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Caristi's Inequality and α-Contraction Mappings

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Abstract. A new Caristi-type inequality is considered and Caristi's fixed point theorem for mappings of complete metric spaces is developed (in both the single- and set-valued cases). On the basis of this development mappings of complete metric spaces which are contractions with respect to a function of two vector arguments are studied. This function is not required to be a metric or even a continuous function. The proved theorems are generalizations of the Banach contraction principle and Nadler's theorem.

Key words: fixed point, set-valued mapping, metric space, contraction mapping.

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Various versions of the contraction mapping have been considered by many authors (see, e.g., the books [1] and [2] and the papers [3] and [4]). One of the generalizations of this principle is Caristi's theorem [5]. The necessary information from the theory of set-valued mappings can be found in [6] and [7].

Caristi's theorem [7]. *Let X be a complete metric space, and let $F\colon X \multimap X$ be a set-valued mapping with closed graph. Suppose that there exists a nonnegative function $\alpha\colon X \to \mathbb{R}_+$ and a nonnegative number c such that, for any point $x \in X$, there is a point $y \in F(x)$ for which Caristi's inequality*

$$\alpha(y) + c\rho(x,y) \leqslant \alpha(x) \tag{K}$$

holds. Then F has a fixed point, i.e., there exists a point $x_* \in X$ such that $F(x_*) \ni x_*$.

In this paper we prove some generalizations of Caristi's theorem and, relying on the obtained results, study contractions with respect to a function of two vector arguments, which is not required to be a metric or even a continuous function.

1. The case of single-valued mappings. Let (X,ρ) be a complete metric space. Suppose given a continuous mapping $f\colon X \to X$ and a function $\alpha\colon X \times X \to \mathbb{R}$.

Definition 1. A mapping $f\colon X \to X$ is called an α-*contraction* (*contraction with respect to a function* α) if it continuous and there exists a number $k \in (0,1)$ such that $\alpha(f(x),f(y)) \leqslant k\alpha(x,y)$ for any $x,y \in X$.

The question arises: Does any α-contraction mapping have a fixed point? The answer to this question is negative, and counterexamples are easy to construct. In what follows, we assume that α is bounded from below and set $\gamma_0 = \inf_{(x,y)\in X\times X} \alpha(x,y)$.

Let us prove the following generalization of Caristi's theorem.

Theorem 1. *If there exists a number $c > 0$ such that*

$$\alpha(f(x),f(y)) + c\rho(x,y) \leqslant \alpha(x,y) \tag{1}$$

for any $x,y \in X$, then, given any point x_0, the successive approximations $x_{n+1} = f(x_n)$ converge to a point x_*, which is the unique fixed point of f, and

$$\rho(x_*,x_0) \leqslant \frac{\alpha(x_0,f(x_0)) - \gamma_0}{c}\,. \tag{2}$$

Proof. Consider the iterative sequence of points starting from x_0. Let $u_n = \alpha(x_n,x_{n+1})$. By the assumptions of the theorem, for any n, we have

$$\rho(x_n,x_{n+1}) \leqslant c^{-1}(\alpha(x_n,x_{n+1}) - \alpha(x_{n+1},x_{n+2})) = c^{-1}(u_n - u_{n+1}). \tag{3}$$

Thus, the sequence $\{u_n\}$ monotonically decreases and is bounded from below; therefore, this is a Cauchy sequence. It is easy to check that, in this case, the sequence $\{x_n\}$ is Cauchy as well. Hence there exists a point x_* which is the limit of the sequence $\{x_n\}$. Since the mapping f is continuous, it follows that x_* is a fixed point of this mapping.

The uniqueness of the fixed point x_* easily follows from inequality (1).

Inequality (3) also implies (2). This completes the proof of the theorem. □

Theorem 1 has the following corollary, which is a generalized form of the Banach contraction principle.

Corollary 1. *Let f be an α-contraction. Suppose that there exists a number $q > 0$ such that $\rho(x, y) \leqslant q\alpha(x, y)$ for any $x, y \in X$. Then*

1. *the mapping f has a unique fixed point x_*;*
2. *for any point $x_0 \in X$, the iterative sequence $\{x_n\}$, where $x_n = f(x_{n-1})$, converges to x_*;*
3. *the following inequality holds:*

$$\rho(x_*, x_0) \leqslant \frac{q(\alpha(x_0, f(x_0)) - \gamma_0)}{1-k}.$$

Proof. Since f is an α-contraction, it follows that $\alpha(f(x), f(y)) + (1-k)\alpha(x, y) \leqslant \alpha(x, y)$. By assumption, we have

$$\alpha(f(x), f(y)) + \frac{1-k}{q}\,\rho(x, y) \leqslant \alpha(x, y).$$

Now the corollary follows from Theorem 1. □

Theorem 1 has a local version. Given a point x_0 in the space X, let $B_R[x_0]$ be the closed ball of radius R centered at this point, and let $f\colon B_R[x_0] \to X$ be a continuous mapping.

