possible to extend the range of values of r for which the interpolation inequality (2.4) still holds? A solution of this problem would complete a picture in studying interpolation inequalities in Lebesgue spaces; it is also related in a certain way to the Gagliardo–Nirenberg inequalities and their generalizations (we refer the reader to [7], [1], [2], [6], [5] and related references therein). This question arises from the theory of PDEs and might provide a procedure for analyzing regularity and obtaining some comparison estimates or important properties of solutions to elliptic and/or parabolic equations in future studies. That is the main motivation for writing this paper.

The following result, Corollary 2.3, asserts that inequality (2.4) holds for an extended range of r but only in the specific case $q_1 = q$. A brief proof is given in Section 3.

Corollary 2.3. *If* $q_1 = q$ *in Theorem* 2.2, *then* (2.4) *is true for any* $r \in [q, p(q/d+1)]$.

However, as we will show in this paper, this interpolation inequality does not hold if $q_1 < q$ and $r < p(q/d+1)$. This leads us to the very important conclusion that the range of r in Theorem 2.2 is optimal. A counterexample in which inequality (2.4) does not hold outside the interval $[q, p(q/d+1)]$ is given in the following theorem.

Theorem 2.4. *Let* $1 < p < d$, *and let* $1 < q < pd/(d-p)$. *Then, for any* $1 \leqslant q_1 < q$, *there exists a sequence* $(u_n)_n \in C_c^\infty(\mathbb{R}^d)$ *such that*

$$\int_{\mathbb{R}^d} |\nabla u_n|^p \, d\mathbf{x} \leqslant 1, \qquad \sup_{B_\rho(\mathbf{z})} \rho^{d(q_1-q)/q} \int_{B_\rho(\mathbf{z})} |u_n|^{q_1} d\mathbf{x} \leqslant 1. \tag{2.5}$$

Moreover, *for any* $r > 0$,

$$\int_{\mathbb{R}^d} |u_n|^r d\mathbf{x} \sim 2^{\frac{d^2(q-q_1)}{q(dp-(d-p)q)}(p(1+q/d)-r)n}, \qquad n \gg 1. \tag{2.6}$$

In particular,

$$\int_{\mathbb{R}^d} |u_n|^r d\mathbf{x} \leqslant C, \qquad n \gg 1, \tag{2.7}$$

if and only if $r \geqslant p(q/d+1)$.

3. Proofs of the Main Results

This section is devoted to proofs of our statements in the previous section. We note that the positive constants C may be different in different inequalities, although they are denoted by the same letter.

Proof of Lemma 2.1. By Poincaré's inequality, for $\rho > 0$ and any $\mathbf{y} \in \mathbb{R}^d$, we have

$$⨍_{B_\rho(\mathbf{y})} |u - (u)_{B_\rho(\mathbf{y})}| \, d\mathbf{x} \leqslant C\rho ⨍_{B_\rho(\mathbf{y})} |\nabla u| \, d\mathbf{x} \quad \text{for any } u \in C_c^1(\mathbb{R}^d), \tag{3.1}$$

where $(f)_\Omega$ denotes the mean value of f over the domain Ω. Taking the supremum of both sides of (3.1) over all $\rho > 0$, we obtain

$$\sup_{\rho>0} \rho^{-1} ⨍_{B_\rho(\mathbf{y})} |u - (u)_{B_\rho(\mathbf{y})}| \, d\mathbf{x} \leqslant C\mathbf{M}(|\nabla u|)(\mathbf{y}) \quad \text{for any } \mathbf{y} \in \mathbb{R}^d. \tag{3.2}$$

By the definition of the sharp maximal function $\mathbf{M}_{\#}$ in (1.2), we can decompose it as

$$[\mathbf{M}_{\#}(u)(\mathbf{y})]^{p(q/d+1)} = \sup_{\rho>0}\left(\rho^{d/q} \fint_{B_\rho(\mathbf{y})} |u-(u)_{B_\rho(\mathbf{y})}|\,d\mathbf{x}\right)^{pq/d}\left(\rho^{-1} \fint_{B_\rho(\mathbf{y})} |u-(u)_{B_\rho(\mathbf{y})}|\,d\mathbf{x}\right)^{p}$$
$$\leqslant C\left(\sup_{\rho>0,\,\mathbf{z}\in\mathbb{R}^d} \rho^{d/q} \fint_{B_\rho(\mathbf{z})} |u|\,d\mathbf{x}\right)^{qp/d}\left(\sup_{\rho>0}\rho^{-1} \fint_{B_\rho(\mathbf{y})} |u-(u)_{B_\rho(\mathbf{y})}|\,d\mathbf{x}\right)^{p}. \tag{3.3}$$

