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Relationship between the Discrete and Resonance Spectrum for the Laplace Operator on a Noncompact Hyperbolic Riemann Surface

D. A. Popov

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Abstract. We consider arbitrary noncompact hyperbolic Riemann surfaces of finite area. For such surfaces, we obtain identities relating the discrete spectrum of the Laplace operator to the resonance spectrum (formed by the poles of the scattering matrix). These identities depend on the choice of a test function. We indicate a class of admissible test functions and consider two examples corresponding to specific choices of the test function.

Key words: Laplace operator, discrete spectrum, resonance spectrum.

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1. Introduction

Let H be the upper half-plane with the Poincaré metric $ds^2 = y^{-2}(dx^2 + dy^2)$. A *cofinite group* is a discrete group $\Gamma \subset \mathrm{PSL}(2,\mathbb{R})$ with noncompact fundamental domain F whose area $|F|$ with respect to the invariant measure $d\nu = y^{-2}\,dx\,dy$ is finite. In what follows, we only deal with cofinite groups Γ.

The Laplace operator $\Delta = y^2(\partial_x^2 + \partial_y^2)$ extends to be a self-adjoint operator on $L^2(F, d\nu)$ with continuous spectrum covering the interval $[1/4, \infty)$ and with discrete spectrum $\{\lambda_n\}$ ($\Delta\varphi_n + \lambda_n\varphi_n = 0$, $0 = \lambda_0 < \lambda_1 \leqslant \lambda_2 \leqslant \cdots$, $\varphi_n \in L^2(F, d\nu)$). Little is known about the structure of the discrete spectrum. In particular, it is not known for what groups Γ this spectrum is infinite.

Selberg posed the following question: what cofinite groups Γ satisfy the Weyl formula

$$N_\Gamma\biggl(T^2 + \frac{1}{4}\biggr) = \#\biggl\{n \biggm| \lambda_n \leqslant T^2 + \frac{1}{4}\biggr\} \simeq \frac{|F|}{4\pi} T^2 \qquad (T \to \infty). \tag{1.1}$$

Nowadays, such groups are said to be *essentially cuspidal*. Formula (1.1) has been proved for a number of groups (see [1]–[4]), in particular, for the congruence subgroups of $\mathrm{SL}(2,\mathbb{Z})$. However, all these groups correspond to nongeneric points in the Teichmüller space [22]. Roelke (e.g., see [1]) conjectured that $N_\Gamma(T^2 + 1/4) \to \infty$ as $T \to \infty$.

The interest in these questions arose in connection with the papers [5], [6]. These papers, as well as [7]–[9], provide a number of sufficient conditions for the Weyl law (1.1) to be violated. Based on these results, Sarnak [2] conjectured that neither the Weyl law nor even the Roelke conjecture holds for generic cofinite groups Γ.

One main approach to studying the spectrum $\{\lambda_n\}$ is based on the Selberg formula, and all the results of the present paper are corollaries of this formula.

The Selberg formula for cofinite groups is given in Sec. 2. Symbolically, it can be written as

$$\sum_{n \geqslant 0} h(r_n) = \Phi_\Gamma[h|\{\lambda_n\}, \{N(P)\}, \varphi] \ \text{ for any } \ h \in \{h\}_S \quad \biggl(\lambda_n = r_n^2 + \frac{1}{4}\biggr), \tag{1.2}$$

where Φ_Γ is a functional on the space $\{h\}_S$ (see Sec. 2). The functional Φ_Γ depends on the spectrum $\{\lambda_n\}$, the set $\{N(P)\}$ of norms of hyperbolic conjugacy classes, the function φ defined by $\varphi(s) = \det\Phi(s)$ (where $\Phi(s)$ is the scattering matrix), and finitely many parameters such as $|F|$, the number of elliptic and parabolic classes, $\operatorname{tr}\Phi(1/2)$, etc. The Selberg formula for a cocompact group Γ reads

$$\sum_{n\geqslant 0} h(r_n) = \Phi_\Gamma[h|\{\lambda_n\},\{N(P)\}]. \tag{1.3}$$

It was shown in [10] that, for strictly hyperbolic groups, the Selberg formula (supplemented by an additional condition on the function h) implies that the spectrum $\{\lambda_n\}$ satisfies equations of the form

$$\sum_{n\geqslant 0} h(r_n) = \widetilde{\Phi}_\Gamma[h|\{\lambda_n\}]. \tag{1.4}$$

The main aim of the present paper is to generalize this result to arbitrary cofinite groups. We show (see Theorem 1) that the following analog of formula (1.4) holds:

$$\sum_{n\geqslant 0} h(r_n) = \widetilde{\Phi}_\Gamma[h|\{\lambda_n\},\{s_\alpha\}]. \tag{1.5}$$

Here $\{s_\alpha\}$ is the set of poles $s_\alpha = \beta_\alpha + i\gamma_\alpha$ of φ such that $\beta_\alpha < 1/2$ and $\gamma_\alpha \neq 0$. This set will be called the *resonance spectrum*. Theorem 1 specifies the explicit form of the functional $\widetilde{\Phi}_\Gamma[h|\{\lambda_n\},\{s_\alpha\}]$. It follows from this theorem that Eq. (1.5) can be rewritten in the form

$$\sum_{n\geqslant 0} h(r_n) + \sum_{\gamma_\alpha > 0} h(\gamma_\alpha) = L[h] + \widetilde{\Phi}'_\Gamma[h|\{r_n\},\{s_\alpha\}] \tag{1.6}$$

and that $|L[h]| \gg |\widetilde{\Phi}'_\Gamma[h|\{r_n\},\{s_\alpha\}]|$ for a fairly broad class of functions h. For such functions h, the relations between the spectra $\{r_n\}$ and $\{s_\alpha\}$ following from the Selberg formula are approximately symmetric with respect to the replacement $\{r_n\} \leftrightarrow \{\gamma_\alpha\}$ (see (5.23)).

Section 6 contains some applications of these results. In particular, we show how to obtain the Selberg–Weyl formula from (1.6) in the form

$$N(r_n < T) + N(0 < \gamma_\alpha < T) = \frac{|F|}{4\pi} T^2 + O(T^{2-\theta}) \qquad (\theta > 0,\ T \to \infty), \tag{1.7}$$

where $N(r_n < T)$ is the number of eigenvalues $\lambda_n \leqslant T^2 + 1/4$.

2. Preliminaries

This section provides some insight into the Selberg formula for cofinite groups and introduces notation to be used in the paper. All the information given here can be found in [1], [3], [12]–[17].

Throughout the paper, we assume that the functions $h(\,\cdot\,)$ belong to the class $\{h\}_S$, that is, satisfy the following conditions:

1. $h(r) = h(-r)$.
2. The function h is holomorphic in the strip $\{r\colon |\operatorname{Im} r| \leqslant 1/2 + \varepsilon,\ \varepsilon > 0\}$.
3. In this strip, one has $|h(r)| = O(1+|r|^2)^{-1-\varepsilon}$ $(|r| \to \infty)$.

By g we denote the Fourier transform of h,

$$g(y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} h(r) e^{-iry}\,dr, \qquad h(r) = \int_{-\infty}^{\infty} e^{iry} g(y)\,dy. \tag{2.1}$$

The numbers r_n, s_n, and $\overline{s}_n$ with a Latin subscript are defined by the formulas

$$\lambda_n = s_n\overline{s}_n, \quad \overline{s}_n = 1 - s_n, \quad s_0 = 1, \quad s_n = \frac{1}{2} + ir_n, \quad \lambda_n = \frac{1}{4} + r_n^2. \tag{2.2}$$

The eigenvalues λ_n in the interval $0 \leqslant \lambda_n < 1/4$ are said to be exceptional; the number M of such eigenvalues is finite. For the exceptional eigenvalues, one has

$$r_n = -i\left(\frac{1}{4} - \lambda_n\right)^{1/2}, \qquad r_0 = -\frac{i}{2}.$$

The eigenvalues $\lambda_n \geqslant 1/4$ will be numbered by a subscript j, so that $\lambda_j = 1/4 + r_j^2$, $r_j \geqslant 0$.