Theorem 2. *Suppose that the mapping f satisfies the following conditions:*

1. *there exists a number $c > 0$ such that $\alpha(f(x), f(y)) + c\rho(x, y) \leqslant \alpha(x, y)$ for any $x, y \in B_R[x_0]$;*
2. *$\alpha(x_0, f(x_0)) \leqslant cR + \gamma_0$.*

Then f has a unique fixed point.

Proof. As in the proof of Theorem 1, consider the iterative sequence of points starting from x_0. It is easy to check that this sequence is well defined, i.e., all points x_n belong to $B_R[x_0]$. Now, as in the proof of Theorem 1, we can show that the sequence $\{x_n\}$ converges and the limit point x_* is a fixed point of f. The uniqueness of this fixed point is obvious. This proves the theorem. □

It is also easy to prove a local theorem for α-contractions.

Corollary 2. *Suppose that there exist numbers $q > 0$ and $k \in (0, 1)$ such that, for any $x, y \in B_R[x_0]$, the following inequalities hold:*

1. *$\rho(x, y) \leqslant q\alpha(x, y)$;*
2. *$\alpha(f(x), f(y)) \leqslant k\alpha(x, y)$.*

If $\alpha(x_0, f(x_0)) \leqslant q(1-k)R + \gamma_0$, then the mapping f has a unique fixed point.

2. The case of set-valued mappings. Let X be a complete metric space. Suppose given a set-valued mapping $F\colon X \multimap X$ with closed graph and a function $\alpha\colon X \times X \to \mathbb{R}$ bounded from below, and let $\gamma_0 = \inf_{(x,y)\in X\times X} \alpha(x, y)$. By analogy with the Hausdorff metric, for subsets A and B of X, we put

$$\alpha_*(A, B) = \sup_{a\in A}\inf_{b\in B} \alpha(a, b) \quad \text{and} \quad h_\alpha(A, B) = \max\{\alpha_*(A, B), \alpha_*(B, A)\}.$$

The following theorem is a development of Caristi's theorem.

Theorem 3. *If there exists a number $c > 0$ such that, given any points $x \in X$ and $y \in F(x)$ and any number $\eta > 0$, there is a point $z \in F(y)$ for which $\alpha(y, z) + c\rho(y, z) \leqslant \alpha(x, y) + \eta$, then F has a fixed point; moreover, for any points $x \in X$ and $y \in F(x)$ and any $\delta > 0$, there exists a* *fixed point* x_* *such that*

$$\rho(y, x_*) < \frac{\alpha(x, y) - \gamma_0}{c} + \delta. \tag{4}$$

Proof. Suppose given any positive number ε. Let us construct a sequence $\{x_n\}_{n=0}^{\infty} \in X$ satisfying the following conditions:

(i) $x_{n+1} \in F(x_n)$;

(ii) $\alpha(x_n, x_{n+1}) + c\rho(x_n, x_{n+1}) < \alpha(x_{n-1}, x_n) + \varepsilon/2^n$ for any $n \geqslant 1$.

Such a sequence can be constructed by induction. We set $\beta_n = \alpha(x_{n-1}, x_n)$ and prove that the sequence $\{\beta_n\}_{n=1}^{\infty}$ converges. For this purpose, we consider the sequence

$$\biggl\{\beta_n + \sum_{j=n}^{\infty} \frac{\varepsilon}{2^j}\biggr\}.$$

Obviously, this sequence monotonically decreases and is bounded from below; therefore, it converges, and hence so does $\{\beta_n\}$.

The sequence $\{x_n\}$ converges as well. This follows from the inequality

$$\rho(x_n, x_{n+p}) < \frac{1}{c}\biggl((\beta_{n+1} - \beta_{n+p+1}) + \sum_{j=n+1}^{n+p} \frac{\varepsilon}{2^j}\biggr),$$

which implies that $\{x_n\}$ is a Cauchy sequence. Since the space X is complete, it follows that $\{x_n\}$ converges to some point x_*, and since $x_{n+1} \in F(x_n)$ and the graph of F is closed, it follows that x_* is a fixed point of the mapping F.

Let us prove inequality (4). Note that, for sufficiently large n, we have

$$\rho(x_1, x_*) < \sum_{j=1}^{n} \rho(x_j, x_{j+1}) + \varepsilon.$$

Therefore,

$$\begin{aligned}\rho(x_1, x_*) &< \frac{1}{c}\biggl((\alpha(x_0, x_1) - \alpha(x_n, x_{n+1}) + \sum_{j=1}^{n} \frac{\varepsilon}{2^j}\biggr) + \varepsilon \\ &< \frac{1}{c}(\alpha(x_0, x_1) - \gamma_0 + \varepsilon) + \varepsilon = \frac{\alpha(x_0, x_1) - \gamma_0}{c} + \frac{\varepsilon(c+1)}{c}.\end{aligned}$$

Since the number ε is arbitrary, we can assume that $\varepsilon(c+1)/c < \delta$, which implies (4). This completes the proof of the theorem. □

Let us prove a fixed point theorem for set-valued α-contractions.