Thanks to (3.2) and (3.3), one also has the following inequality between the Hardy–Littlewood maximal function $\mathbf{M}$ and the sharp maximal function $\mathbf{M}_{\#}$:

$$[\mathbf{M}_{\#}(u)(\mathbf{y})]^{p(q/d+1)} \leqslant C\left(\sup_{B_\rho(\mathbf{z})} \rho^{d/q} \fint_{B_\rho(\mathbf{z})} |u|\,d\mathbf{x}\right)^{qp/d} [\mathbf{M}(|\nabla u|)(\mathbf{y})]^p. \tag{3.4}$$

It is well known [9] that, for any $s\in(1,\infty)$ and $f\in L^s(\mathbb{R}^d)$, there exists a positive constant C such that the following inequalities hold:

$$C^{-1}\int_{\mathbb{R}^d} |f|^s d\mathbf{x} \leqslant \int_{\mathbb{R}^d} |\mathbf{M}_{\#}(f)|^s d\mathbf{x} \leqslant C\int_{\mathbb{R}^d} |f|^s d\mathbf{x}, \tag{3.5}$$
$$C^{-1}\int_{\mathbb{R}^d} |f|^s d\mathbf{x} \leqslant \int_{\mathbb{R}^d} |\mathbf{M}(f)|^s d\mathbf{x} \leqslant C\int_{\mathbb{R}^d} |f|^s d\mathbf{x}. \tag{3.6}$$

It follows from (3.4), (3.5), and (3.6) that

$$\int_{\mathbb{R}^d} |u|^{p(q/d+1)} d\mathbf{x} \leqslant C\int_{\mathbb{R}^d} [\mathbf{M}_{\#}(u)(\mathbf{x})]^{p(q/d+1)} d\mathbf{x}$$
$$\leqslant C\int_{\mathbb{R}^d} [\mathbf{M}(|\nabla u|)(\mathbf{x})]^p \left(\sup_{B_\rho(\mathbf{z})} \rho^{d/q} \fint_{B_\rho(\mathbf{z})} |u(\mathbf{y})|\,d\mathbf{y}\right)^{qp/d} d\mathbf{x}$$
$$\leqslant C\int_{\mathbb{R}^d} |\nabla u|^p \left(\sup_{B_\rho(\mathbf{z})} \rho^{d/q} \fint_{B_\rho(\mathbf{z})} |u(\mathbf{y})|\,d\mathbf{y}\right)^{qp/d} d\mathbf{x}$$
$$\leqslant C\left(\sup_{B_\rho(\mathbf{z})} \rho^{d/q} \fint_{B_\rho(\mathbf{z})} |u|\,d\mathbf{x}\right)^{qp/d} \int_{\mathbb{R}^d} |\nabla u|^p d\mathbf{x},$$

which completes our proof. □

Proof of Theorem 2.2. According to the strengthened Sobolev inequality, for all $u\in W^{1,p}(\mathbb{R}^d)$, one has

$$\left(\int_{\mathbb{R}^d} |u|^{pd/(d-p)} d\mathbf{x}\right)^{(d-p)/(dp)} \leqslant C\left(\int_{\mathbb{R}^d} |\nabla u|^p d\mathbf{x}\right)^{1/p},$$