When considering the Selberg formula, we restrict ourselves to the case of the trivial one-dimensional representation $\chi\colon \Gamma \to \mathbb{C}$, $\chi(\gamma) = 1$ for all $\gamma \in \Gamma$. Then the Selberg formula reads

$$\sum_{n\geqslant 0} h(r_n) = H[h] + S_R[h] + S_P[g] + \mathscr{P}[h|\varphi] \tag{2.3}$$

for every $h \in \{h\}_S$ and determines the form of the functional Φ_Γ in (1.2).

Let us give the definitions of the objects occurring on the right-hand side in (2.3). First,

$$H[h] = \frac{|F|}{4\pi} \int_{-\infty}^{\infty} r \tanh \pi r h(r)\, dr. \tag{2.4}$$

Second, $S_R[h]$ is the contribution of the conjugacy classes (in Γ) of elliptic elements (the contribution of elliptic conjugacy classes), and

$$S_R[h] = \sum_{\{R\}} \sum_{k=1}^{p-1} \frac{1}{p \sin \pi k/p} \int_{-\infty}^{\infty} h(r) \frac{e^{-2\pi k r/p}}{1 + e^{-2\pi r}}\, dr, \tag{2.5}$$

where the sum is over the set $\{R\}$ of primitive elliptic conjugacy classes and $p = p(R)$ is the order of a class R. The number $|\{R\}|$ of elliptic conjugacy classes and their maximum order are finite.

The third term on the right-hand side in (2.3) is the contribution of hyperbolic conjugacy classes; it is given by the formula

$$S_P[g] = \sum_{\{P_0\}} \sum_{k=1}^{\infty} \frac{\ln N(P_0)}{N(P_0)^{k/2} - N(P_0)^{-k/2}}\, g(k \ln N(P_0)), \tag{2.6}$$

where the sum is over the set of primitive classes P_0 and $N(P_0)$ is the norm of a class P_0. Recall that every hyperbolic element $\gamma \in \Gamma$ is conjugate in $\mathrm{SL}(2,\mathbb{R})$ to the transformation $z \to N(P)z$, $N(P) > 1$, $P = P_0^k$ $(k \geqslant 1)$, where P is an arbitrary hyperbolic class and $N(P) = N(P_0)^k$. In what follows, we write

$$B_0 = \min_{\{P\}} N(P), \quad B_0 > 1, \qquad b_0 = \ln B_0. \tag{2.7}$$

The last term $\mathscr{P}[h|\varphi]$ on the right-hand side in (2.3) is the contribution of parabolic conjugacy classes (the contribution of the continuous spectrum), and

$$\begin{aligned}\mathscr{P}[h|\varphi] = \frac{1}{4\pi} \int_{-\infty}^{\infty} h(r)\, \frac{\varphi'}{\varphi}\left(\frac{1}{2} + ir\right) dr - \frac{n}{2\pi} \int_{-\infty}^{\infty} h(r)\, \frac{\Gamma'}{\Gamma}(1 + ir)\, dr \\ - \frac{h(0)}{4}\left(n - \operatorname{tr} \Phi\left(\frac{1}{2}\right)\right) - n g(0) \ln 2.\end{aligned} \tag{2.8}$$

Here $\Phi(s)$ is the $n \times n$ matrix of free terms in the Eisenstein series (the scattering matrix),

$$\varphi(s) = \det \Phi(s), \tag{2.9}$$

n is the number of primitive parabolic conjugacy classes (the number of pairwise nonequivalent parabolic points in $\overline{F}$), and $\Gamma(\,\cdot\,)$ is the gamma function.

The properties of φ are described in [1], [3], [12], [17]. This is a meromorphic function satisfying the functional equations

$$\varphi(s)\varphi(1 - s) = 1, \qquad \varphi(s) = \widetilde{\varphi(\tilde{s})}, \tag{2.10}$$

where the tilde stands for complex conjugation. The function φ is holomorphic in the half-plane $\operatorname{Re} s > 1/2$ except for finitely many poles on the interval $(1/2, 1]$. The poles s_α of φ with $\operatorname{Re} s_\alpha < 1/2$ lie in the strip $-\nu_0 < \operatorname{Re} s < 1/2$ symmetrically with respect to the real axis, and

$$\sum_\alpha \left(\frac{1}{2} - \beta_\alpha\right) |s_\alpha|^{-2} = C_\Gamma < \infty, \qquad \sum_{0 < \gamma_\alpha \leqslant x} 1 \leqslant A_\Gamma x^2. \tag{2.11}$$

Throughout the following, Greek subscripts are used to number the poles of φ.

The Selberg zeta function $Z(\,\cdot\,)$ for cofinite groups is defined in the same way as for cocompact groups; for $\operatorname{Re} s > 1$, one has

$$Z(s) = \prod_{\{P_0\}} \prod_{k=1}^{\infty} [1 - N(P_0)^{-k-s}].$$

This function has an analytic continuation into the entire plane of the variable $s = \sigma + it$ as a meromorphic function and satisfies a functional equation of the form $Z(1-s) = A(s)Z(s)$. An explicit expression for the factor $A(s)$ can be found in the papers [1, 12], which give a complete description of all the zeros and poles of Z, their multiplicities being indicated. Let us present this description and simultaneously introduce a numbering to be used for the nontrivial zeros of the Selberg zeta function.

The nontrivial zeros of $Z(\,\cdot\,)$ are

1. The zeros s_j on the critical line $\operatorname{Re} s = 1/2$. They are arranged symmetrically with respect to the real axis, and one has the corresponding eigenvalues

$$\lambda_j = s_j(1-s_j), \qquad s_j = 1/2 + r_j \qquad (j \geqslant 0).$$

2. The zeros $s_m \in (0,1)$, $m = 1, \dots, M_1$. They are arranged symmetrically with respect to the point $s = 1/2$, and one has the corresponding eigenvalues

$$\lambda_m = s_m(1-s_m), \qquad s_m = \sigma_m.$$

3. The zeros

$$s_\alpha = \beta_\alpha + i\gamma_\alpha, \qquad -\nu_0 < \beta_\alpha < 1/2,$$

at the poles of the function φ in (2.9). These zeros are arranged symmetrically with respect to the real axis.

4. The zeros $s_\nu = \sigma_\nu$, $1/2 < \sigma_\nu \leqslant 1$, $\nu = 0, 1, \dots, M_2 - 1$, at the poles of φ. One has the corresponding eigenvalues $\lambda_\nu = \sigma_\nu(1-\sigma_\nu)$ $(\nu \neq 0)$ and $\lambda_0 = 0$ $(\sigma_0 = 1)$. The poles of $Z(\,\cdot\,)$ lie at the points $s = -l + 1/2$, $l = 0, 1, \dots$, and the *trivial zeros of* $Z(\,\cdot\,)$ lie at the points $s = -l$, $l = 0, 1, \dots$.

The numbers λ_m, λ_ν, and λ_j exhaust the whole discrete spectrum, and so $\{\lambda_n\} = \{\lambda_m\} \cup \{\lambda_\nu\} \cup \{\lambda_j\}$.