Theorem 4. *Let* $F\colon X \multimap X$ *be a set-valued mapping with closed graph. Suppose that the following conditions hold*:

1. *there exists a number* $q > 0$ *such that* $\rho(x, y) \leqslant q\alpha(x, y)$ *for any* $x, y \in X$;
2. *there exists a number* $k \in (0, 1)$ *such that*

$$h_\alpha(F(x), F(y)) \leqslant k\alpha(x, y) \tag{5}$$

for any $x, y \in X$.

Then the mapping F *has a fixed point, and for any number* $\delta > 0$, *there exists a fixed point* x_* *such that*

$$\rho(y, x_*) < \frac{qk\,(\alpha(x, y) - \gamma_0)}{1 - k} + \delta.$$

Proof. Take $x \in X$ and $y \in F(x)$. By virtue of inequality (5) and the definition of the function h_α, for any positive number η, there exists a point $z \in F(y)$ such that $\alpha(y, z) \leqslant k\alpha(x, y) + \eta$. Since the number k belongs to $(0,1)$, we can represent it in the form $k = 1/(1+c)$, where $c > 0$. Then

$$\alpha(y,z) + c\alpha(y,z) = \alpha(y,z) + \frac{1-k}{k}\,\alpha(y,z) \leqslant \alpha(x,y) + \eta.$$

Therefore, by the assumptions of the theorem, we have

$$\alpha(y,z) + \frac{1-k}{qk}\,\rho(y,z) \leqslant \alpha(x,y) + \eta.$$

Now the required assertion follows from Theorem 3. □

This theorem is a generalization of Nadler's theorem (see, e.g., [9]).

Consider a local version of Theorem 3. Given an $x_0 \in X$, let $B_R[x_0]$ be the closed ball of radius R centered at x_0, and let $F\colon B_R[x_0] \multimap X$ be a set-valued mapping with closed graph such that $B_R[x_0] \cap F(x_0) \neq \varnothing$.

We also assume that the following condition holds: *there exists a number* $c > 0$ *such that, for any points* $x \in X$ *and* $y \in F(x) \cap B_R[x_0]$ *and any number* $\eta > 0$, *there is a point* $z \in F(y)$ *for which*

$$\alpha(y,z) + c\rho(y,z) \leqslant \alpha(x,y) + \eta. \tag{6}$$

Theorem 5. *Suppose that a set-valued mapping* F *satisfies condition* (6). *If there exists a point* $x_1 \in B_R[x_0] \cap F(x_0)$ *for which* $\alpha(x_0,x_1) < c(R - \rho(x_0,x_1)) + \gamma_0$, *then the mapping* F *has a fixed point.*

This theorem is proved by the same scheme as a similar theorem for single-valued mappings.

By analogy with Theorem 4, a local fixed point theorem for set-valued α-contractions can be proved.

Nemytskii proved a well-known fixed point theorem for mappings on a compact metric space which are contractions in a weak sense (see [9]). We generalize this theorem to the case of set-valued weak α-contractions.

Let X be a compact metric space. Suppose given a lower semicontinuous function $\alpha\colon X\times X \to \mathbb{R}$ and a set-valued mapping $F\colon X \multimap X$ with closed graph.

Theorem 6. *If* $\alpha_*(y, F(y)) < \alpha(x,y)$ *for any* $x \in X$ *and* $y \in F(x)$, $x \neq y$, *then the mapping* F *has a fixed point.*

Proof. Consider the function $\beta\colon X \to \mathbb{R}$ defined by

$$\beta(x) = \min_{u \in F(x)} \alpha(x,u).$$

This function is well defined and lower semicontinuous; therefore, it attains its minimum value on the set X. We put $\beta(x_*) = \min_{x\in X} \beta(x)$. Let us show that $x_* \in F(x_*)$. Suppose that, on the contrary, $x_* \notin F(x_*)$. Then $\beta(x_*) = \alpha(x_*,y_*)$ for $y_* \neq x_*$. We have $\alpha_*(y_*,F(y_*)) < \alpha(x_*,y_*) = \beta(x_*)$. Thus, there exists a point $z_* \in F(y_*)$ such that $\alpha(y_*,z_*) < \alpha(x_*,y_*)$, whence $\beta(y_*) < \beta(x_*)$. This contradiction proves the theorem. □

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