which implies that if (2.2) holds, then

$$\int_{\mathbb{R}^d} |u|^{pd/(d-p)} d\mathbf{x} \leqslant C. \tag{3.7}$$

Thanks to Hölder's inequality, for all u satisfying (2.3), we have

$$\sup_{B_\rho(\mathbf{z})} \rho^{d/q} \fint_{B_\rho(\mathbf{z})} |u|\,d\mathbf{x} \leqslant C \sup_{B_\rho(\mathbf{z})} \rho^{d(1-q)/q} \int_{B_\rho(\mathbf{z})} |u|\,d\mathbf{x}$$
$$\leqslant C \sup_{B_\rho(\mathbf{z})} \rho^{d(1-q)/q}\cdot\rho^{d(q_1-1)/q_1}\left(\int_{B_\rho(\mathbf{z})} |u|^{q_1} d\mathbf{x}\right)^{1/q_1}$$
$$\leqslant C\left(\sup_{B_\rho(\mathbf{z})} \rho^{d(q_1-q)/q} \int_{B_\rho(\mathbf{z})} |u|^{q_1} d\mathbf{x}\right)^{1/q_1} \leqslant C$$

for any $1 \leqslant q_1 \leqslant q$. Using Lemma 2.1, one obtains

$$\int_{\mathbb{R}^d} |u|^{p(q/d+1)} d\mathbf{x} \leqslant C. \tag{3.8}$$

It is easy to see that, for any $r \in [p(q/d+1), pd/(d-p)]$, there exists a $\theta \in [0,1]$ such that

$$\frac{1}{r} = \frac{\theta d}{p(q+d)} + \frac{(1-\theta)(d-p)}{pd}.$$

Applying Hölder's inequality, one obtains

$$\begin{aligned}\int_{\mathbb{R}^d} |u|^r d\mathbf{x} &= \int_{\mathbb{R}^d} |u|^{r\theta} |u|^{r(1-\theta)} d\mathbf{x} \\ &\leqslant \left(\int_{\mathbb{R}^d} [|u|^{r\theta}]^{p(q/d+1)/(r\theta)} d\mathbf{x} \right)^{r\theta d/(p(q+d))} \left(\int_{\mathbb{R}^d} [|u|^{r(1-\theta)}]^{pd/(r(1-\theta)(d-p))} d\mathbf{x} \right)^{r(1-\theta)(d-p)/(pd)} \\ &\leqslant \left(\int_{\mathbb{R}^d} |u|^{p(q/d+1)} d\mathbf{x} \right)^{r\theta d/(p(q+d))} \left(\int_{\mathbb{R}^d} |u|^{pd/(d-p)} d\mathbf{x} \right)^{r(1-\theta)(d-p)/(pd)}. \end{aligned} \tag{3.9}$$

It should be mentioned that the obtained inequality (3.9) is also an interpolation inequality in Lebesgue spaces; such inequalities were studied in [9, 10].

Combining (3.7), (3.8), and (3.9), we obtain the proof of Theorem 2.2. □

Next, let us give a proof of Corollary 2.3.

Proof of Corollary 2.3. If $q_1 = q$ in Theorem 2.2, assumption (2.3) becomes

$$\sup_{B_\rho(\mathbf{z})} \int_{B_\rho(\mathbf{z})} |u|^q d\mathbf{x} \leqslant 1,$$

and it follows that

$$\int_{\mathbb{R}^n} |u|^q d\mathbf{x} \leqslant 1. \tag{3.10}$$

The fulfillment of estimate (2.4) for any $r \in [q, p(q/d+1)]$ follows from (3.8), (3.10), and a repeatedly applied interpolation inequality in Lebesgue space of the form (3.9). □

Proof of Theorem 2.4. Suppose given functions $\phi \in C_c^\infty(\mathbb{R}^{d-1}, [0,1])$ and $\eta \in C_c^\infty(\mathbb{R}, [0,1])$ satisfying the conditions

$$\phi(\mathbf{x}) = \begin{cases} 1, & \mathbf{x} \in B_1(0), \\ 0, & \mathbf{x} \in B_2(0)^c, \end{cases} \qquad \eta(t) = \begin{cases} 1, & t \in (-1,1), \\ 0, & t \in (-2,2)^c, \end{cases}$$

$|\nabla \phi(\mathbf{x})| \leqslant 1$ for any $\mathbf{x} \in \mathbb{R}^{d-1}$, and $|\eta'(t)| \leqslant 1$ for any $t \in \mathbb{R}$.