3. Explicit Formula for $S_P[g]$

In analytic number theory, the term *explicit formulas* refers to formulas representing the object of study by a series over zeros and poles of the corresponding analytic function. The main example is given by the explicit formula representing the Chebyshev function $\Psi(\,\cdot\,)$ by a series over the nontrivial zeros of the Riemann zeta function. In our setting, the function (see [14], [15])

$$\Lambda^\Gamma(P) = \frac{\ln N(P_0)}{1 - N(P)^{-1}} \tag{3.1}$$

is an analog of the Mangoldt function $\Lambda(\,\cdot\,)$, and

$$\Psi^\Gamma(x) = \sum_{B_0 \leqslant N(P) \leqslant x} \Lambda^\Gamma(P) \tag{3.2}$$

is the corresponding analog of the Chebyshev function. Based on the results in [15], [16], let us present a definitive version of an explicit formula for the function

$$\Psi_1^\Gamma(x) = \int_{B_0}^{x} \Psi^\Gamma(\xi)\, d\xi. \tag{3.3}$$

This formula reads

$$\begin{aligned} \Psi_1^\Gamma(x) &= \Sigma_{R,\Delta}(x) + \Sigma_{R,\varphi}(x) + \Psi_{1,0}^\Gamma + \Delta_R(x), \\ \Delta_R(x) &= O\biggl(\frac{x^2 \ln x}{R}\biggr), \qquad x \geqslant 2,\ R \to \infty. \end{aligned} \tag{3.4}$$

The functions $\Sigma_{R,\Delta}(x)$ and $\Sigma_{R,\varphi}(x)$ on the right-hand side in (3.4) are given by the formulas

$$\Sigma_{R,\Delta}(x) = \sum_{0\leqslant r_j\leqslant R} \frac{x^{1+s_j}}{s_j(1+s_j)} + \frac{x^{1+\tilde{s}_j}}{\tilde{s}_j(1+\tilde{s}_j)} + \sum_{1/2<s_m<1}\biggl(\frac{x^{1+s_m}}{s_m(1+s_m)} + \frac{x^{1+\overline{s}_m}}{\overline{s}_m(1+\overline{s}_m)}\biggr) + \sum_{\nu=0}^{M_2-1}\frac{x^{1+s_\nu}}{s_\nu(1+s_\nu)}\,, \tag{3.5}$$

$$\Sigma_{R,\varphi}(x) = \sum_{(\alpha,R)} \frac{x^{1+s_\alpha}}{s_\alpha(1+s_\alpha)} + \frac{x^{1+\tilde{s}_\alpha}}{\tilde{s}_\alpha(1+\tilde{s}_\alpha)} \qquad (s_\alpha = \beta_\alpha + i\gamma_\alpha), \tag{3.6}$$

while $\Psi^{r}_{1,0}(x)$ is the contribution of the poles and trivial zeros of $Z(s)$ (see (3.10)). Just as above, the tilde stands for complex conjugation, $\overline{s} = 1-s$, and the summation in $\sum_{(\alpha,R)}$ is over all the poles s_α of φ such that

$$\beta_\alpha < 1/2, \quad \gamma_\alpha > 0, \quad 0 < \gamma_\alpha < R. \tag{3.7}$$

Strictly speaking, formula (3.4) is lacking in the book [16], which only contains the result of passing to the limit as $R\to\infty$ in (3.4), while the proof of formula (3.4) itself is barely outlined. In this connection, note that, by reproducing the scheme of proof of Propositions 5.7–5.10 in [15], one can prove (3.4) with the weaker remainder estimate

$$\Delta_R(x) = O\biggl(\frac{x^3\ln x}{\ln R}\biggr), \qquad x\geqslant 2, \tag{3.8}$$

which suffices for our aims.

Formula (3.4) can be proved by standard methods of analytic number theory based on the integral representation [15]

$$\Psi_1^\Gamma(x) = \frac{1}{2\pi i}\int_{\sigma_1-i\infty}^{\sigma_1+i\infty}\frac{x^{s+1}}{s(s+1)}\,\frac{Z'}{Z}(s)\,ds, \qquad \sigma_1 > 1. \tag{3.9}$$

The first two terms on the right-hand side in (3.4) are the sum of residues of the integrand in the rectangular domain with vertices $\sigma_1 \pm iR$ and $-A\pm iR$ ($A\to\infty$). The residues at the points $s = 0,-1$ must be considered separately (see [15]), and their contribution is included in $\Psi^\Gamma_{1,0}(x)$. This function includes the contributions of the poles and trivial zeros of $Z(\,\cdot\,)$, and

$$\Psi^\Gamma_{1,0}(x) = -\frac{2}{3}x^{3/2}(n - \operatorname{tr}\Phi(1/2)) + \widetilde{\Psi}^\Gamma_{1,0}(x). \tag{3.10}$$

The first term on the right-hand side in this formula is the contribution of the pole at $s = 1/2$. The function $\Psi^\Gamma_{1,0}(x)$ can be explicitly evaluated based on the results in [1], [12], and $\widetilde{\Psi}^\Gamma_{1,0}(\,\cdot\,)$ is a function differentiable for $x\geqslant 2$ and such that

$$\widetilde{\Psi}^\Gamma_{1,0}(x) = C_1^\Gamma x\ln x + O(x), \qquad \frac{d\widetilde{\Psi}^\Gamma_{1,0}}{dx}(x) = C_1^\Gamma\ln x + C_2^\Gamma + O(x^{-1}). \tag{3.11}$$

For example, if $\Gamma = \mathrm{SL}(2,\mathbb{Z})$ and $B_0 = \frac14(3+\sqrt5\,)^2 \simeq 6.8541$, then

$$\widetilde{\Psi}^\Gamma_{1,0}(x) = C_1x\ln x + C_2x + C_3 + C_4x\ln(1-x^{-1}) + C_5\ln(1-x^{-1}) + C_6x^{-1/2} + C_7(1+x^{-1})\ln\frac{1+x^{-1/2}}{1-x^{-1/2}}\,,$$

where the C_i are some constants. Let us introduce the function

$$F(x) = \frac{d}{dx}\,\frac{g(\ln x)}{\sqrt{x}}\,. \tag{3.12}$$

Lemma 1 (an explicit formula for $S_P[g]$)**.** *Let $B > 2$, let $B \geqslant B_0$, and let $h \in \{h\}_S$ be a function such that*

$$\int_{b_0}^{\infty} e^{y/2} y[|g(y)| + |g^{(1)}(y)| + |g^{(2)}(y)|]\, dy = C_\Gamma[g] < \infty, \qquad g^{(3)} \in L^1(b_0, \infty). \tag{3.13}$$

Then for any cofinite group Γ one has

$$S_P[g] = S_P^\infty[g] + S_{\mathrm{ex}}[g] + S_0[g], \tag{3.14}$$

where

$$S_0[g] = -\int_B^\infty F(x)\, \frac{d}{dx}\, \Psi_{1,0}^\Gamma(x)\, dx - \int_{B_0}^B \Psi^\Gamma(x) F(x)\, dx, \tag{3.15}$$

$$S_{\mathrm{ex}}[g] = -\sum_{1/2<s_m<1} \int_B^\infty \left(\frac{x^{s_m}}{s_m} + \frac{x^{\overline{s}_m}}{\overline{s}_m}\right) F(x)\, dx - \sum_{\nu=0}^{M_2-1} \int_B^\infty \frac{x^{s_\nu}}{s_\nu}\, F(x)\, dx, \tag{3.16}$$

$$S_P^\infty[g] = -\sum_{(\alpha)} \int_B^\infty \left(\frac{x^{s_\alpha}}{s_\alpha} + \frac{x^{\tilde{s}_\alpha}}{\tilde{s}_\alpha}\right) F(x)\, dx - \sum_{r_j \geqslant 0} \int_B^\infty \left(\frac{x^{s_j}}{s_j} + \frac{x^{\overline{s}_j}}{\overline{s}_j}\right) F(x)\, dx, \tag{3.17}$$

and the symbol $\sum_{(\alpha)}$ stands for a sum over the poles $s_\alpha = \beta_\alpha + i\gamma_\alpha$ of φ such that $\beta_\alpha < 1/2$ and $\gamma_\alpha > 0$ (see (3.7)*).*

Here and in what follows, $\sum_{r_j \geqslant 0} + \sum_{(\alpha)} = \lim_{R\to\infty}\bigl(\sum_{r_j \leqslant R} + \sum_{|\gamma_\alpha| \leqslant R}\bigr)$.