Let us choose a parameter $\theta \in (0, 10^{-10d})$ and a number $n \geqslant 100/\theta$. Then, for any fixed $k \geqslant 10/\theta$, we define a sequence $\{a_{k,j}\}$ by

$$a_{k,j} = 2^{k-1} + 1 + j 2^{k-1-\theta(k-1)}, \qquad 1 \leqslant j \leqslant 2^{\theta(k-1)} - 3.$$

It is easy to check that, for any $1 \leqslant j \leqslant 2^{\theta(k-1)} - 3$, one has

$$a_{k,j} \in (2^{k-1} + 2, 2^k - 2).$$

Consider the functions χ_n and σ_n defined by

$$\chi_n(t) = \sum_{k \geqslant 10\theta^{-1}}^{n} \sum_{j=1}^{2^{\theta(k-1)}-3} \eta(t - a_{k,j}) \quad \text{and} \quad \sigma_n(t) = \chi_n(t) + \chi_n(-t).$$

Note that, since

$$\operatorname{supp}(\eta(\,\cdot\, - a_{k,j})) \cap \operatorname{supp}(\eta(\,\cdot\, - a_{k',j'})) = \varnothing \quad \text{for any } (k,j) \neq (k',j'),$$

it follows that, for any $r > 0$,

$$\sup_{\rho>0,\, t_0\in\mathbb{R}} \left(\rho^{-\theta} \int_{t_0-\rho}^{t_0+\rho} \sigma_n(t)^r dt \right) \sim 1 \quad \text{for all } n \geqslant 100/\theta, \tag{3.11}$$

$$\int_{\mathbb{R}} \sigma_n(t)^r dt \sim 2^{\theta n} \quad \text{for all } n \geqslant 100/\theta, \tag{3.12}$$

and

$$\int_{\mathbb{R}} |\sigma_n'(t)|^r dt \sim 2^{\theta n} \quad \text{for all } n \geqslant 100/\theta. \tag{3.13}$$

Indeed, relation (3.12) follows from the estimates

$$\begin{aligned}
\int_{\mathbb{R}} \sigma_n(t)^r dt &\sim \sum_{k\geqslant 10\theta^{-1}}^{n} \sum_{j=1}^{2^{\theta(k-1)}-3} \int_{\mathbb{R}} \eta\,(t - a_{k,j})^r\, dt \\
&\sim \sum_{k\geqslant 10\theta^{-1}}^{n} \sum_{j=1}^{2^{\theta(k-1)}-3} \int_{\mathbb{R}} \eta\,(t)^r\, dt \\
&\sim \sum_{k\geqslant 10\theta^{-1}}^{n} \sum_{j=1}^{2^{\theta(k-1)}-3} 1 \;\sim\; 2^{\theta n}.
\end{aligned}$$

The proof of (3.13) is similar. To prove (3.11), it suffices to prove a similar relation in which the supremum is taken over ρ large enough (because the function σ_n is bounded). Note that

$$\sup_{\rho>2^{100}, t_0\in\mathbb{R}} \left(\rho^{-\theta} \int_{t_0-\rho}^{t_0+\rho} \sigma_n(t)^r dt \right) \sim \sup_{\rho>2^{100}, t_0\in\mathbb{R}} \sum_{k\geqslant 10\theta^{-1}}^{n} \sum_{j=1}^{2^{\theta(k-1)}-3} \left(\rho^{-\theta} \int_{t_0-\rho}^{t_0+\rho} \eta\,(t - a_{k,j})^r\, dt \right). \tag{3.14}$$

For each $\rho > 2^{100}$, choose $m \in \mathbb{N}$ such that $\rho \sim 2^m$. We obtain

$$\sup_{t_0\in\mathbb{R}} \sum_{k\geqslant 10\theta^{-1}}^{n} \sum_{j=1}^{2^{\theta(k-1)}-3} \int_{t_0-\rho}^{t_0+\rho} \eta\,(t - a_{k,j})^r\, dt \sim \#\left\{ a_{k,j} \in [-\rho,\rho] \right\} \sim \sum_{k=2}^{m} \left(2^{\theta(k-1)} - 3 \right) \sim 2^{\theta m} \sim \rho^{\theta}. \tag{3.15}$$

where # denotes the cardinality of a set. Combining (3.14) with (3.15), we obtain (3.11).

Now, for $n > 100/\theta$, we define the following sequence of functions:

$$u_n(t,\mathbf{x}) = 2^{-\alpha dn/q} \sigma_n(2^{-\alpha n} t) \phi(2^{-\alpha n}\mathbf{x}) \quad \text{for each } (t,\mathbf{x}) \in \mathbb{R} \times \mathbb{R}^{d-1},$$

where $\theta = d(q - q_1)/q \in (0, 10^{-10d})$ and $\alpha = d(q - q_1)/(dp - (d-p)q)$.

Let us prove that u_n satisfies (2.5) and (2.6).

(i) First, we compute

$$\begin{aligned}\int_{\mathbb{R}^d} |\nabla u_n|^p d\mathbf{x}\, dt &\sim 2^{-p\alpha dn/q} \int_{\mathbb{R}} |\sigma_n'(2^{-\alpha n}t)|^p 2^{-p\alpha n} dt \int_{\mathbb{R}^{d-1}} \phi(2^{-\alpha n}\mathbf{x})^p d\mathbf{x} \\ &\quad + 2^{-p\alpha dn/q} \int_{\mathbb{R}} |\sigma_n(2^{-\alpha n}t)|^p dt \int_{\mathbb{R}^{d-1}} |\nabla\phi(2^{-\alpha n}\mathbf{x})|^p 2^{-p\alpha n} d\mathbf{x} \\ &= 2^{-p\alpha dn/q} \int_{\mathbb{R}} |\sigma_n'(t)|^p 2^{\alpha n - p\alpha n} dt \int_{\mathbb{R}^{d-1}} \phi(\mathbf{x})^p 2^{(d-1)\alpha n} d\mathbf{x} \\ &\quad + 2^{-p\alpha dn/q} \int_{\mathbb{R}} |\sigma_n(t)|^p 2^{\alpha n} dt \int_{\mathbb{R}^{d-1}} |\nabla\phi(\mathbf{x})|^p 2^{(d-1)\alpha n - p\alpha n} d\mathbf{x} \\ &= 2^{n(-p\alpha d/q + d\alpha - p\alpha)} \Bigg(\int_{\mathbb{R}} |\sigma_n'(t)|^p dt \int_{\mathbb{R}^{d-1}} \phi(\mathbf{x})^p d\mathbf{x} \\ &\qquad + \int_{\mathbb{R}} |\sigma_n(t)|^p dt \int_{\mathbb{R}^{d-1}} |\nabla\phi(\mathbf{x})|^p d\mathbf{x} \Bigg). \end{aligned} \tag{3.16}$$

It is easy to see that, for θ and α chosen above, we have $-\frac{p\alpha d}{q} + d\alpha - p\alpha + \theta = 0$, and from (3.12), (3.13), and (3.16) we obtain

$$\int_{\mathbb{R}^d} |\nabla u_n|^p \sim 2^{-p\alpha dn/q + d\alpha n - p\alpha n + \theta n} = 1 \quad \text{for any } n > 100/\theta, \tag{3.17}$$

which implies the first inequality of (2.5).

(ii) By changing variables inside the integrals, one obtains

$$\begin{aligned} \sup_{B_\rho(s,y)\subset\mathbb{R}^d} &\left(\rho^{-\theta} \int_{B_\rho(s,y)} |u_n(t,\mathbf{x})|^{q_1} dt\, d\mathbf{x}\right) \sim \sup_{\rho>0} \left(\rho^{-\theta} \int_{B_\rho(0)} |u_n(t,\mathbf{x})|^{q_1} dt\, d\mathbf{x}\right) \\ &\sim 2^{-q_1\alpha dn/q} \sup_{\rho>0} \left(\rho^{-\theta} \int_{-\rho}^{\rho} \sigma_n(2^{-\alpha n}t)^{q_1} dt \int_{|\mathbf{x}|<\rho} \phi(2^{-\alpha n}\mathbf{x})^{q_1} d\mathbf{x}\right) \\ &\sim 2^{-q_1\alpha dn/q} \sup_{\rho>0} \left(\rho^{-\theta} \int_{-2^{-\alpha n}\rho}^{2^{-\alpha n}\rho} \sigma_n(t)^{q_1} 2^{\alpha n} dt \int_{|\mathbf{x}|<2^{-\alpha n}\rho} \phi(\mathbf{x})^{q_1} 2^{(d-1)\alpha n} d\mathbf{x}\right) \\ &\sim 2^{-q_1\alpha dn/q} \sup_{\rho>2^{\alpha n}} \left(\rho^{-\theta} \int_{-2^{-\alpha n}\rho}^{2^{-\alpha n}\rho} \sigma_n(t)^{q_1} 2^{\alpha n} dt \int_{|\mathbf{x}|<2^{-\alpha n}\rho} \phi(\mathbf{x})^{q_1} 2^{(d-1)\alpha n} d\mathbf{x}\right) \\ &\overset{(3.12)}{\sim} 2^{-q_1\alpha dn/q} 2^{\alpha dn} 2^{-\alpha\theta n} = 1. \end{aligned}$$