Proof. An analog of Lemma 1 for strictly hyperbolic groups was proved in [10]. Let us rewrite (2.6) in the form

$$S_P[g] = \sum_{\{P\}} \Lambda^\Gamma(P) f(N(P)), \qquad f(x) = \frac{g(\ln x)}{\sqrt{x}}\,.$$

By Abel's partial summation formula,

$$\sum_{B_0 \leqslant N(P) \leqslant x} \Lambda^\Gamma(P) f(N(P)) = \Psi^\Gamma(x) f(x) - \int_{B_0}^x \Psi^\Gamma(\xi) F(\xi)\, d\xi.$$

In view of the estimates (see [15], [16])

$$\Psi^\Gamma(x) = O(x), \quad \Psi_1^\Gamma(x) = O(x^2), \qquad x \to \infty,$$

and condition (3.13), we obtain

$$S_P[g] = -\int_{B_0}^\infty \Psi^\Gamma(x) F(x)\, dx = -\int_B^\infty \frac{d\Psi_1^\Gamma}{dx}(x) F(x)\, dx - \int_{B_0}^B \Psi^\Gamma(x) F(x)\, dx.$$

Further, we integrate by parts and find that

$$S_P[g] = \Psi_1^\Gamma(B) F(B) - \int_{B_0}^B \Psi^\Gamma(x) F(x)\, dx + \int_B^\infty \Psi_1^\Gamma(x)\, \frac{dF}{dx}(x)\, dx. \tag{3.18}$$

Let us use formula (3.4). We obtain

$$\begin{aligned} S_P[g] = \int_B^\infty [\Sigma_{R,\Delta}(x) + \Sigma_{R,\varphi}(x) + \Psi_{1,0}^\Gamma(x)]\, \frac{dF}{dx}(x)\, dx \\ + \Psi_1^\Gamma(B) F(B) - \int_{B_0}^B \Psi^\Gamma(x) F(x)\, dx + \int_B^\infty \frac{dF}{dx}(x) \Delta_R(x)\, dx, \end{aligned}$$

whence it follows that

$$S_P[g] = -\Psi_1^\Gamma(B)F(B) + \Delta_R(B)F(B)$$
$$- \int_B^\infty \frac{d}{dx}[\Sigma_{R,\Delta}(x) + \Sigma_{R,\varphi}(x) + \Psi_{1,0}^\Gamma(x)]F(x)\,dx$$
$$+ \Psi_1^\Gamma(B)F(B) - \int_{B_0}^B \Psi^\Gamma(x)F(x)\,dx + \int_B^\infty \frac{dF}{dx}(x)\Delta_R(x)\,dx.$$

We differentiate, change the order of summation and integration, and then pass to the limit as $R \to \infty$, thus obtaining the desired result. The proof of Lemma 1 is complete. □

Formally, one can obtain Eq. (3.14) by substituting an explicit formula for $\Psi_1^\Gamma(x)$ (see [16]) into (3.18) with subsequent integration by parts and changing the order of integration and summation.

Note that if we use the estimate (3.9), then the entire derivation remains valid except that $e^{y/2}$ must be replaced with $e^{(k-3/2)y}$ in condition (3.13).

Lemma 2. *Let the assumptions of Lemma* 1 *be satisfied. Then*

$$\sum_{n\geqslant 0} h(r_n) = H[h] + S_R[h] + S_P^\infty[g] + S_{\mathrm{ex}}[g] + S_0[g] + \mathscr{P}[h|\varphi]. \tag{3.19}$$

Here

$$S_P^\infty[g] = -\sum_{r_j\geqslant 0} \frac{1}{r_j^2 + 1/4} \int_b^\infty (\cos(r_j y) + 2r_j \sin(r_j y)) f(y)\,dy$$
$$- \sum_{(\alpha)} \int_B^\infty \left(\frac{x^{s_\alpha}}{s_\alpha} + \frac{x^{\tilde{s}_\alpha}}{\tilde{s}_\alpha}\right) F(x)\,dx, \tag{3.20}$$
$$f(y) = -\frac{1}{2}g(y) + g^{(1)}(y), \qquad b = \ln B,$$

and the remaining terms on the right-hand side in (3.19) *have been defined above.*

Proof. The proof amounts to the substitution of the expression (3.14) into the Selberg formula (2.3) with regard to the remark that

$$\left(\frac{x^{s_j}}{s_j} + \frac{x^{\tilde{s}_j}}{\tilde{s}_j}\right) F(x)\,dx = \frac{1}{1/4 + r_j^2} (\cos(r_j y) + 2r_j \sin(r_j y)) f(y)\,dy \qquad (\tilde{s}_j = \overline{s}_j)$$

for $y = \ln x$. The proof of Lemma 2 is complete. □

4. Explicit Formula for $\mathscr{P}[h|\varphi]$

We introduce the notation

$$J[h|\varphi] = \frac{1}{4\pi} \int_{-\infty}^\infty h(r)\, \frac{\varphi'}{\varphi}\left(\frac{1}{2} + ir\right) dr, \tag{4.1}$$

$$\Delta\mathscr{P}[h|\varphi] = -\frac{n}{2\pi} \int_{-\infty}^\infty h(r)\, \frac{\Gamma'}{\Gamma}(1 + ir)\,dr - \frac{h(0)}{4}(n - \operatorname{tr}\Phi(1/2)) - ng(0)\ln 2 \tag{4.2}$$

and represent $\mathscr{P}[h|\varphi]$ (2.8) in the form

$$\mathscr{P}[h|\varphi] = J[h|\varphi] + \Delta\mathscr{P}[h|\varphi]. \tag{4.3}$$

As was already noted, the main properties of the function φ in (2.9) are indicated in [1], [3], [12]. According to these papers,

$$\varphi(s) = \sqrt{\pi}\left(\frac{\Gamma(s - 1/2)}{\Gamma(s)}\right)^n \sum_{n=1}^\infty \frac{a_n}{b_n^{2s}}, \qquad a_1 \neq 0,\ 0 < b_1 < b_2 < \dots, \tag{4.4}$$

for $\sigma = \operatorname{Re} s > 1$. By the functional equation (2.10), if s_α is a pole of φ, then so is $\tilde{s}_\alpha$, while $1 - s_\alpha$ and $1 - \tilde{s}_\alpha$ are zeros of φ. The logarithmic derivative of φ occurring in (4.1) has the simple fraction decomposition [17]

$$\frac{\varphi'}{\varphi}(s) = \sum_\nu \left(\frac{1}{s-1+s_\nu} - \frac{1}{s-s_\nu} \right) + \sum_{\beta_\alpha < 1/2} \left(\frac{1}{s-1+\tilde{s}_\alpha} - \frac{1}{s-s_\alpha} \right) - 2\ln b_1. \tag{4.5}$$