Thus,

$$\sup_{B_\rho(s,y)\subset\mathbb{R}^d} \rho^{-\theta} \int_{B_\rho(s,y)} |u_n|^{q_1} d\mathbf{x} \sim 1 \quad \text{for any } n > 100/\theta,$$

which implies the second inequality of (2.5).

(iii) For each $r > 0$, we have

$$\begin{aligned} \int_{\mathbb{R}^d} |u_n(t,\mathbf{x})|^r dt\, d\mathbf{x} &= 2^{-r\alpha dn/q} \int_{\mathbb{R}} \sigma_n(2^{-\alpha n}t)^r dt \int_{\mathbb{R}^{d-1}} \phi(2^{-\alpha n}\mathbf{x})^r d\mathbf{x} \\ &= 2^{-r\alpha dn/q + d\alpha n} \int_{\mathbb{R}} \sigma_n(t)^r dt \int_{\mathbb{R}^{d-1}} \phi(\mathbf{x})^r d\mathbf{x} \\ &\overset{(3.12)}{\sim} 2^{-r\alpha dn/q + d\alpha n + \theta n} = 2^{\frac{d^2(q-q_1)}{q(dp-(d-p)q)}(p(1+q/d)-r)n}, \end{aligned}$$

which implies (2.6).

Finally, relation (2.6) shows that estimate (2.7) holds if and only if $r \geqslant p(q/d+1)$. □

References

[1] D. R. Adams, *Lecture Notes on L^p-Potential Theory*, Dept. of Math., University of Umea, 1981.

[2] D. R. Adams and L. I. Hedberg, *Functions Spaces and Potential Theory*, Springer-Verlag, Berlin–Heidelberg, 1996.

[3] R. A. Adams and J. J. F. Fourier, *Sobolev spaces*, 2*nd edition*, Pure and Applied Mathematics, vol. 140, Elsevier/Academic, Amsterdam, 2003.

[4] H. Brezis, *Functional Analysis*, *Sobolev Spaces and Partial Differential Equations*, Universitext, Springer, New York, 2011.

[5] N.-A. Dao, J.-I. Díaz, and Q.-H. Nguyen, "Generalized Gagliardo-Nirenberg inequalities using Lorentz spaces, BMO, Hölder spaces and fractional Sobolev spaces", *Nonlinear Anal.*, **173** (2018), 146–153.

[6] D. S. McCormick, J. C. Robinson, and J. L. Rodrigo, "Generalised Gagliardo–Nirenberg inequalities using weak Lebesgue spaces and BMO", *Milan J. Math.*, **81**:2 (2013), 265–289.

[7] L. Nirenberg, "On elliptic partial differential equations", *Ann. Scuola Norm. Sup. Pisa*, **13** (1959), 115–162.

[8] G. Patalucci and A. Pisante, "Improved Sobolev embeddings, profile decomposition, and concentration compactness for fractional Sobolev spaces", *Calc. Var. Partial Differ. Equ.*, **50**:3–4 (2014), 799–829.

[9] E. M. Stein, *Harmonic Analysis*: *Real-Variable Methods*, *Orthogonality*, *and Oscillatory Integrals*, Princeton Univ. Press, Princeton, NJ, 1993.

[10] E. M. Stein and R. Shakarchi, *Functional Analysis*: *Introduction to Further Topics in Analysis*, Princeton Univ. Press, Princeton, NJ, 2011.

[11] J. Van Schaftingen, "Interpolation inequalities between Sobolev and Morrey–Campanato spaces: A common gateway to concentration-compactness and Gagliardo–Nirenberg interpolation inequalities", *Port. Math.*, **71**:3 (2014), 159–175.

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