The summation in $\sum_\mu$ is over all poles s_ν of φ such that $1/2 < s_\nu \leqslant 1$. If $s = 1/2 + ir$, then $(r \in \mathbb{R})$

$$\begin{aligned} &\frac{1}{s-1+s_\nu} - \frac{1}{s-s_\nu} = -(1-2s_\nu)\frac{1}{r^2 + (s_\nu - 1/2)^2} > 0 \qquad (1/2 < s_\nu \leqslant 1), \\ &\frac{1}{s-1+\tilde{s}_\alpha} - \frac{1}{s-s_\alpha} = \frac{1}{ir - a_\alpha^{(1)}} - \frac{1}{ir - a_\alpha^{(2)}} = -\frac{1-2\beta_\alpha}{(r-\gamma_\alpha)^2 + (\beta_\alpha - 1/2)^2} < 0. \end{aligned} \tag{4.6}$$

In the last formula,

$$a_\alpha^{(1)} = 1/2 - \tilde{s}_\alpha, \qquad a_\alpha^{(2)} = s_\alpha - 1/2. \tag{4.7}$$

It was proved in [1], [12] that the series

$$\sum_{\beta_\alpha < 1/2} \frac{1-2\beta_\alpha}{(s-\gamma_\alpha)^2 + (\beta_\alpha - 1/2)^2}$$

converges uniformly on compact sets. We substitute the expressions (4.5) and (4.6) into (4.1), change the order of integration and summation, and obtain

$$J[h|\varphi] = J_0[h|\varphi] + J_1[h|\varphi], \tag{4.8}$$

where

$$J_0[h|\varphi] = -\sum_\nu (1-2s_\nu)\frac{1}{4\pi}\int_{-\infty}^{\infty} \frac{h(r)\,dr}{r^2 + (s_\nu - 1/2)^2} - 2g(0)\ln 2, \tag{4.9}$$

$$J_1[h|\varphi] = \sum_{\beta_\alpha < 1/2} (I(a_\alpha^{(1)}) - I(a_\alpha^{(2)})) \tag{4.10}$$

and

$$I(a) = \frac{1}{4\pi}\int_{-\infty}^{\infty} \frac{h(r)}{ir - a}\,dr.$$

By Parseval's identity,

$$I(a) = \frac{1}{2}\int_{-\infty}^{\infty} g(y)\hat{f}(-y)\,dy,$$

where

$$\hat{f}(-y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{iry}\frac{1}{ir-a}\,dr = \frac{1}{\pi}\int_0^{\infty} \frac{r\sin(ry)}{r^2+a^2}\,dr - \frac{a}{\pi}\int_0^{\infty} \frac{\cos(ry)}{r^2+a^2}\,dr.$$

Since (see [18])

$$\begin{aligned} &\int_0^{\infty} \frac{r\sin(ry)}{r^2+a^2}\,dr = \frac{\pi}{2}e^{-ay}, \qquad \operatorname{Re} a > 0,\ y > 0, \\ &\int_0^{\infty} \frac{r\cos(ry)}{r^2+a^2}\,dr = \frac{\pi}{2a}e^{-ay}, \qquad \operatorname{Re} a > 0,\ y \geqslant 0, \end{aligned}$$

it follows that

$$\begin{aligned} &\hat{f}(-y) = \frac{1}{2}\operatorname{sgn} y\, e^{-|y|a} - \frac{1}{2}e^{-|y|a}, \qquad \operatorname{Re} a > 0, \\ &\hat{f}(-y) = \frac{1}{2}\operatorname{sgn} y\, e^{|y|a} + \frac{1}{2}e^{|y|a}, \qquad \operatorname{Re} a < 0. \end{aligned}$$

It remains to note that $\operatorname{Re} a_\alpha^{(1)} > 0$, $\operatorname{Re} a_\alpha^{(2)} < 0$, $g(y) = g(-y)$, and hence

$$I(a_\alpha^{(1)}) = -\frac{1}{2}\int_0^\infty g(y)e^{-ya_\alpha^{(1)}}\,dy, \qquad I(a_\alpha^{(2)}) = \frac{1}{2}\int_0^\infty g(y)e^{ya_\alpha^{(2)}}\,dy.$$

As a result, we obtain

$$I(a_\alpha^{(1)}) - I(a_\alpha^{(2)}) = -\frac{1}{2}\int_0^\infty g(y)[e^{-y/2+\tilde{s}_\alpha y} + e^{-y/2+s_\alpha y}]\,dy. \tag{4.11}$$

We substitute this expression into (4.10), take into account our definition of $\sum_{(\alpha)}$ ($\sum_{(\alpha)} = \sum_{\beta_\alpha<1/2,\ \gamma_\alpha>0}$), and arrive at the relation

$$\begin{aligned} J_1[h|\varphi] &= -\sum_{(\alpha)}\int_0^\infty g(y)[e^{-y/2+\tilde{s}_\alpha y} + e^{-y/2+s_\alpha y}]\,dy \\ &= -2\sum_{(\alpha)}\int_0^\infty g(y)e^{(\beta_\alpha-1/2)y}\cos(\gamma_\alpha y)\,dy. \end{aligned} \tag{4.12}$$

Lemma 3 (an explicit formula for $\mathscr{P}[h|\varphi]$). *For every cofinite group Γ and every $h \in \{h_S\}$, one has*

$$\mathscr{P}[h|\varphi] = J_1[h|\varphi] + \Delta\mathscr{P}[h|\varphi] + J_0[h|\varphi]. \tag{4.13}$$

Here $J_1[h|\varphi]$ is defined in (4.12), *$\Delta\mathscr{P}[h|\varphi]$ is defined in* (4.2), *and $J_0[h|\varphi]$ is defined in* (4.9).

Proof. The proof amounts to the substitution of the right-hand sides of (4.8) and (4.12) into (4.3). The series $J_1[h|\varphi]$ converges in view of (4.13). □

5. Main Theorem

Theorem 1 proved in the section specifies the explicit form of the functional $\widetilde{\Phi}_\Gamma$ in formula (1.5). First, let us transform the expression (3.20) for $S_P^\infty[g]$.

Lemma 4. *Let the assumptions of Lemma* 1 *be satisfied, and let $\lambda_n \neq 1/4$. Then*

$$S_P^\infty[g] = W[g] + S_P^1[g|\Delta] + S_P^2[g|\varphi] + S_P^3[g|\varphi] - J_1[h|\varphi]. \tag{5.1}$$

Here $J_1[h|\varphi]$ is defined in (4.12), *and*

$$W[g] = -2f(b)\biggl[\sum_{r_j\geqslant 0}\frac{\cos(r_j b)}{r_j^2+1/4} + \sum_{(\alpha)}\frac{\cos(b\gamma_\alpha)}{\gamma_\alpha^2}e^{(\beta_\alpha-1/2)b}\biggr], \tag{5.2}$$

$$S_P^1[g|\Delta] = -\sum_{r_j>0}\frac{1}{r_j^2+1/4}\int_b^\infty \cos(r_j y)\Bigl[-\frac{1}{2}g(y) + 2g^{(2)}(y)\Bigr]\,dy, \tag{5.3}$$

$$\begin{aligned} S_P^2[g|\varphi] = 2\sum_{(\alpha)}\biggl[&g(b)e^{(\beta_\alpha-1/2)b}\sin(\gamma_\alpha b)\Bigl(\frac{\gamma_\alpha}{\beta_\alpha^2+\gamma_\alpha^2} - \frac{1}{\gamma_\alpha}\Bigr) \\ &+ \frac{(\beta_\alpha-1/2)}{\gamma_\alpha^2}g(0) - g(b)\frac{\beta_\alpha^3\cos(b\gamma_\alpha)}{\gamma_\alpha^2(\beta_\alpha^2+\gamma_\alpha^2)}\,e^{(\beta_\alpha-1/2)b}\biggr], \end{aligned} \tag{5.4}$$

$$S_P^3[g|\varphi] = 2\sum_{(\alpha)}\gamma_\alpha^{-2}\int_0^b \frac{d^2}{dy^2}\bigl(e^{(\beta_\alpha-1/2)y}g(y)\bigr)\cos(\gamma_\alpha y)\,dy. \tag{5.5}$$

If $\lambda_n = 1/4$, then one must add $-4k\int_0^b f(y)\,dy$ to the right-hand side of (5.1), *where k is the multiplicity of the eigenvalue $\lambda_n = 1/4$.*

Proof. Consider the integral

$$A_\alpha = \int_B^\infty \left(\frac{x^{s_\alpha}}{s_\alpha} + \frac{x^{\tilde{s}_\alpha}}{\tilde{s}_\alpha} \right) F(x)\, dx,$$

occurring on the right-hand side in (3.20). By definition (3.12) of the function F,

$$A_\alpha = -\frac{2g(b)}{s_\alpha \tilde{s}_\alpha} e^{(\beta_\alpha - 1/2)b} (\beta_\alpha \cos(b\gamma_\alpha) + \gamma_\alpha \sin(b\gamma_\alpha)) - 2\int_b^\infty e^{(\beta_\alpha-1/2)y} \cos(\gamma_\alpha y) g(y)\, dy. \qquad (5.6)$$

It follows from (3.20) in view of (4.12) and (5.6) that

$$S_P^\infty[g] = -\sum_{r_j>0} \frac{1}{(r_j^2+1/4)r_j} \int_b^\infty (\cos(r_j y) + 2r_j \sin(r_j y)) f(y)\, dy + \sum_\alpha B_\alpha - J_1[h|\varphi], \qquad (5.7)$$

where

$$B_\alpha = \frac{2\psi_\alpha(b)}{\beta_\alpha^2 + \gamma_\alpha^2} (\beta_\alpha \cos(b\gamma_\alpha) + \gamma_\alpha \sin(b\gamma_\alpha)) - 2\int_0^b \psi_\alpha(y) \cos(y\gamma_\alpha)\, dy,$$
$$\psi_\alpha(y) = e^{(\beta_\alpha-1/2)y} g(y). \qquad (5.8)$$

We integrate by parts and obtain

$$B_\alpha = 2\cos(b\gamma_\alpha) C_\alpha + 2\sin(b\gamma_\alpha)\psi_\alpha(b) \left(\frac{\gamma_\alpha}{\gamma_\alpha^2 + \beta_\alpha^2} - \frac{1}{\gamma_\alpha} \right) + \frac{2}{\gamma_\alpha^2} \psi_\alpha^{(1)}(0)$$
$$+ \frac{2}{\gamma_\alpha^2} \int_0^b \psi_\alpha^{(2)}(y) \cos(\gamma_\alpha y)\, dy, \qquad C_\alpha = \frac{\psi_\alpha(b)\beta_\alpha}{\beta_\alpha^2+\gamma_\alpha^2} - \frac{\psi_\alpha^{(1)}(b)}{\gamma_\alpha^2}. \qquad (5.9)$$

We take into account definition (5.8) of the function $\psi_\alpha(y)$, use the relation $g^{(1)}(b) = f(b) + g(b)/2$, where $f(y)$ was defined in Lemma 2, and find that

$$C_\alpha = -\frac{f(b) e^{(\beta_\alpha-1/2)b}}{\gamma_\alpha^2} + \psi_\alpha(b) \left(-\frac{\beta_\alpha^3}{\gamma_\alpha^2(\gamma_\alpha^2+\beta_\alpha^2)} \right) \qquad (5.10)$$

and hence

$$B_\alpha = 2\cos(b\gamma_\alpha) \left(-\frac{f(b)e^{(\beta_\alpha-1/2)b}}{\gamma_\alpha^2} - \psi_\alpha(b) \frac{\beta_\alpha^3}{\gamma_\alpha^2(\gamma_\alpha^2+\beta_\alpha^2)} \right)$$
$$+ 2\sin(b\gamma_\alpha)\psi_\alpha(b) \left(\frac{\gamma_\alpha}{\beta_\alpha^2+\gamma_\alpha^2} - \frac{1}{\gamma_\alpha} \right) + \frac{2}{\gamma_\alpha^2} \psi_\alpha^{(1)}(0) + \frac{2}{\gamma_\alpha^2} \int_0^b \psi_\alpha^{(2)} \cos(\gamma_\alpha y)\, dy. \qquad (5.11)$$

Note that since $g^{(1)}(0) = 0$, it follows that

$$\frac{2}{\gamma_\alpha^2} \psi_\alpha^{(1)}(0) = \frac{2(\beta_\alpha - 1/2)}{\gamma_\alpha^2} g(0). \qquad (5.12)$$

of the function f,

$$\int_b^\infty \big(\cos(r_j y) + 2r_j \sin(r_j y) \big) f(y)\, dy$$
$$= 2f(b)\cos(r_j b) + \int_b^\infty \left(-\frac{1}{2} g(y) + 2g^{(2)}(y) \right) \cos(r_j y)\, dy. \qquad (5.13)$$

To obtain the desired result (5.1), it remains to substitute the expressions (5.13) and (5.11) into (5.7). We integrate by parts and find that

$$S_P^1[g|\Delta] = \sum_{r_j>0} \frac{1}{(r_j^2+1/4)r_j} \Bigg\{ \sin(r_j b) \left[-\frac{1}{2} g(b) + 2g^{(2)}(b) \right]$$
$$+ \int_b^\infty \sin(r_j y) \left[-\frac{1}{2} g^{(1)}(y) + 2g^{(3)}(y) \right] dy \Bigg\}. \qquad (5.14)$$

Since

$$\sum_{(\alpha)} \frac{(\beta_\alpha - 1/2)}{\gamma_\alpha^2} = C_\Gamma < \infty, \quad \sum_{(\alpha)} \gamma_\alpha^{-3} < \infty, \quad \sum_{r_j > 0} r_j^{-3} < \infty, \quad g^{(3)} \in L^1[0, \infty], \tag{5.15}$$

we see that the absolute convergence of the series $S_P^1[g|\Delta]$ and $S_P^2[g|\varphi]$ follows from (5.14) and (5.4) and that it suffices to integrate by parts to prove the absolute convergence of the series $S_P^3[g|\varphi]$. Now the convergence of the series $W[g]$ follows from formula (5.1), and $W[g]$ includes all nonabsolutely convergent series on the right-hand side in (5.1).

The proof of Lemma 4 is complete. □

Corollary 1. *For any $b > b_0$ and any cofinite group Γ, the series in the definition of $W[g]$ converges; i.e.,*

$$\sum_{r_j \geqslant 0} \frac{\cos(r_j b)}{r_j^2 + 1/4} + \sum_{(\alpha)} \frac{\cos(b\gamma_\alpha)}{\gamma_\alpha^2} e^{(\beta_\alpha - 1/2)b} = C_\Gamma(b) < \infty. \tag{5.16}$$

Now we are in a position to prove the following theorem.

Theorem 1. *Let a function $h \in \{h\}_S$ satisfy condition* (3.13). *Then*

$$\sum_{n \geqslant 0} h(r_n) = H[h] + G[h] + S_P^1[g|\Delta] + S_P^2[g|\varphi] + S_P^3[g|\varphi] + M[g] \tag{5.17}$$

for any cofinite group Γ and any $B \geqslant B_0$, $B > 2$.

Here

$$G[h] = -\frac{n}{2\pi} \int_{-\infty}^{+\infty} h(r)\psi(1 + ir)\, dr \qquad \left(\psi(x) = \frac{\Gamma'}{\Gamma}(x)\right), \tag{5.18}$$

$$M[g] = W[g] + S_{\mathrm{ex}}[g] + S_R[h] + S_0[g] - \sum_\nu \frac{1 - 2s_\nu}{4\pi} \int_{-\infty}^{+\infty} \frac{h(r)\, dr}{r^2 + (s_\nu - 1/2)^2}$$
$$- \left(n - \operatorname{tr}\Phi\left(\frac{1}{2}\right)\right)\frac{h(0)}{4} - g(0) n \ln 2, \tag{5.19}$$

where the summation in $\sum_\mu$ is over the poles s_ν of φ such that $1/2 < s_\nu \leqslant 1$ $(s_\mu = \sigma_\nu)$.

Proof. The proof amounts to the substitution of (5.1) and (4.13) into (3.19). Note that the expression $J_1[h|\varphi]$ occurs in (5.1) and (4.13) with opposite signs. The proof of Theorem 1 is complete. □

The expression $S_P^1[g|\Delta]$ in (5.17) depends on the behavior of $g(y)$ for $y > b$, while $S_P^3[g|\varphi]$ depend on the behavior of the same function for $0 < y < b$. Since

$$\int_b^\infty g^{(2)}(y)\cos(r_j y)\, dy = -\frac{1}{2} r_j^2 h(r_j) - \int_0^b g^{(2)}(y)\cos(r_j y)\, dy,$$
$$\frac{1}{2}\int_b^\infty g(y)\cos(r_j y)\, dy = \frac{1}{4} h(r_j) - \frac{1}{2}\int_0^b g(y)\cos(r_j y)\, dy,$$

we obtain

$$S_P^1[g|\Delta] = \sum_{r_j > 0} h(r_j) + \sum_{r_j > 0} \frac{1}{r_j^2 + 1/4}\left[\int_0^b \left[-\frac{1}{2} g(y) + 2g^{(2)}(y)\right]\cos(r_j y)\, dy\right], \tag{5.20}$$

which permits rewriting (5.17) in the form

$$\sum_{\lambda_n < 1/4} h(r_n) = H[h] + G[h] + \widetilde{S}_P^1[g|\Delta] + S_P^2[g|\varphi] + S_P^3[g|\varphi] + M[g], \tag{5.21}$$

where

$$\widetilde{S}^1_P[g|\Delta] = \frac{1}{2}\sum_{r_j>0}\frac{1}{(r_j^2+1/4)r_j}\left[-g(b)\sin(r_jb) + \int_0^b g^{(1)}(y)\sin(r_jy)\,dy\right]$$
$$+ 2\sum_{r_j>0}\frac{1}{r_j^2+1/4}\int_0^b g^{(2)}(y)\cos(r_jy)\,dy. \qquad (5.22)$$

Let us transform $S^3_P[g|\varphi]$ in a similar way. The expression (5.5) contains the integral

$$\int_0^b e^{(\beta_\alpha-1/2)y}g^{(2)}(y)\cos(\gamma_\alpha y)\,dy = \int_0^b (e^{(\beta_\alpha-1/2)y}-1)g^{(2)}(y)\cos(\gamma_\alpha y)\,dy$$
$$-\frac{1}{2}\gamma_\alpha^2 h(\gamma_\alpha) - \int_b^\infty g^{(2)}(y)\cos(\gamma_\alpha y)\,dy.$$

We use this expression in (5.5), substitute the result into (5.17), and arrive at the following assertion.

Corollary 2. *Under the assumptions of Theorem* 1, *for every cofinite group one has*

$$\sum_{n\geqslant 0} h(r_n) + \sum_{(\alpha)} h(\gamma_\alpha) = H[h] + G[h] + R[h] + M[g], \qquad (5.23)$$

where

$$R[h] = S^1_P[g|\Delta] + S^2_P[g|\varphi] + \Delta S^3_P[g|\varphi], \qquad (5.24)$$

$$\Delta S^3_P[g|\varphi] = 2\sum_{(\alpha)}\frac{(\beta_\alpha-1/2)^2}{\gamma_\alpha^2}\int_0^b e^{(\beta_\alpha-1/2)y}g(y)\cos(\gamma_\alpha y)\,dy$$
$$+ 4\sum_{(\alpha)}\frac{\beta_\alpha-1/2}{\gamma_\alpha^2}\int_0^b e^{(\beta_\alpha-1/2)y}g^{(1)}(y)\cos(\gamma_\alpha y)\,dy$$
$$+ 2\sum_{(\alpha)}\gamma_\alpha^{-2}\int_0^b (e^{(\beta_\alpha-1/2)y}-1)g^{(2)}(y)\cos(\gamma_\alpha y)\,dy$$
$$- 2\sum_{(\alpha)}\gamma_\alpha^{-2}\int_b^\infty g^{(2)}(y)\cos(\gamma_\alpha y)\,dy. \qquad (5.25)$$

Equation (5.23) specifies the explicit form of the functionals $L[h]$ and $\widetilde{\Phi}_\Gamma[h|\{r_j\},\{\gamma_\alpha\}]$ in formula (1.6) with $L[h] = H[h] + G[h]$.

6. Some Applications

The study of the asymptotics as $t \to 0$ of the series $\sum_n e^{-\lambda_n t}$ is a standard object of spectral theory. It was proved in the book [17] that

$$\sum_{n\geqslant 0} e^{-tr_n^2} - \frac{1}{4\pi}\int_{-\infty}^{+\infty} e^{-tr^2}\frac{\varphi'}{\varphi}\left(\frac{1}{2}+ir\right)dr = \frac{|F|}{4\pi t} - \frac{n\ln(1/t)}{4\sqrt{\pi t}} - \frac{\gamma n}{4\sqrt{\pi t}} + O(1) \qquad (6.1)$$

as $t \to 0$ for any cofinite group Γ.

Let us use formula (5.17) and successively compute the asymptotics as $t \to 0$ of all terms on the right-hand side in this relation for the case in which

$$h(r) = e^{-tr^2}, \qquad g(y) = \frac{1}{2\sqrt{\pi t}}e^{-y^2/4t}. \qquad (6.2)$$

We omit the details of calculations and present the final result

$$\sum_{n\geqslant 0} e^{-tr_n^2} = \frac{|F|}{4\pi t} - \frac{n\ln(1/t)}{4\sqrt{\pi t}} - \sum_{(\alpha)} e^{-t\gamma_\alpha^2} + Q(t) + \frac{C}{\sqrt{\pi t}} + O(1). \qquad (6.3)$$

Here $Q(t)$ is given by the relations

$$\begin{aligned} Q(t) &= -2\sum_{(\alpha)} \frac{\beta_\alpha - 1/2}{\gamma_\alpha} F_\alpha(t) + \frac{3}{\sqrt{\pi t}} \sum_{(\alpha)} \frac{\beta_\alpha - 1/2}{\gamma_\alpha^2} G_\alpha(t) = O(t^{-1/2}), \\ F_\alpha(t) &= \frac{i}{4} e^{-\gamma_\alpha^2 t} [\operatorname{erf}(-i\gamma_\alpha \sqrt{t}\,) - \operatorname{erf}(i\gamma_\alpha \sqrt{t}\,)], \\ G_\alpha(t) &= e^{-\gamma_\alpha^2 t/2} [D_{-4}(-i\sqrt{2t}\,\gamma_\alpha) + D_{-4}(i\sqrt{2t}\,\gamma_\alpha)]. \end{aligned} \tag{6.4}$$

We use the standard notation in [20]; erf z is the error function, and $D_\nu(z)$ is the parabolic cylinder function. Since

$$D_{-m-1}(z) = \frac{\pi}{2} \frac{(-1)^m}{m!} e^{-z^2/4} \frac{d^m}{dz^m} \left(e^{z^2/4} \operatorname{erfc} \frac{z}{\sqrt{2}} \right),$$

we see that $Q(t)$ can be expressed via the error function.

Equation (6.3) is consistent with (5.23) and specifies the explicit form of the first two terms of the asymptotics as $t \to 0$ of $R[h]$ with $h(r) = \exp(-tr^2)$. The first and second terms on the right-hand side in (6.3) correspond to the asymptotics of $H[h]$ and $G[h]$, respectively, while the third and fourth terms correspond to the asymptotics of the function $S_P^3[g|\varphi]$ defined in (5.5).

As a second application, let us show that formula (5.17) implies the Selberg–Weyl formula (1.7). Selberg [3] proved that

$$N(T) - \frac{1}{4\pi} \int_{-T}^{T} \frac{\varphi'}{\varphi} \left(\frac{1}{2} + ir \right) dr \sim \frac{|F|}{4\pi} T^2 \qquad (T \to \infty) \tag{6.5}$$

for any cofinite group Γ. The following refinement of this formula was obtained in [12]:

$$N(T) - \frac{1}{4\pi} \int_{-T}^{T} \frac{\varphi'}{\varphi} \left(\frac{1}{2} + ir \right) dr = \frac{|F|}{4\pi} T^2 - \frac{n}{\pi} T \ln T + CT + O\left(\frac{T}{\ln T} \right). \tag{6.6}$$

One can use (4.5) to prove that

$$N(T) + M(T) = \frac{|F|}{4\pi} T^2 + O(T \ln T), \qquad M(T) = \#\{\alpha \mid 0 < \gamma_\alpha < T\}. \tag{6.7}$$

Theorem 1 readily implies a weakened version of this formula. To this end, consider the case in which

$$\begin{aligned} h(r) = h_{\alpha,T}(r) &= \int_{-\infty}^{\infty} \chi_T(s) \delta_\alpha(r-s)\, ds, \\ \chi_T(r) = \begin{cases} 1, & |r| < T, \\ 0, & |r| > T, \end{cases} & \qquad \delta_\alpha(r) = \sqrt{\frac{\alpha}{\pi}}\, e^{-\alpha r^2}, \end{aligned} \tag{6.8}$$

and hence

$$g(y) = \frac{1}{\pi} \frac{\sin(Ty)}{y}\, e^{-y^2/4\alpha}. \tag{6.9}$$

Let us use Eq. (5.23). It can be shown that one can pass to the limit as $\alpha \to \infty$ on both sides in this equation. This justifies the application of Eq. (5.23) for the case in which

$$h(r) = \chi_T(r), \qquad g(y) = \frac{1}{\pi} \frac{\sin(Ty)}{y}. \tag{6.10}$$

In this case, all the integrals on the right-hand side in (5.23) can readily be computed. Using the estimates

$$\sum_{|\gamma_\alpha - T| < \delta} 1 \ll T\delta, \qquad \sum_{|r_j - T| < \delta} 1 \ll T\delta \qquad (\delta \gtrsim T^{1/2}) \tag{6.11}$$

and taking $\delta \simeq T^{1/2}$, we find that

$$N(T) + M(T) = \frac{|F|}{4\pi} T^2 + O(T^{3/2}), \tag{6.12}$$

and this formula can be refined.

Inclosing, note that, in the case of the modular group ($\Gamma = \mathrm{SL}(2,\mathbb{Z})$), one has $s_\alpha = \rho_{\alpha/2}$, where the ρ_α are the nontrivial zeros of the Riemann zeta function. Using the results of the present paper, the author has been able to prove (see [21]) that the distribution of primes can be reconstructed from the discrete spectrum of the Laplace operator for $\Gamma = \mathrm{SL}(2,\mathbb{Z})$.

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References

[1] A. B. Venkov, "Spectral theory of automorphic functions," Trudy Mat. Inst. Steklov, **153** (1981), 3–171; English transl.: Proc. Steklov Inst. Math., **153** (1982), 1–163.

[2] P. Sarnak, "Spectra of hyperbolic surfaces," Bull. Amer. Math. Soc., **40**:4 (2003), 441–478.

[3] A. Selberg, Harmonic Analysis, Teil 2, Vorlesungniederschrift, Göttingen, 1954.

[4] A. B. Venkov, "On essentially cuspidal noncongruence subgroups of $\mathrm{PSL}(2,\mathbb{Z})$," J. Funct. Analys., **92**:1 (1990), 1–7.

[5] J. M. Deshouillers, H. Iwaniec, R. S. Phillips, and P. Sarnak, "Maass cusp forms," Proc. Nat. Acad. Sci. USA, **82**:11 (1985), 3533–3534.

[6] R. S. Phillips and P. Sarnak, "On cusp forms for co-finite subgroups of $\mathrm{PSL}(2,\mathbb{R})$," Invent. Math., **80**:2 (1985), 339–364.

[7] S. A. Wolpert, "Spectral limits for hyperbolic surfaces I, II," Invent. Math., **108**:1 (1992), 67–89, 91–129.

[8] S. A. Wolpert, "Disappearance of cups forms in special families," Ann. Math., **139**:2 (1994), 239–291.

[9] W. Luo, "Nonvanishing of L-values and the Weil law," Ann. Math., **154**:2 (2001), 477–502.

[10] D. A. Popov, "On the Selberg trace formula for strictly hyperbolic groups," Funkts. Anal. Prilozhen., **47**:4 (2013), 53–66; English transl.: Functional Anal. Appl., **47**:4 (2013), 290–301.

[11] D. A. Popov, Selberg formula for cofinite groups and the Roelke conjecture, `https://arxiv.org/abs/1705.05612v4`.

[12] A. B. Venkov, "Remainder term in the Weyl–Selberg asymptotic formula," Zap. Nauchn. Sem. LOMI, **91** (1979), 5–24; English transl.: J. Soviet Math., **17**:5 (1981), 2083–2097.

[13] A. B. Venkov, "Spectral theory of automorphic functions, the Selberg zeta-function, and some problems of analytic number theory and mathematical physics," Uspekhi Mat. Nauk, **34**:3 (1979), 69–135; English transl.: Russian Math. Surveys, **34**:3 (1979), 79–153.

[14] D. A. Hejhal, "The Selberg trace formula and the Riemann zeta function," Duke Math. J., **43**:3 (1976), 441–482.

[15] D. A. Hejhal, The Selberg Trace Formula for $\mathrm{PSL}(2,\mathbb{R})$, vol. 1, Lect. Notes in Math., vol. 548, Springer-Verlag, Berlin–New York, 1976.

[16] D. A. Hejhal, The Selberg Trace Formula for $\mathrm{PSL}(2,\mathbb{R})$, vol. 2, Lect. Notes in Math., vol. 1001, Springer-Verlag, Berlin, 1983.

[17] H. Iwaniec, Introduction to the Spectral Theory of Automorphic Forms, Revista Matemática Iberoamericana, Madrid, 1995.

[18] I. S. Gradshtein and I. M. Ryzhik, Tables of Integrals, Series, and Products, Academic Press, New York–London, 1965.

[19] H. Bateman and A. Erdélyi, Higher Transcendental Functions, vol. 2, McGraw–Hill, New York–Toronto–London, 1953.

[20] M. Abramowitz and I. Stegun, eds., Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, National Bureau of Standards, Washington, D.C., 1964.

[21] D. A. Popov, “Discrete spectrum of the Laplace operator on the fundamental domain of the modular group and the Chebyshev psi function,” Izv. Ross. Akad. Nauk Ser. Mat., **83**:5 (2019).
[22] W. Abikoff, The real analytic theory of Teichmüller space, Lect. Notes in Math., vol. 820, Springer-Verlag, Berlin, 1980.

Lomonosov Moscow State University, Moscow, Russia
Belozersky Research Institute of Physical-Chemical Biology,
Moscow State University, Moscow, Russia
e-mail: Popov-Kupavna@yandex.ru

Translated by V. E. Nazaikinskii